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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 42

LONDON:
PRINTED AND PUBLISHED FOR THE SOCIETY BY
C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C. 2

1937

BIBL.
LONDIN.
UNIV.

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PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

NULLSTELLEN EPSTEINSCHER ZETAFUNKTIONEN

Von H. Kober.

[Received 22 January, 1936.—Read 20 February, 1936.]

1. Das Ziel dieser Arbeit ist der Beweis des folgenden

Hauptsatzes. *Es sei* $Q(x, y) = ax^2+bxy+cy^2$ *irgend eine reelle, positiv-definite quadratische Form,* $s = \sigma+it$, *für* $\sigma > 1$ *sei*

$$Z(s) = \sum_{m,\,n}' \{Q(m, n)\}^{-s} = \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ m^2+n^2>0}}^{\infty} \{Q(m, n)\}^{-s},$$

dann hat $Z(s)$ *auf der kritischen Geraden* $s = \frac{1}{2}+it$ *unendlich viele Nullstellen.*

Der Beweis stützt sich unter anderem auf folgenden

Hilfssatz. *Es sei* $-\Delta = b^2-4ac$ *die Discriminante von* $Q(x, y)$, $\epsilon > 0$, *für* $\Re(z) > 0$ *sei*

$$\Theta(z) = \sum_{m,\,n} e^{-2\pi\Delta^{-\frac{1}{2}}Q(m,\,n)\,z} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-2\pi\Delta^{-\frac{1}{2}}Q(m,\,n)\,z};$$

(I) *ist mindestens eine der drei Zahlen* $a\Delta^{-\frac{1}{2}}$, $b\Delta^{-\frac{1}{2}}$, $c\Delta^{-\frac{1}{2}}$ *irrational, so ist für* $\epsilon \to 0$

$$\Theta(e^{i(\frac{1}{2}\pi-\epsilon)}) = o(\epsilon^{-1});$$

(II) *sind alle drei Zahlen rational und ist k die kleinste positive Zahl, für welche folgende drei Zahlen:*

$$\alpha = ka\Delta^{-\frac{1}{2}}, \quad \tilde{\beta} = kb\Delta^{-\frac{1}{2}}, \quad \gamma = kc\Delta^{-\frac{1}{2}}$$

gleichzeitig ganz sind, so ist k ganz und $\equiv 0$ *oder* 2 (mod 4), *ist ferner η eine bestimmte komplexe Zahl mit absolutem Wert* $= 1$, *so ist*

$$\begin{aligned}\Theta(e^{i(\frac{1}{2}\pi-\epsilon)}) &= \eta\sqrt{\left(\frac{2}{k}\right)}\frac{1}{\epsilon}+O\left(\frac{1}{\sqrt{\epsilon}}\right) \quad \textit{für} \quad k\equiv 0 \pmod 4,\\ &= \qquad\qquad\qquad O\left(\frac{1}{\sqrt{\epsilon}}\right) \quad \textit{für} \quad k\equiv 2 \pmod 4.\end{aligned}$$

Dass k rational ist, folgt aus $k = \alpha:(a\Delta^{-\frac{1}{2}})$, dass k ganz ist, aus

$$4\alpha\gamma-\tilde{\beta}^2 = k^2\Delta^{-1}(4ac-b^2) = k^2;$$

ferner folgt aus der letzten Gleichung, dass k und $\tilde{\beta}$ gerade sein müssen, also $k \equiv 0$ oder 2 (mod 4), $\tilde{\beta} = 2\beta$, und aus der Minimaleigenschaft von k, dass $(\alpha, \tilde{\beta}, \gamma) = 1$ und $(\alpha, \beta, \gamma) = 1$ ist. Potter und Titchmarsh* haben bereits bewiesen, dass im allgemeinen $Z(s)$ unendlich viele Nullstellen auf der Geraden $\sigma = \frac{1}{2}$ hat, und dass in allen diesen Fällen die Anzahl dieser Nullstellen im Intervall $0 < t \leqslant T$

$$N_0(T) > KT^{\frac{1}{2}-\epsilon}$$

für jedes $\epsilon > 0$ und $T > T_0(\epsilon)$, $K = K(\epsilon)$ ist. Folgende Fälle sind durch ihren Beweis nicht erfasst bezw. nicht allgemein erfasst:

(1) $a\Delta^{-\frac{1}{2}}$ und $c\Delta^{-\frac{1}{2}}$ rational, $b\Delta^{-\frac{1}{2}}$ irrational.

(2) Für den Fall, dass a, b und c ganz, $(a, b, c) = 1$, $\Delta^{\frac{1}{2}}$ ganz und $\equiv 0$ (mod 4) ist, haben sie ein Kriterium aufgestellt, das in jedem einzelnen Falle durchzurechnen ist. Das Kriterium lässt sich nun aber sehr vereinfachen und dann allgemein anwenden.

Der Beweis des Hauptsatzes erfolgt hier auf einem etwas anderen, einfacheren Wege; das Verfahren entspricht der ursprünglichen, von Hardy für $\zeta(s)$ entwickelten Methode in der ersten† von Landau vorgenommenen Abänderung und beruht auf der Untersuchung der Laplace-

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* H. S. A. Potter and E. C. Titchmarsh, "The zeros of Epstein's zeta functions", *Proc. London Math. Soc.* (2), 39 (1935), 372–384.

† E. Landau, "Über die Hardysche Entdeckung unendlich vieler Nullstellen . . .", *Math. Annalen*, 76 (1915), 212–243, s. 229 u.f., bes. 232.

Transformation von $|Z(\frac{1}{2}+it)|$ und der Betrachtung der Θ-Reihe in einer gewissen Nachbarschaft des singulären Punktes $z=i$. Es hat auch den Vorzug, dass in den Fällen $\Delta > 2\pi^2 a^2$ der Satz bereits ganz kurz aus einer groben Abschätzung von Θ folgt, ohne Benutzung des Hilfssatzes; es hat den grossen *Nachteil*, dass es die Abschätzung für $N_0(T)$ nach unten nicht liefert*.

Der Beweis wäre bedeutend leichter, wenn die für $\zeta(s)$ gültige Beziehung

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T |\zeta(\tfrac{1}{2}+it)|\, dt = \infty,$$

welche aus Inghams Untersuchungen† folgt, auch für $Z(s)$ gelten würde. Bisher ist aber, ausser im Falle $Q=x^2+y^2$, nicht einmal etwas bekannt über‡

$$\int_0^T |Z(\tfrac{1}{2}+it)|^2\, dt.$$

2. Es sei λ *irgend eine* durch die Form $Q(m, n)$ darstellbare Zahl, $r(\lambda)$ die Anzahl der Darstellungen, dann ist§

$$\int_0^T \lambda^{\frac{1}{2}+it} Z(\tfrac{1}{2}+it)\, dt = r(\lambda)\, T + O(T^{\frac{1}{2}+\delta})$$

für jedes $\delta > 0$; mit Hilfe partieller Integration folgt für $\epsilon \to 0$

$$\int_0^\infty e^{-\epsilon t} \lambda^{\frac{1}{2}+it} Z(\tfrac{1}{2}+it)\, dt = \int_1^\infty + \int_0^1 = \frac{r(\lambda)}{\epsilon} + O(\epsilon^{-\frac{1}{2}-\delta});$$

$$\int_0^\infty e^{-\epsilon t} |Z(\tfrac{1}{2}+it)|\, dt \geqslant r(\lambda)\, \lambda^{-\frac{1}{2}} \epsilon^{-1} + O(\epsilon^{-\frac{1}{2}-\delta}). \tag{1}$$

3. Es ist auf Grund der für $\sigma > 0$ gültigen Gleichung

$$\Gamma(s)(2\pi Q \Delta^{-\frac{1}{2}})^{-s} = \int_0^\infty x^{s-1} e^{-2\pi\Delta^{-\frac{1}{2}} Q(m,\, n)\, x}\, dx:$$

$$Z(s)\,\Gamma(s)(2\pi\Delta^{-\frac{1}{2}})^{-s} = \int_0^\infty x^{s-1} \sum_{m,\, n}{}' e^{-2\pi\Delta^{-\frac{1}{2}} Q x}\, dx = \int_0^\infty x^{s-1} \{\Theta(x) - 1\}\, dx$$

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* Den Beweis hatte ich im wesentlichen unmittelbar vor dem Erscheinen der P.-T. Arbeit fertig gestellt mit erheblicher Hilfe von Dr. K. Silberberg, Breslau, ich spreche hier meinen besten Dank aus. Ferner danke ich Dr. H. Heilbronn, Cambridge.

† A. E. Ingham, "Mean value theorems and the Riemann zeta-function", *Quart. J. of Math.* (Oxford), 4 (1933), Theorem A u. 1, 278 u.f.

‡ Mit diesem Mittelwerte beschäftigt sich eine weitere Arbeit der Verfassers.

§ Ähnlich zu beweisen wie bei Landau, 242, oder Potter und Titchmarsh, Lemma 6.

für $\sigma > 1$, also nach Mellin's Umkehrungsformel für $\sigma > 1$

$$\Theta(x) - 1 = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} Z(s)\, \Gamma(s) (2\pi \Delta^{-\frac{1}{2}} x)^{-s}\, ds.$$

Man verschiebe den Integrationsweg in die Gerade $\sigma = \frac{1}{2}$ und berücksichtige, dass

$$Z(s) \sim 2\pi \Delta^{-\frac{1}{2}} (s-1)^{-1} \text{ für } s \to 1; \quad |\Gamma(s)| = O(|t|^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi |t|}) \text{ für } |t| \to \infty$$

ist, und erhält so die für $\Re(x) > 0$ gültige Gleichung

$$\Theta(x) - 1 - \frac{1}{x} = \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} Z(s)\, \Gamma(s) (2\pi \Delta^{-\frac{1}{2}} x)^{-s}\, ds.$$

Man setze
$$x = \exp\left(\tfrac{1}{2}\pi i - i\epsilon\right),$$
$$\rho(t) = (2\pi)^{-\frac{1}{2} - it} \Delta^{\frac{1}{4} + \frac{1}{2} it} Z(\tfrac{1}{2} + it)\, \Gamma(\tfrac{1}{2} + it),$$

dann ist, auf Grund der Funktionalgleichung von $Z(s)$, $\rho(t) = \rho(-t)$, also für reelles t reell, ferner auf Grund einer bekannten asymptotischen Darstellung von $|\Gamma(\sigma + it)|$

$$|\rho(t)| = \Delta^{\frac{1}{4}} |Z(\tfrac{1}{2} + it)|\, e^{-\frac{1}{2}\pi |t|} \left\{1 + O\left(\frac{1}{t^2}\right)\right\} \quad \text{für} \quad |t| \to \infty;$$

$$\Theta(e^{\frac{1}{2}\pi i - i\epsilon})\, e^{\frac{1}{4}\pi i - \frac{1}{2}\epsilon i} - 2\cos\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\epsilon\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(t)\, e^{(\frac{1}{2}\pi - \epsilon)t}\, dt$$
$$= \frac{1}{2\pi} \int_{0}^{\infty} \rho(t)\, e^{(\frac{1}{2}\pi - \epsilon)t}\, dt + O(1).$$

Wäre nun der Hauptsatz falsch, dann gäbe es ein $t_0 > 0$, so dass $\rho(t)$ für $t > t_0$ ein festes Vorzeichen hätte, und aus den letzten beiden Gleichungen würde dann unschwer folgen:

$$\int_0^{\infty} |Z(\tfrac{1}{2} + it)|\, e^{-\epsilon t}\, dt = 2\pi \Delta^{-\frac{1}{4}} |\Theta(e^{\frac{1}{2}\pi i - i\epsilon})| + O(1). \tag{2}$$

4. *Der Fall* $\Delta > 2\pi^2 a^2$, d.h. $a < \Delta^{\frac{1}{2}} 2^{-\frac{1}{2}} \pi^{-1}$. Es ist

$$\Theta(e^{\frac{1}{2}\pi i - i\epsilon}) = \sum_n e^{-(\pi/2a)\Delta^{\frac{1}{2}} e^{\frac{1}{2}\pi i - i\epsilon} n^2} \sum_m e^{-2\pi a \Delta^{-\frac{1}{2}} e^{\frac{1}{2}\pi i - i\epsilon} \{m + (b/2a) n\}^2} \tag{3}$$
$$= \left(\frac{\Delta}{4a^2}\right)^{\frac{1}{4}} e^{-\frac{1}{4}\pi i + \frac{1}{2}\epsilon i} \sum_n e^{-(\pi/2a)\Delta^{\frac{1}{2}} e^{\frac{1}{2}\pi i - i\epsilon} n^2}$$
$$\times \sum_m e^{-(\pi/2a)\Delta^{\frac{1}{2}} e^{-\frac{1}{2}\pi i + i\epsilon} m^2} \cos\frac{\pi}{a} bmn$$

auf Grund einer bekannten Transformation der ϑ-Reihe*. Die letzte innere Summe ist, gleichmässig in n, absolut

$$\leqslant \sum_m \exp\left(-m^2 \Delta^{\frac{1}{2}} \frac{\pi}{2a} \sin \epsilon\right) = \Delta^{-\frac{1}{4}}(2a)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} + O(1), \tag{3, 1}$$

die äussere ebenfalls, also würde aus (2) folgen: Für $\epsilon \to 0$ ist

$$\int_0^\infty |Z(\tfrac{1}{2}+it)| e^{-\epsilon t} dt \leqslant 2\pi \left(\frac{2a}{\Delta}\right)^{\frac{1}{2}} \epsilon^{-1} + O(\epsilon^{-\frac{1}{2}}),$$

im Widerspruch zu (1) für $\lambda = a$, also $r(\lambda) \geqslant 2$, im Falle $\Delta > 2\pi^2 a^2$.

5. Zur Erfassung aller Fälle ist die scharfe Abschätzung des Hilfssatzes (siehe 1) für Θ notwendig. Es sei $a\Delta^{-\frac{1}{2}}$ irrational, also auch $\xi = \Delta^{\frac{1}{2}}(2a)^{-1}$. Dann ist† gleichmässig in ϑ

$$s_N(\xi, \vartheta) = \sum_{\nu=1}^{N} e^{\nu^2 \pi i \xi} \cos 2\pi\nu\vartheta = o(N) \quad \text{für} \quad N \to \infty.$$

Wenn l eine Quadratzahl ist, sei $a_l = \cos(2\pi\vartheta\sqrt{l}) \exp(l\pi i \xi)$, sonst $= 0$, also

$$S(n) = \sum_{l=1}^{n} a_l = o(\sqrt{n}). \tag{4, 1}$$

Setzt man $\vartheta = nb(2a)^{-1}$,

$$q = \exp(-2\pi\xi \sin \tfrac{1}{2}\epsilon\, e^{\frac{1}{2}\epsilon i}) = 1 - \pi\Delta^{\frac{1}{2}}(2a)^{-1}\epsilon + O(\epsilon^2), \tag{4, 2}$$

so ist die letzte innere Summe von (3)

$$= \sum_m e^{m^2 \pi i \xi} q^{m^2} \cos 2m\pi\vartheta = \sum_m a_m q^m = 1 + 2\sum_1^\infty a_n q^n = 1 + 2(1-q)\sum_1^\infty S_n q^n$$

$$= o\{|1-q|(1-|q|)^{-\frac{3}{2}}\} = o(\epsilon^{-\frac{1}{2}})$$

unter Berücksichtigung von (4, 1) u. (4, 2). Die äussere Summe in (3) ist wegen (3, 1) $= O(\epsilon^{-\frac{1}{2}})$, so folgt

$$\Theta\{\exp(\tfrac{1}{2}\pi i - i\epsilon)\} = o(\epsilon^{-1}). \tag{4}$$

Falls $c\Delta^{-\frac{1}{2}}$ irrational ist, vertausche man a und c; sind aber $a\Delta^{-\frac{1}{2}}$ und $c\Delta^{-\frac{1}{2}}$ rational, $b\Delta^{-\frac{1}{2}}$ irrational, so bilde man die Θ-Reihe über der zu $Q(x, y)$ äquivalenten Form $\tilde{Q}(x', y')$, in welche Q durch die Substitution $x = x'$, $y = x' + y'$ übergeht und deren erster Koefficient $\tilde{a} = a + b + c$

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* Siehe, z.B., Landau, *Handbuch*, § 127.

† G. H. Hardy and J. E. Littlewood, *Acta Math.*, 37 (1914), 193–238.

sicher zu $\Delta^{\frac{1}{2}}$ in keinem rationalen Verhältnis steht. So ist der (I) Teil des Hilfssatzes bewiesen.

6. Es seien jetzt $a\Delta^{-\frac{1}{2}}$, $b\Delta^{-\frac{1}{2}}$ und $c\Delta^{-\frac{1}{2}}$ rational, k, α, β, γ mögen die im Abschnitt 1 festgelegte Bedeutung haben, es sei $k \equiv 2 \pmod 4$. Statt über Q bilde man die Θ-Reihe über

$$\tilde{Q}(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2 = k\Delta^{-\frac{1}{2}}\, Q(x, y),$$

deren Discriminante $-k^2$ ist. Es ist*

$$S(n) = \sum_{\tilde{Q} \leqslant n} \exp(-2\pi i \tilde{Q} k^{-1}) = O(\sqrt{n}),$$

hieraus folgt durch partielle Summation ähnlich wie im Teil 5

$$\Theta\{\exp(\tfrac{1}{2}\pi i - i\epsilon)\} = O(\epsilon^{-\frac{1}{2}}) \quad [k \equiv 2 \pmod 4], \tag{5}$$

also die zweite Behauptung des (II) Teiles des Hilfssatzes.

Es sei nun $k \equiv 0 \pmod 4$. Dann ist*

$$S(n) = \sum_{\tilde{Q} \leqslant n} \exp(-2\pi i \tilde{Q} k^{-1}) = 2\pi \sigma_k k^{-2} n + O(\sqrt{n}), \tag{6, 1}$$

$$|\sigma_k|^2 = \sum_{m=1}^{k} \sum_{n=1}^{k} e^{-2\pi i \{\tilde{Q}(m, n)/k\}} \tag{6, 2}$$

für
$$\alpha m + \beta n \equiv 0 \pmod{\tfrac{1}{2}k}, \quad \beta m + \gamma n \equiv 0 \pmod{\tfrac{1}{2}k}.$$

Es wird bewiesen werden, dass $|\sigma_k|^2 = 2k$, also $\sigma_k = \eta \sqrt{(2k)}$ ist, wobei $|\eta| = 1$, und hieraus folgt durch partielle Summation die vorletzte Behauptung des Hilfssatzes,

$$\Theta\{\exp(\tfrac{1}{2}\pi i - i\epsilon)\} = \eta \left(\frac{2}{k}\right)^{\frac{1}{2}} \epsilon^{-1} + O(\epsilon^{-\frac{1}{2}}) \quad [k \equiv 0 \pmod 4]. \tag{6}$$

7. $|\sigma_k|^2 = 2k$. *Beweis*: Jedes Glied der Summe (6, 2) ist 1, da $k \mid \tilde{Q}$. Denn zunächst ist auf Grund der Bedingungen für m und n in (6, 2) $\tilde{Q}(m, n) = m(\alpha m + \beta n) + n(\beta m + \gamma n)$ durch $\frac{1}{2}k$ teilbar. Es sei $k = P \cdot 2^r$, P ungerade, also $r \geqslant 2$. Um $k \mid \tilde{Q}$ zu beweisen, braucht man also nur $2^r \mid \tilde{Q}$ zu zeigen. Da (siehe 1) $\alpha\gamma - \beta^2 = \frac{1}{4}k^2$ gerade ist und $(\alpha, \beta, \gamma) = 1$, muss mindestens eine der beiden Zahlen α und γ ungerade sein, etwa α. Nun ist

$$\alpha\tilde{Q} = \alpha m(\alpha m + \beta n) + n(\alpha\beta m + \alpha\gamma n) = (\alpha m + \beta n)^2 + \tfrac{1}{4}k^2 n^2.$$

Es ist daher, mit Rücksicht auf die Bedingungen von (6, 2), jedes der beiden Glieder der rechten Seite durch $\frac{1}{4}k^2$, also durch 2^{2r-2}, also auch durch 2^r teilbar, da $2r-2 \geqslant r$; daher ist $2^r \mid \tilde{Q}$, $k \mid \tilde{Q}$.

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* Potter und Titchmarsh, Lemma 7 und 8.

Die Anzahl der zulässigen Wertepaare m, n in (6, 2) ist $2k$. Es sei nämlich $(\beta, \frac{1}{2}k) = v \geqslant 1$, dann folgt aus $\alpha\gamma - \beta^2 = \frac{1}{4}k^2$ und $(\alpha, \beta, \gamma) = 1$, dass v, α, β, γ folgende Form haben müssen:

$$(7,1) \quad v = fg, \quad \alpha = f^2\alpha_1, \quad \beta = fg\beta_1, \quad \gamma = g^2\gamma_1, \quad k = 2fgk_1, \quad \alpha_1\gamma_1 - \beta_1{}^2 = k_1{}^2,$$

wobei

$$(7,2) \quad (f, g) = (\alpha_1, g) = (\gamma_1, f) = 1, \quad (\beta_1, k_1) = (\alpha_1, k_1) = (\gamma_1, k_1) = 1.$$

Die Kongruenzbedingungen von (6, 2) lauten jetzt

$$(7,3) \qquad \alpha_1 fm + \beta_1 gn \equiv 0 \pmod{gk_1}, \quad \beta_1 fm + \gamma_1 gn \equiv 0 \pmod{fk_1}.$$

Hieraus und aus (7, 2) sowie aus $1 \leqslant m \leqslant k$, $1 \leqslant n \leqslant k$ folgt:

$$(7,4) \qquad m = gm_1, \quad n = fn_1; \quad 1 \leqslant m_1 \leqslant 2fk_1, \quad 1 \leqslant n_1 \leqslant 2gk_1,$$

$$(7,5) \quad f(\alpha_1 m_1 + \beta_1 n_1) \equiv 0 \pmod{k_1}, \quad g(\beta_1 m_1 + \gamma_1 n_1) \equiv 0 \pmod{k_1}.$$

Nun folgt aber aus (7, 5), (7, 2) und aus der Gleichung

$$(7,6) \qquad \gamma_1(\alpha_1 m_1 + \beta_1 n_1)^2 - \alpha_1(\beta_1 m_1 + \gamma_1 n_1)^2 = k_1{}^2(\alpha_1 m_1{}^2 - \gamma_1 n_1{}^2):$$

$$(7,7) \quad g(\alpha_1 m_1 + \beta_1 n_1) \equiv 0 \pmod{k_1}, \quad f(\beta_1 m_1 + \gamma_1 n_1) \equiv 0 \pmod{k_1}.$$

Da aber $(f, g) = 1$ ist, folgt aus (7, 5) und (7, 7)

$$(7,8) \quad \text{(A)} \ \alpha_1 m_1 + \beta_1 n_1 \equiv 0 \pmod{k_1}, \quad \text{(B)} \ \beta_1 m_1 + \gamma_1 n_1 \equiv 0 \pmod{k_1},$$

ferner ergibt sich aus (7, 6) und (7, 2) die völlige Gleichwertigkeit der letzten beiden Kongruenzen; man betrachte etwa die erste, es sei u eine Lösung von $\alpha_1 u \equiv 1 \pmod{k_1}$, dann ist (A) gleichwertig mit $m_1 \equiv -\beta_1 u n_1 \pmod{k_1}$, diese Kongruenz aber hat für jedes n_1 in m_1 genau $2f$ Lösungen wegen (7, 4), für n_1 sind $2gk_1$ Werte zulässig, also ist die Gesamtzahl der zulässigen Wertepaare von m_1, n_1 und also auch von m, n gleich $2f \cdot 2gk_1 = 2k$.

8. *Anwendung auf* $Z(s)$. Aus (2) und (4) folgt sofort der gewünschte Widerspruch zu (1), ebenso aus (2) und (5). Im Falle $k \equiv 0 \pmod 4$ würde aus (1), (2), und (6) folgen

$$(8) \qquad \frac{r(\lambda)}{\sqrt{\lambda}} \frac{1}{\epsilon} + o\left(\frac{1}{\epsilon}\right) \leqslant \int_0^\infty e^{-\epsilon t} \left| \tilde{Z}(\tfrac{1}{2}+it) \right| dt \leqslant \frac{2\pi}{\sqrt{k}} \left(\frac{2}{k}\right)^{\frac{1}{2}} \frac{1}{\epsilon} + o\left(\frac{1}{\epsilon}\right),$$

wenn man anstelle von $Z(s)$ die Funktion $\tilde{Z} = k^{-s} \Delta^{\frac{1}{2}s} Z(s)$ betrachtet, da $\tilde{Q}$ die Discriminante $-k^2$ hat.

Der Widerspruch ergibt sich also sofort, falls

$$r(\lambda)\,\lambda^{-\frac{1}{2}} > 2\pi\sqrt{2}\,k^{-1} \tag{8, 1}$$

ist. Wählt man $\lambda = a$, $r(\lambda) \geqslant 2$, die Form als eine reduzierte, also $a \leqslant k 3^{-\frac{1}{2}}$, so ist die Bedingung (8, 1) sicher erfüllt für $k > 11{,}7 > 2\pi^2\,3^{-\frac{1}{2}}$. Es bleiben also nur die beiden Fälle $k = 4$ und $k = 8$ übrig, die entsprechenden Formen sind

$$\tilde{Q}_1(x, y) = x^2 + 4y^2; \quad \tilde{Q}_2(x, y) = 4x^2 + 4xy + 5y^2, \quad \tilde{Q}_3 = x^2 + 16y^2.$$

Aber für $Z_{Q_1}(s)$ ist der Satz auch richtig*, im Falle von $\tilde{Q}_2$ wähle man $\lambda = 5$, $r(\lambda)$ also $= 4$, dann ist die Bedingung (8, 1) wieder erfüllt, ebenso wie für $\lambda = 1$ im Falle von $\tilde{Q}_3$. So ist der Hauptsatz bewiesen.

9 Frobenstr., Breslau.

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* Potter und Titchmarsh, 383, Zeile 16.

SOME FUNCTIONS WHICH ARE SELF-RECIPROCAL IN THE HANKEL-TRANSFORM

By R. S. VARMA.

[Received 20 April, 1936.—Read 23 April, 1936.]

1. The object of this paper is to investigate some parabolic cylinder functions which are self-reciprocal in the Hankel-transform. Two such functions are obtained in § 3 by the help of the infinite integral involving the parabolic cylinder function evaluated in § 2. Again, we know that a generalization of the parabolic cylinder function is given by Sonine's polynomial*

$$T_m{}^n(x) = \sum_{r=0}^{n} \frac{(-1)^r x^{n-r}}{r!\,(n-r)!\,\Gamma(n+m-r+1)}, \tag{1.1}$$

since, for $m = -\frac{1}{2}$ and $m = \frac{1}{2}$, (1.1) reduces to

$$T^n_{-\frac{1}{2}}(x^2) = \frac{2^n}{2n!\,\sqrt{\pi}}\, e^{\frac{1}{2}x^2} D_{2n}(x\sqrt{2}) \tag{1.11}$$

and

$$xT_{\frac{1}{2}}{}^n(x^2) = \frac{2^{n+\frac{1}{2}}}{(2n+1)!\,\sqrt{\pi}}\, e^{\frac{1}{2}x^2} D_{2n+1}(x\sqrt{2}), \tag{1.12}$$

respectively. In § 4, an operational image for Sonine's polynomial and also an infinite integral involving it are given. This infinite integral is then used in § 5 to illustrate the method of § 3 in showing that the function $x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2} T_\nu{}^n(x^2)$ is self-reciprocal in the Hankel-transform when n is even and that it is self-reciprocal in the skew Hankel-transform when n is odd, a theorem proved otherwise by Wilson†.

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* N. J. Sonine, "Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries", *Math. Annalen*, 16 (1880), 1–80.

† B. M. Wilson, "On an extension of Milne's integral equation", *Messenger of Math.*, 53 (1923–4), 157–160.

2. Let
$$f(s)=\int_0^\infty x^{s-1}e^{\frac14 x^2}D_n(x)\,dx. \tag{2.1}$$

Using Whittaker's integral* for $D_n(x)$, viz.
$$D_n(x)=\frac{1}{\Gamma(-n)}e^{-\frac14 x^2}\int_0^\infty e^{-tx-\frac12 t^2}t^{-n-1}\,dt \qquad [R(n)<0],$$

we have
$$\begin{aligned} f(s) &= \frac{1}{\Gamma(-n)}\int_0^\infty x^{s-1}\,dx\int_0^\infty e^{-tx-\frac12 t^2}t^{-n-1}\,dt \\ &= \frac{1}{\Gamma(-n)}\int_0^\infty e^{-\frac12 t^2}t^{-n-1}\,dt\int_0^\infty x^{s-1}e^{-tx}\,dx \\ &= \frac{\Gamma(s)}{\Gamma(-n)}\int_0^\infty t^{-s-n-1}e^{-\frac12 t^2}\,dt \qquad [R(s)>0] \\ &= \frac{\Gamma(s)}{2^{\frac12 n+\frac12 s+1}\Gamma(-n)}\int_0^\infty e^{-\rho}\rho^{-\frac12 n-\frac12 s-1}\,d\rho \\ &= \frac{\Gamma(s)\,\Gamma(-\frac12 n-\frac12 s)}{2^{\frac12 n+\frac12 s+1}\Gamma(-n)}, \qquad [R(n+s)<0], \end{aligned}$$

since
$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt \quad [R(x)>0],$$

the change in the order of integration being obviously permissible.

Hence we have
$$f(s)=\frac{\Gamma(s)\,\Gamma(-\frac12 n-\frac12 s)}{2^{\frac12 n+\frac12 s+1}\Gamma(-n)}, \tag{2.2}$$

where $0<R(s)<R(-n)$.

3. If $x^\lambda e^{\frac14 x^2}D_n(x)$ is R_ν, so that†
$$x^\lambda e^{\frac14 x^2}D_n(x)=\int_0^\infty \sqrt{(xy)}\,J_\nu(xy)\,y^\lambda e^{\frac14 y^2}D_n(y)\,dy, \tag{3.1}$$

we have
$$\begin{aligned} f(\lambda+s) &= \int_0^\infty x^{\lambda+s-1}e^{\frac14 x^2}D_n(x)\,dx \\ &= \int_0^\infty x^{s-1}\,dx\int_0^\infty \sqrt{(xy)}\,J_\nu(xy)\,y^\lambda e^{\frac14 y^2}D_n(y)\,dy \\ &= \int_0^\infty y^{\lambda+\frac12}e^{\frac14 y^2}D_n(y)\,dy\int_0^\infty x^{s-\frac12}J_\nu(xy)\,dx. \end{aligned} \tag{3.2}$$

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* E. T. Whittaker, "On the functions associated with the parabolic cylinder in harmonic analysis", *Proc. London Math. Soc.* (1), 35 (1903), 417–427.

† We say that a function is R_ν as an abbreviation of the statement that it is self-reciprocal in the Hankel-transform of order ν; likewise, that it is R_c or R_s if it is self-reciprocal in the cosine or sine transform (when $\nu=\mp\frac12$); and finally that $f(x)$ is skew R_ν if it is reciprocal to $-f(x)$.

The change in the order of integration is permissible, as proved below, if $R(s) < 0$, $R(s+\nu) > \frac{1}{2}$, $R(\lambda-s) > 0$, $R(\lambda+\nu) > \frac{1}{2}$, and $R(\lambda+n+1) < 0$. Since*

$$\int_0^\infty x^q J_n(ax)\,dx = \frac{2^q}{a^{q+1}} \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}q+\frac{1}{2})}{\Gamma(\frac{1}{2}n-\frac{1}{2}q+\frac{1}{2})} \qquad [R(q) < \tfrac{1}{2},\ R(n+q+1) > 0], \quad (3.3)$$

$$f(\lambda+s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})} \int_0^\infty y^{\lambda-s} e^{\frac{1}{4}y^2} D_n(y)\,dy,$$

we find that (3.1) is equivalent to the functional equation

$$f(\lambda+s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})} f(\lambda+1-s). \qquad (3.4)$$

To justify the change in the order of integration in (3.2), suppose that

$$\theta(y) = y^{\lambda+\frac{1}{2}} e^{\frac{1}{4}y^2} D_n(y) \int_0^\infty x^{s-\frac{1}{2}} J_\nu(xy)\,dx$$

and that
$$\phi(x) = x^{s-\frac{1}{2}} \int_0^A y^{\lambda+\frac{1}{2}} e^{\frac{1}{4}y^2} D_n(y) J_\nu(xy)\,dy,$$

where A is small.

By the help of (3.3), we have

$$\theta(y) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})} y^{\lambda-s} e^{\frac{1}{4}y^2} D_n(y) \qquad [R(s) < 1,\ R(\nu+s+\tfrac{1}{2}) > 0].$$

We see that $\theta(y)$ converges uniformly in $y \geqslant 0$, provided that $R(\lambda-s) > 0$.

Since, when n is not a positive integer,

$$D_n(y) = e^{-\frac{1}{4}y^2} \sum_{m=0}^\infty \frac{\Gamma(\frac{1}{2}m-\frac{1}{2}n)}{m!\,\Gamma(-n)} (\sqrt{2})^{m-n-2}(-y)^m,$$

and, since for all values of ν,

$$J_\nu(y) = \sum_{r=0}^\infty \frac{(-)^r y^{\nu+2r}}{2^{\nu+2r} r!\,\Gamma(\nu+r+1)},$$

$\phi(x)$ also converges uniformly in $x \geqslant 0$, provided that $R(s+\nu) > \frac{1}{2}$ and $R(\lambda+\nu) > -\frac{3}{2}$.

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* H. Weber, *Journal für Math.*, 69 (1869), 222–237.

Again, consider the integral

$$I=\int_T^\infty x^{s-1}\,dx\int_{T'}^\infty |\sqrt{(xy)}\,J_\nu(xy)\,y^\lambda\,e^{\frac{1}{4}y^2}\,D_n(y)|\,dy,$$

where T and T' are large.

Since, for large values of y,

$$|D_n(y)|=O(e^{-\frac{1}{4}y^2}y^n),\qquad |J_\nu(y)|=O(y^{-\frac{1}{2}}),$$

I is less than a constant multiple of

$$\int_T^\infty x^{s-1}\,dx\int_{T'}^\infty y^{\lambda+n}\,dy,$$

which tends to zero when $R(s)<0$ and $R(\lambda+n+1)<0$. Hence the inversion is justified under the conditions stated above.

Substituting the values of $f(\lambda+s)$ and $f(\lambda+1-s)$ given by (2.2), we have

$$\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\lambda)\,\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\lambda+\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}n-\tfrac{1}{2}s-\tfrac{1}{2}\lambda)\,\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}s+\tfrac{3}{4})$$
$$=\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\,\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}n-\tfrac{1}{2}\lambda-\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}\lambda-\tfrac{1}{2}s+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}\lambda+1-\tfrac{1}{2}s).$$

This is satisfied by taking

either $\lambda=\nu+\frac{1}{2}$ and $\lambda+1=-1-n-\lambda$,

so that $\lambda=\nu+\frac{1}{2}$ and $n=-2\nu-3$, (3.5)

or $\lambda+1=\nu+\frac{1}{2}$ and $\lambda=-1-n-\lambda$,

so that $\lambda=\nu-\frac{1}{2}$ and $n=-2\nu$. (3.6)

The relation (3.5) shows that $x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-3}(x)$ is R_ν, that is,

$$x^{\nu+\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu-3}(x)=\int_0^\infty \sqrt{(xy)}\,J_\nu(xy)\,y^{\nu+\frac{1}{2}}\,e^{\frac{1}{4}y^2}\,D_{-2\nu-3}(y)\,dy \tag{3.7}$$

$[R(\nu)>0]$,

and, by the theory of analytic continuation, the range for ν may be extended to $R(\nu)>-1$.

The relation (3.6) gives that $x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu}(x)$ is also R_ν, that is,

$$x^{\nu-\frac{1}{2}}e^{\frac{1}{4}x^2}D_{-2\nu}(x)=\int_0^\infty \sqrt{(xy)}\,J_\nu(xy)\,y^{\nu-\frac{1}{2}}\,e^{\frac{1}{4}y^2}\,D_{-2\nu}(y)\,dy \tag{3.8}$$

$[R(\nu)>\frac{1}{2}]$,

and, by the theory of analytic continuation, the range for ν may be extended to $R(\nu) > -\frac{1}{2}$.

Putting $\nu = -\frac{1}{2}$ in (3.7), we get

$$e^{\frac{1}{4}x^2} D_{-2}(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty e^{\frac{1}{4}y^2} D_{-2}(y) \cos xy \, dy. \tag{3.9}$$

It will be interesting to note that the result (3.9) is a particular case of Bailey's theorem* that

$$x^{\frac{1}{2}-\nu} \int_0^\infty \frac{t^{1-2\nu} e^{-\frac{1}{2}x^2t^2}}{1+t^2} \, dt \tag{3.10}$$

is R_ν. When $\nu = -\frac{1}{2}$, this theorem, combined with Whittaker's integral†

$$D_n(z) = \frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} e^{-\frac{1}{4}z^2} z^n \int_0^\infty e^{-u} u^{-\frac{1}{2}n-\frac{1}{2}} \left(1+\frac{2u}{z^2}\right)^{\frac{1}{2}n} du \quad [R(n) < 1],$$

shows that the function

$$x \int_0^\infty \frac{t^2 e^{-\frac{1}{2}x^2t^2}}{1+t^2} \, dt = \sqrt{\left(\frac{\pi}{2}\right)} e^{\frac{1}{4}x^2} D_{-2}(x)$$

is R_c.

4. In the following integral, due to Sonine‡,

$$\int_0^\infty e^{ky} e^{-y} y^m T_m^n(y) \, dy = \frac{k^n}{n!\,(1-k)^{m+n+1}} \quad [R(k) < 1],$$

put $k = -p+\frac{1}{2}$; we get

$$\int_0^\infty e^{-py} e^{-\frac{1}{2}y} y^m T_m^n(y) \, dy = \frac{(\frac{1}{2}-p)^n}{n!\,(\frac{1}{2}+p)^{m+n+1}} \tag{4.1}$$

$$[R(p) > -\tfrac{1}{2}].$$

Since§ $f(y) \doteqdot \phi(p),$

when $p \int_0^\infty e^{-py} f(y) \, dy = \phi(p),$

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* W. N. Bailey, "A note on an integral due to Ramanujan", *Journal London Math. Soc.*, 6 (1931), 216–17.

† Whittaker, *loc. cit.*

‡ Sonine, *loc. cit.*

§ See, *e.g.*, S. Goldstein, *Proc. London Math. Soc.* (2), 34 (1932), 103–104.

provided that the integral converges, (4.1) gives

$$e^{-\frac{1}{2}y}\, y^m\, T_m{}^n(y) \doteqdot \frac{p(\frac{1}{2}-p)^n}{n!\,(\frac{1}{2}+p)^{m+n+1}}. \tag{4.2}$$

Recently Goldstein* has shown that if $\phi_1(p)$ and $\phi_2(p)$ are the operational representations of $f_1(y)$ and $f_2(y)$ respectively, we have

$$\int_0^\infty \frac{\phi_1(y)\, f_2(y)}{y}\, dy = \int_0^\infty \frac{\phi_2(y)\, f_1(y)}{y}\, dy.$$

Using (4.2) and the known result

$$\frac{y^l}{\Gamma(l+1)} \doteqdot \frac{1}{p^l} \quad [R(l) > -1]$$

in the above theorem of Goldstein, we get

$$\int_0^\infty e^{-\frac{1}{2}y}\, y^{m-l-1}\, T_m{}^n(y)\, dy$$

$$= \frac{1}{n!\,\Gamma(l+1)} \int_0^\infty \frac{p^l(\frac{1}{2}-p)^n}{(\frac{1}{2}+p)^{m+n+1}}\, dp$$

$$= \frac{2^{m-l}}{n!\,\Gamma(l+1)} \int_0^\infty \frac{u^l(1-u)^n}{(1+u)^{m+n+1}}\, du$$

$$= \frac{2^{m-l}}{n!\,\Gamma(l+1)} \sum_{r=0}^{n} (-1)^r\, {}_nC_r \int_0^\infty \frac{u^{l+r}}{(1+u)^{m+n+1}}\, du$$

$$= \frac{2^{m-l}}{n!\,\Gamma(l+1)\,\Gamma(m+n+1)} \sum_{r=0}^{n} (-1)^r\, {}_nC_r\, \Gamma(l+r+1)\, \Gamma(m+n-l-r).$$

By a change of the variable, we obtain

$$\int_0^\infty x^s\, e^{-\frac{1}{2}x^2}\, T_m{}^n(x^2)\, dx$$

$$= \frac{2^{\frac{1}{2}s-\frac{1}{2}}}{n!\,\Gamma(m-\frac{1}{2}s+\frac{1}{2})\,\Gamma(m+n+1)}$$

$$\times \sum_{r=0}^{n} (-1)^r\, {}_nC_r\, \Gamma(m+r-\tfrac{1}{2}s+\tfrac{1}{2})\, \Gamma(n-r+\tfrac{1}{2}s+\tfrac{1}{2}) \tag{4.3}$$

$$[R(s) > -1].$$

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* S. Goldstein, "Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function", *Proc. London Math. Soc.* (2), 34 (1932), 103–125.

Putting $m = -\frac{1}{2}$ and $m = \frac{1}{2}$, we get

$$\int_0^\infty x^s D_{2n}(x\sqrt{2})\,dx$$

$$= \sqrt{\pi}\, 2^{-n+\frac{1}{2}s-\frac{1}{2}} \frac{2n\,!}{n\,!\,\Gamma(-\frac{1}{2}s)\,\Gamma(n+\frac{1}{2})} \sum_{r=0}^{n} (-1)^r\, {}_nC_r\, \Gamma(r-\tfrac{1}{2}s)\, \Gamma(n-r+\tfrac{1}{2}s+\tfrac{1}{2})$$

$$[R(s) > -1]$$

and

$$\int_0^\infty x^{s-1} D_{2n+1}(x\sqrt{2})\,dx$$

$$= \sqrt{\pi}\, 2^{-n-\frac{1}{2}s-1} \frac{(2n+1)\,!}{n!\,\Gamma(1-\frac{1}{2}s)\,\Gamma(n+\frac{3}{2})}$$

$$\times \sum_{r=0}^{n} (-1)^r\, {}_nC_r\, \Gamma(r-\tfrac{1}{2}s+1)\, \Gamma(n-r+\tfrac{1}{2}s+\tfrac{1}{2})$$

$$[R(s) > -1].$$

5. To prove Wilson's theorem, let

$$f(s) = \int_0^\infty x^s e^{-\frac{1}{2}x^2} T_m{}^n(x^2)\,dx \qquad (5.1)$$

and suppose that $x^\lambda e^{-\frac{1}{2}x^2} T_m{}^n(x^2)$ is R_ν; then

$$f(\lambda+s) = \int_0^\infty x^{s+\lambda} e^{-\frac{1}{2}x^2} T_m{}^n(x^2)\,dx$$

$$= \int_0^\infty x^s\,dx \int_0^\infty \sqrt{(xy)}\, J_\nu(xy)\, y^\lambda e^{-\frac{1}{2}y^2} T_m{}^n(y^2)\,dy$$

$$= \int_0^\infty y^{\lambda+\frac{1}{2}} e^{-\frac{1}{2}y^2} T_m{}^n(y^2)\,dy \int_0^\infty x^{s+\frac{1}{2}} J_\nu(xy)\,dx, \qquad (5.2)$$

the change in the order of integration being permissible if $R(\lambda) > -1$, $R(s) < -1$, $R(s+\nu+\frac{3}{2}) > 0$, and $R(\lambda+\nu+\frac{1}{2}) > 0$. Hence, by (3.3),

$$f(\lambda+s) = 2^{s+\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{3}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{1}{4})} \int_0^\infty y^{\lambda-s-1} e^{-\frac{1}{2}y^2} T_m{}^n(y^2)\,dy.$$

With the notation of (5.1), this gives

$$f(\lambda+s) = 2^{s+\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{3}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{1}{4})} f(\lambda-s-1).$$

Now (4.3) shows that we require the equation

$$\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}s+\tfrac{1}{4})\,\Gamma(m+r-\tfrac{1}{2}s-\tfrac{1}{2}\lambda+\tfrac{1}{2})\,\Gamma(n-r+\tfrac{1}{2}s+\tfrac{1}{2}\lambda+\tfrac{1}{2})\,\Gamma(m-\tfrac{1}{2}\lambda+\tfrac{1}{2}s+1)$$
$$=\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{3}{4})\,\Gamma(m+n-r-\tfrac{1}{2}\lambda+\tfrac{1}{2}s+1)\,\Gamma(r+\tfrac{1}{2}\lambda-\tfrac{1}{2}s)\,\Gamma(m+\tfrac{1}{2}-\tfrac{1}{2}\lambda-\tfrac{1}{2}s)$$

to be true for all values of r; hence we must have

$$\tfrac{1}{2}\nu+\tfrac{1}{4}=m+\tfrac{1}{2}-\tfrac{1}{2}\lambda \quad \text{and} \quad m-\tfrac{1}{2}\lambda+\tfrac{1}{2}=\tfrac{1}{2}\lambda,$$

which gives $$m=\nu \quad \text{and} \quad \lambda=\nu+\tfrac{1}{2}.$$

On considering the expressions for $f(\lambda+s)$ and $f(\lambda-s-1)$, we therefore deduce that

$$x^{\nu+\frac{1}{2}}\,e^{-\frac{1}{2}x^2}\,T_\nu{}^n(x^2)=(-1)^n\int_0^\infty \sqrt{(xy)}\,J_\nu(xy)\,y^{\nu+\frac{1}{2}}\,e^{-\frac{1}{2}y^2}\,T_\nu{}^n(y^2)\,dy \tag{5.3}$$
$$[R(\nu)>-\tfrac{1}{2}]$$

is R_ν or skew R_ν according as n is even or odd. By analytic continuation the range for ν may be extended to $R(\nu)>-1$.

As a special case* (5.3), combined with (1.11) and (1.12), gives Milne's result that $D_{2n}(x\sqrt{2})$ is R_c or skew R_c according as n is an even or an odd integer, and that $D_{2n+1}(x\sqrt{2})$ is R_s or skew R_s according as n is an even or an odd integer.

It remains to justify the change in the order of integration in (5.2). Consider

$$\psi(y)=y^{\lambda+\frac{1}{2}}\,e^{-\frac{1}{2}y^2}\,T_m{}^n(y^2)\int_0^\infty x^{s+\frac{1}{2}}\,J_\nu(xy)\,dx$$
$$=2^{s+\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{3}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{1}{4})}\,y^{\lambda-s-1}\,e^{-\frac{1}{2}y^2}\,T_m{}^n(y^2).$$

It is evident that $\psi(y)$ is uniformly convergent in $y\geqslant 0$ if $R(s)<0$, $R(s+\nu+\frac{3}{2})>0$, and $R(\lambda-s-1)>0$.

Again, consider the integral

$$I=\int_T^\infty x^s\,dx\int_0^\infty \left|\sqrt{(xy)}\,J_\nu(xy)\,y^\lambda\,e^{-\frac{1}{2}y^2}\,T_m{}^n(y^2)\right|dy,$$

where T is large.

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* A. Milne, "On the equation of the parabolic cylinder function", *Proc. Edin. Math. Soc.*, 32 (1914), 2–14.

Since, for large values of x,

$$|J_\nu(x)| = O(x^{-\frac{1}{2}}),$$

I does not exceed a constant multiple of

$$\int_T^\infty x^s\,dx \int_0^\infty |e^{-\frac{1}{2}y^2} y^\lambda T_m{}^n(y^2)|\,dy,$$

which, considering the integral (4.3), tends to zero if $R(s) < -1$ and and $R(\lambda) > -1$.

Hence the inversion is justified under the conditions stated above.

My best thanks are due to the Referee for helpful criticisms and suggestions.

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ON THE KANTOR GROUP OF A SET OF POINTS IN A PLANE

By Patrick Du Val.

[Received 15 May, 1936.—Read 18 June, 1936.]

1. Some years ago I published in these *Proceedings* a paper[†] whose main content was the representation of certain groups of Cremona transformations in a plane and in three dimensions by means of certain crystallographic groups in Euclidean and Minkowskian space. The details of these correspondences were there pretty fully worked out, with their applications to the theory of the twenty-seven lines on a cubic surface, and to θ-functions of three and four variables, but the real reason why there should be such correspondences was so little recognised at that time that it seems appropriate, as a prelude to a fairly wide extension of the theory, to give a more rational account of the matter already published, without, however, spending time over the details of the particular groups and their applications, for which the reader is referred to the paper above named.

It may be pointed out that investigations along very parallel lines to those here developed have been carried out by Coble[‡]; the present discussion was, however, quite independent, and may be regarded as not entirely redundant, the more so as Coble's work has not attained a wide familiarity in this country. It may be said, on the whole, that what he does algebraically is here done geometrically; and I hope to show what a much clearer idea of the same theory the geometrical interpretation gives, and what a much more powerful instrument we have in it, more especially since it enables us to make use of the considerable body of crystallographic theory made available by Coxeter, to which we shall refer later.

[†] P. Du Val, "On the directrices of a set of points in a plane", *Proc. London Math. Soc.* (2), 35 (1933), 23–74. In this I used (for reasons which escape me but which certainly were not good) the term "directrices" for the exceptional curves based on the set.

[‡] A. B. Coble, "Point sets and allied Cremona groups", *Trans. American Math. Soc.* (1), 16 (1915), 155–198; (2), 17 (1916), 345–385; (3), 18 (1917), 331–372.

2. We consider a set of points in a plane, ϵ in number, say $(A_1, \ldots, A_\epsilon)$, or briefly (A). We consider linear systems of curves in the plane, which are complete with regard to assigned base conditions, *i.e.* which consist of all curves of certain order having certain multiplicities at certain points; we say that such a system is based on the set (A) if its assigned base points are all included in the set (A). If the system consists of curves of order m, having multiplicity i_κ at A_κ, we indicate it by the characteristic

$$m,(i_1, \ldots, i_\epsilon), \text{ or briefly } m,(i).$$

Every such characteristic is regarded as defining a linear system, which is said to be merely virtual if there are in fact no curves satisfying the assigned conditions.

The addition and subtraction of linear systems in the ordinary sense (where the sum of two systems is that to which belongs every composite curve consisting of a curve of the one system and a curve of the other) clearly corresponds to the addition and subtraction of corresponding elements in their characteristics. Since the correspondence between characteristics and linear systems is one-one, we may interpret all this by saying

> *The linear systems based on ϵ points can be represented as an integer vector field in space $R_{\epsilon+1}$ of $\epsilon+1$ dimensions, the components of the vector being the elements in the characteristic of the linear system.*

Heavy type will indicate the vectors of the field, and the same symbols, enclosed within $||$, the corresponding systems of curves in π.

3. If $|\mathbf{f}|$, $|\mathbf{g}|$ are two systems with characteristics $m,(i)$, $n,(j)$, respectively, then the variable curve of $|\mathbf{f}|$ meets the variable curve of $|\mathbf{g}|$ in

$$\mathbf{f}\cdot\mathbf{g} = mn - \sum_{\kappa=1}^{\epsilon} (i_\kappa j_\kappa) \tag{3.1}$$

variable points, provided that m, n are high enough for both systems to contain variable curves, and to satisfy no base conditions other than those assigned in the characteristics. In any case this expression will be called the intersection number of the two systems. The intersection number

$$\mathbf{f}\cdot\mathbf{f} = m^2 - \sum_{\kappa=1}^{\epsilon} (i_\kappa^2) \tag{3.2}$$

of any system with itself is called the grade of the system; it will commonly be indicated by ν, or more precisely by $\nu(\mathbf{f})$.

Since (3.1) is a bilinear form in the components of the vectors $\mathbf{f}$, $\mathbf{g}$, it is clearly natural to interpret it as their scalar product, and as defining the metrical properties of the vector space $R_{\epsilon+1}$. Since (3.2) consists of a sum of squares, one with positive and the rest with negative coefficients,

> *The metric of the space $R_{\epsilon+1}$ is Minkowskian, i.e. it is of the type whose four-dimensional case is used in the relativity theory.*

(Writers on relativity seem commonly to call this metric Galileian, but for this term there is no reasonable justification.) We may conveniently adopt the terminology already familiar in this connection, and call a vector timelike or spacelike according as its length is real or pure imaginary, *i.e.* according as the corresponding linear system has positive or negative grade. We recall that

> *The prime perpendicular to a timelike vector contains a Euclidean metric (with the trivial difference that all lengths are pure imaginary); the prime perpendicular to a spacelike vector contains a Minkowskian metric.*

4. Among the curves based on a set (A) we count the neighbourhood $\mathbf{a}_\kappa$ of A_κ; a point of this neighbourhood is defined by a direction at A_κ, all curves which pass through A_κ and have the same tangent there being held to meet $\mathbf{a}_\kappa$ in the same point. In order that $\mathbf{a}_\kappa$ may have intersection number i_κ with $m,(i)$ (as is evidently required), we must attribute to it the characteristic $0,(-\delta_\kappa)$, where (δ_κ) is a row of numbers of which the κ-th is 1 and the rest are all 0. This necessitates the further convention that

> *To increase by h the multiplicity of any system at A_κ is the same thing as to subtract $h\mathbf{a}_\kappa$ from the system.*

5. Every linear system $|\mathbf{f}| = m,(i)$ defines what is called its adjoint system $|\mathbf{f}'| = m',(i')$, where $m' = m-3$, $i_\kappa' = i_\kappa - 1$. This can be written

$$|\mathbf{f}'| = |\mathbf{f}-\mathbf{k}|, \tag{5.1}$$

where $|\mathbf{k}|$ has the characteristic $3,(1, \ldots, 1)$, or briefly $3,(1^\epsilon)$, *i.e.* it is the system (whether actual or merely virtual) of cubics with simple base points at all of the set (A). We may call $|\mathbf{k}|$ the anticanonical system.

We define the numerical character of a linear system $|\mathbf{f}| = m, (i)$:

$$\sigma(\mathbf{f}) = \mathbf{k}\cdot\mathbf{f} = 3m - \sum_{\kappa=1}^{\epsilon} (i_\kappa), \tag{5.2}$$

which we call the anticanonical order of the system. In terms of this and the character $\nu(\mathbf{f})$ already defined (the grade), the genus π of the general curve of the system, and its virtual freedom ρ (*i.e.* the freedom calculated on the assumption that the base conditions are all independent) are given by

$$\left.\begin{aligned} 2\rho &= \nu+\sigma, \\ 2\pi-2 &= \nu-\sigma; \end{aligned}\right\} \tag{5.3}$$

or, if we like to regard ν, π as fundamental, we have

$$\left.\begin{aligned} \rho &= \nu-\pi+1, \\ \sigma &= \nu-2\pi+2, \end{aligned}\right\} \tag{5.4}$$

of which the first expresses the Riemann-Roch theorem, and the second the fact that $|\mathbf{k}|$ traces on the general curve of $|\mathbf{f}|$ the difference between its characteristic and canonical series.

The net $|\mathbf{l}| = 1, (0^\epsilon)$ of lines in the plane has $\nu = 1$, $\sigma = 3$, $\pi = 0$, $\rho = 2$; while the neighbourhood $\mathbf{a}_\kappa$ of the base point A_κ is a complete system $|\mathbf{a}_\kappa| = 0, (-\delta_\kappa)$, having $\nu = -1$, $\sigma = 1$, $\pi = \rho = 0$.

6. Consider any Cremona transformation by which the plane is transformed into another plane; we confine our attention, of course, to transformations based on the set (A), *i.e.* in which the lines of the new plane correspond to a homaloidal net $|\mathbf{l}'|$ based on (A). The neighbourhoods of those of the points (A) which are base points (of multiplicity greater than 0) of this net are transformed into curves, which are called exceptional curves, in the new plane; while an equal number of exceptional curves in the old plane are transformed into the neighbourhoods of points in the new. These points, together with the transforms of those of the points (A) which are not base points (or are base points of multiplicity 0) of $|\mathbf{l}'|$, form a set of points, ϵ in number, which we may call $(A_1', \ldots, A_\epsilon')$, in any order, and which is said to be congruent to (A) under the transformation. We can simplify the statement by including the neighbourhoods of the base points among the exceptional curves based on them; then we can say that

> *In a Cremona transformation based on* (A) *the neighbourhoods* $(\mathbf{a})$ *are transformed into a set of* ϵ *exceptional curves, and some set of* ϵ *exceptional curves is transformed into the neighbourhoods* $(\mathbf{a}')$ *of a set of points* (A') *congruent to* (A).

The two sets of exceptional curves may have some members in common, or they may coincide completely; in the last case the transformation is a projectivity, and the set (A') is a permutation of the set (A).

> *Every system based on* (A) *is transformed into a system based on* (A'), *for which the characters* ν, σ, π, ρ *have the same values as for the original system; these are accordingly Cremona invariants, and so is the intersection number of any two systems.*

In particular, every homaloidal net has the characters $\nu=1$, $\sigma=3$, $\pi=0$, $\rho=2$, and every exceptional curve has the characters $\nu=-1$, $\sigma=1$, $\pi=\rho=0$. Homaloidal nets and exceptional curves based on (A) we shall, for brevity, call H-nets and E-curves.

7. Regarded as a transformation in the vector field in $R_{\epsilon+1}$, what we have done is to take a new base of fundamental vectors (or unit vectors) in terms of which to express all the vectors of the field. If $m,(i)$ is the characteristic of a linear system $|\mathbf{f}|$ in the old plane, then

$$\mathbf{f}=m\mathbf{l}-\sum_{\kappa=1}^{\epsilon}(i_\kappa \mathbf{a}_\kappa),$$

that is, m, (i) are the components of $\mathbf{f}$ in terms of a set of vectors $\mathbf{l},(-\mathbf{a})$. The characteristic $m',(i')$ of the transformed system, however, consists of the components of the same vector in terms of the new set of unit vectors $\mathbf{l}',(-\mathbf{a}')$; thus

> *A Cremona transformation based on* (A) *corresponds to a change of coordinates in the vector space* $R_{\epsilon+1}$.

On account of the invariance of (3.1), (5.2),

> *This transformation is orthogonal, and leaves unchanged the components* $3,(1^\epsilon)$ *of the vector* $\mathbf{k}$.

The explicit equations of transformation are

$$\left.\begin{aligned} m' &= nm-\sum_{\kappa=1}^{\epsilon}(j_\kappa i_\kappa), \\ i_\kappa' &= s_\kappa m-\sum_{\lambda=1}^{\epsilon}(h_{\kappa\lambda} i_\lambda), \end{aligned}\right\} \qquad (7.1)$$

where $n,(j)$ is the characteristic of $|l'|$, and $s_\kappa,(h_{\kappa 1},\ldots,h_{\kappa\epsilon})$, or briefly $s_\kappa,(h_\kappa)$, that of $|\mathbf{a}_\kappa'|$. Since $|\mathbf{l}'|$ is an H-net, and $\mathbf{a}_\kappa'$ an E-curve, and since the general line of the new plane does not go through any of the points (A'), nor have the neighbourhoods of two of these any common point, we have

$$\left.\begin{array}{c} n^2 - \sum_{\kappa=1}^{\epsilon} (j_\kappa^2) = 1, \quad 3n - \sum_{\kappa=1}^{\epsilon} (j_\kappa) = 3, \\ s_\kappa^2 - \sum_{\lambda=1}^{\epsilon} (h_{\kappa\lambda}^2) = -1, \quad 3s_\kappa - \sum_{\lambda=1}^{\epsilon} (h_{\kappa\lambda}) = 1, \\ ns_\kappa - \sum_{\lambda=1}^{\epsilon} (j_\lambda h_{\kappa\lambda}) = 0, \\ s_\kappa s_\lambda - \sum_{\mu=1}^{\epsilon} (h_{\kappa\mu} h_{\lambda\mu}) = 0 \quad (\lambda \neq \kappa), \end{array}\right\} \tag{7.2}$$

of which the four quadratic relations express that the matrix

$$\begin{pmatrix} n & -j_1 & -j_2 & \cdots & -j_\epsilon \\ s_1 & -h_{11} & -h_{12} & \cdots & -h_{1\epsilon} \\ s_2 & -h_{21} & -h_{22} & \cdots & -h_{2\epsilon} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_\epsilon & -h_{\epsilon 1} & -h_{\epsilon 2} & \cdots & -h_{\epsilon\epsilon} \end{pmatrix},$$

or briefly
$$\begin{pmatrix} n & -j \\ s & -h \end{pmatrix},$$

is orthogonal; while the two linear relations express the invariance of the vector **k**.

> *The transformations* (7.1) *evidently form a group. We call it the Kantor group of the set* (A), *and indicate it by* $\Gamma(\epsilon, 2)$. *It is our main topic.*

[The 2 in the symbol $\Gamma(\epsilon, 2)$ refers to the fact that we are working in the plane, since an analogous group is defined by a set of points in any number of dimensions.] We shall refer to the group briefly as Γ.

8. The orthogonal transformations of the group Γ can equally well be regarded as motions (either positive or negative, *i.e.*, including operations such as reflexions which reverse the orientation of space) acting in the vector space. These motions leave fixed the origin O of coordinates, and the vector **k**; they accordingly leave fixed every point of the line k through

O parallel to $\mathbf{k}$; and they leave fixed as a whole each of the pencil of parallel primes R_ϵ^σ whose equations are

$$3m - \sum_{\kappa=1}^{\epsilon} (i_\kappa) = \sigma,$$

and each of the family of concentric spheres S_ϵ^ν whose equations are

$$m^2 - \sum_{\kappa=1}^{\epsilon} (i_\kappa^2) = \nu,$$

where $m, (i)$ are now interpreted as coordinates in $R_{\epsilon+1}$, and σ, ν as parameters giving the variable members of the two systems. In R_ϵ^σ is a fixed point L^σ, its intersection with k. The intersection of R_ϵ^σ, S_ϵ^ν is a sphere $S_{\epsilon-1}^{\sigma,\nu}$, with centre L^σ, which is itself invariant under Γ. Of course, the sphere S_ϵ^ν in Minkowskian space is a hyperboloid, of two sheets or one according as its radius is timelike or spacelike, *i.e.* as ν is positive or negative. R_ϵ^σ on the other hand need not be Minkowskian, but is Euclidean or Minkowskian according as the vector $\mathbf{k}$ perpendicular to it is timelike or spacelike; since, however,

$$\mathbf{k} \cdot \mathbf{k} = 9 - \epsilon, \tag{8.1}$$

The metric in R_ϵ^σ *is Euclidean if* $\epsilon < 9$, *Minkowskian if* $\epsilon > 9$.

The case $\epsilon = 9$ is more subtle, and will be treated separately. The radius of $S_{\epsilon-1}^{\sigma,\nu}$ is easily found; for that of S_ϵ^ν is $\sqrt{\nu}$, while the perpendicular from 0 on to R_ϵ^σ is clearly $\sigma/\sqrt{(9-\epsilon)}$; hence the radius $r_{\sigma,\nu}$ of $S_{\epsilon-1}^{\sigma,\nu}$ is given by

$$r_{\sigma,\nu}^2 = \nu - \frac{\sigma^2}{9-\epsilon}. \tag{8.2}$$

Still taking $m, (i)$ to be coordinates, so that a linear system $|\mathbf{f}|$ is represented not only by the vector $\mathbf{f}$ but by the point whose radius vector from O is $\mathbf{f}$, we see that the group Γ leaves invariant not only the line k and the family of primes and spheres just described, but the lattice of integer points. Whether it contains all motions which leave invariant all this, is not at this stage evident.

> *Since a sphere in Euclidean space can only contain a finite number of lattice points, the group* $\Gamma(\epsilon, 2)$ *is finite for* $\epsilon < 9$.

9. Consider now under what circumstances an operation of Γ can be the reflexion in a prime. Suppose the prime to be that through O perpen-

dicular to the vector **c**; then every vector **f** gives rise to a vector

$$\mathbf{f}' = \mathbf{f} - 2\mathbf{c}\,\frac{\mathbf{c}\cdot\mathbf{f}}{\mathbf{c}\cdot\mathbf{c}}, \tag{9.1}$$

since $\mathbf{cc}\cdot\mathbf{f}/\mathbf{c}\cdot\mathbf{c}$ is the directed component of **f** parallel to **c**, and this is reversed by the reflexion. We may write this

$$\mathbf{f}' = \left(1 - 2\,\frac{\mathbf{cc}\cdot}{\mathbf{c}\cdot\mathbf{c}}\right)\mathbf{f}, \tag{9.2}$$

regarding the expression in brackets as an operator acting on the variable vector **f**; the expression $\mathbf{cc}\cdot$ occurring in it is a linear vector operator, or tensor of the second rank, whose components are the squares and products of those of **c**. In order that this may give an integer vector $\mathbf{f}'$ for every integer vector **f**, it is clear that $\mathbf{c}\cdot\mathbf{c}$ must be a common factor of all the components of $2\mathbf{cc}\cdot$; hence, replacing **c** [as we may without altering the operation (9.2)] by the least integer vector parallel to it, *i.e.* dividing its components by their highest common factor, and calling the new vector **c**, we see that $\mathbf{c}\cdot\mathbf{c}$ must be a factor of 2. But in order that **k** may be left unchanged by the operation, we must evidently have $\mathbf{k}\cdot\mathbf{c} = 0$; and from (5.3) or (5.4) (since ρ, π are integers) we have, for any integer vector, $\nu \equiv \sigma \pmod 2$, so that **c** cannot have $\nu = \pm 1$ while it has $\sigma = 0$; we are thus led to consider vectors **c** for which $\sigma = 0$, $\nu = \pm 2$.

Now if $\mathbf{k}\cdot\mathbf{c} = 0$, $\mathbf{c}\cdot\mathbf{c} = 2$, the operation $(1 - \mathbf{cc}\cdot)$ is indeed a reflexion, and leaves invariant everything that we want it to leave invariant; but it does not belong to the Kantor group Γ, since (as can easily be seen) it transforms every vector for which the coordinate m is positive into one for which m is negative, and there is certainly no Cremona transformation which does this. Hence

> *Every reflexion included in the Kantor group is an operation of the form* $(1 + \mathbf{cc}\cdot)$, *where* **c** *is a vector for which* $\sigma = 0$, $\nu = -2$, *and consequently* $\pi = 0$, $\rho = -1$.

The last result means that the system $|\mathbf{c}|$ is merely virtual, except for some special position of the base points. It can never, of course, contain more than one curve, for if there were two they would have to meet in -2 points, besides the assigned intersections in (A). If the base points are so specialized that there is an actual curve of $|\mathbf{c}|$, then the operation $(1 + \mathbf{cc}\cdot)$, applied to any linear system which does not contain the curve **c** as part, gives us merely the same system again, with the addition of a multiple of **c** as fixed part; this we shall call a degenerate Cremona transformation, if the operation is one which would give a proper Cremona transformation for a sufficiently general configuration (A).

The vectors $\mathbf{c}$ here considered correspond to what Coble calls discriminating conditions, or D-conditions. We shall call them D-vectors.

10. We continue to study the operations of the form $(1+\mathbf{cc}\cdot)$, where $\sigma(\mathbf{c})=0$, $\nu(\mathbf{c})=-2$. In the first place, if $\mathbf{c}$, $\mathbf{d}$ are any two vectors of this kind, the angle between them is given by

$$\cos c \wedge d = \frac{\mathbf{c}\cdot\mathbf{d}}{\sqrt{\{(\mathbf{c}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{d})\}}} = -\frac{\mathbf{c}\cdot\mathbf{d}}{2}, \tag{10.1}$$

the negative sign being necessary to make the cosine of an acute angle positive. In particular, if the angle is circular (the angle between two vectors in Minkowskian space can be either circular or hyperbolic, according as the plane joining them is Euclidean or Minkowskian), its only possible values are 0, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$; $\frac{2}{3}\pi$, and π. Only the first (or last) three of these angles represent really different cases, since either $\mathbf{c}$ or $\mathbf{d}$ may be reversed in sign without altering the corresponding operation.

We can verify directly that if $\mathbf{c}\cdot\mathbf{d}=0$ the two operations commute, which is what we expect of the reflexions in perpendicular primes; we have, in fact,

$$(1+\mathbf{cc}\cdot)(1+\mathbf{dd}\cdot)=1+\mathbf{cc}\cdot+\mathbf{dd}\cdot=(1+\mathbf{dd}\cdot)(1+\mathbf{cc}\cdot). \tag{10.2}$$

On the other hand, if $\mathbf{c}\cdot\mathbf{d}=1$, putting $\mathbf{c}+\mathbf{d}+\mathbf{e}=0$, we have $\mathbf{d}\cdot\mathbf{e}=\mathbf{e}\cdot\mathbf{c}=1$, and

$$(1+\mathbf{cc}\cdot)(1+\mathbf{dd}\cdot)(1+\mathbf{cc}\cdot)=1+\mathbf{ee}\cdot, \tag{10.3}$$

i.e. of the three operations $(1+\mathbf{cc}\cdot)$, $(1+\mathbf{dd}\cdot)$, $(1+\mathbf{ee}\cdot)$, the transform of any one by either of the others is the third, as we expect of the reflexions in three primes inclined at angles of $\frac{1}{3}\pi$.

11. Two types of D-vector are of particular importance to us. These correspond to the virtual difference of the neighbourhoods of two base points, and the virtual system of lines passing through three base points. The former is actual if one base point is in the neighbourhood of the other, the latter if the three base points are collinear.

The operation corresponding to the reflexion is, in the former case, simply the interchange of the two base points (*i.e.* the renaming of A_κ, A_λ as A_λ', A_κ' respectively), and in the latter the quadratic transformation for which the homaloidal net $|\mathbf{l}'|$ consists of conics with base points at the three points in question, and each side of this triangle is transformed into the neighbourhood of the opposite vertex.

We shall call the operation of the former kind $[\kappa\lambda]$, that of the latter kind $[\kappa\lambda\mu]$, the symbols inside the brackets being the suffixes of the base points concerned; and, without risk of confusion, we may use the same symbols for the primes of these reflexions.

If we assume that the points (A) are sufficiently general for none of the D-vectors to correspond to actual curves, then the proof that every Cremona transformation is a product of transformations of the form $[\kappa\lambda\mu]$ can be carried through in its simplest form, since no point A_λ is in the neighbourhood of another A_κ, and the same is true of the congruent set (A') at every stage of the reduction, $\mathbf{a}_\kappa' - \mathbf{a}_\lambda'$ being also a D-vector. Even if the configuration of points (A) is more special, since we allow as operations of the Kantor group, and degenerate Cremona transformations, those operations which would be Cremona transformations if the base points were sufficiently general, the resolution of any operation of Γ into operations of the form $[\kappa\lambda\mu]$ is exactly the same as if the base points were more general, the only difference being that some of the factors may be degenerate. We arrive at the conclusion that Γ is generated by the operations $[\kappa\lambda\mu]$; but this is not quite what we want. Every operation $[\kappa\lambda\mu]$ is the transform of a particular one, say [123], under a suitable permutation of the symbols $(1, \ldots, \epsilon)$; so that Γ is generated by adjoining [123] to the symmetric group on these symbols; and, since as the symmetric group itself is generated by the interchanges of consecutive symbols, we have

The group $\Gamma(\epsilon, 2)$ *is generated by the operations (reflexions in primes)* [123], [12], [23], ..., $[(\epsilon-1)\,\epsilon]$.

12. We here recall some results of Coxeter's on groups generated by reflexions†, and make use of his notation. A group in Euclidean space of ϵ dimensions, generated by reflexions in ϵ primes through the origin, the angle between any two of which is either $\frac{1}{2}\pi$ or $\frac{1}{3}\pi$, we shall call a finite Coxeter group. He represents it by a figure of ϵ dots, representing the ϵ primes, any two of them being linked by a line if the angle between the corresponding primes is $\frac{1}{3}\pi$, and unlinked if the angle is $\frac{1}{2}\pi$. He shows that such a diagram or graph consists either of a single chain of dots, in which consecutive dots are linked, or of three chains, say of n, p, q links respectively, having an end dot in common; and that the condition for the configuration of primes indicated by this latter graph to exist in Euclidean

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† H. S. M. Coxeter, "Groups whose fundamental regions are simplexes", *Journal London Math. Soc.*, 6 (1931), 132–136; "Discrete groups generated by reflexions", *Annals of Math.*, 5 (1934), 588–621; and "The complete enumeration of finite groups of the form $R_i^2 = (R_i R_j)^{k_{ij}} = 1$", *Journal London Math. Soc.*, 10 (1935), 21–25.

space, the primes having the origin in common and no further point, is

$$npq < n+p+q+2; \tag{12.1}$$

the group generated by the first configuration of primes is called $[3^{\epsilon-1}]$ ($\epsilon-1$ being the number of links in the chain, *i.e.* the number of angles of $\frac{1}{3}\pi$ amongst the primes); the other is called $[3^{n,p,q}]$† and its dimension number is

$$\epsilon = n+p+q+1. \tag{12.2}$$

(12.1) is satisfied

(i) if $q=0$, for all n, p (this is the same configuration and the same group as $[3^{n+p}]$);

(ii) if $p=q=1$, for all n;

(iii) by $(n, p, q) = (2, 2, 1)$, $(3, 2, 1)$, or $(4, 2, 1)$.

We consider also the group generated by reflexions in the faces of a simplex in Euclidean space of $\epsilon-1$ dimensions, where again every angle between two primes of the figure is either $\frac{1}{2}\pi$ or $\frac{1}{3}\pi$. This we call an infinite Coxeter group; it may be regarded as the limit of the finite Coxeter group when the common point of all the primes recedes to infinity, and they are then cut by a prime perpendicular to all of them. If we regard the finite Coxeter group as acting on a sphere in the ϵ-dimensional space (with centre at the origin), its generating operations are reflexions in the faces of a spherical simplex; and the infinite Coxeter group may be regarded as the limit of this when the radius of the sphere becomes infinite, the simplex remaining finite. There are some cases which do not concern us, but the possible simplexes of the kind which would be represented by the graph of three chains meeting in a point are those for which

$$npq = n+p+q+2, \tag{12.3}$$

i.e., the cases

$$(n, p, q) = (5, 2, 1), \quad (3, 3, 1), \quad \text{or} \quad (2, 2, 2).$$

Finally, I have shown in the paper referred to above that, if

$$npq > n+p+q+2, \tag{12.4}$$

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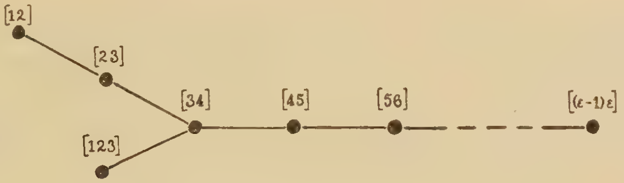
† This particular notation, instead of $\begin{bmatrix}3^n\\3^p\\3^q\end{bmatrix}$, was introduced by Coxeter, "Finite groups generated by reflections and their subgroups generated by reflections", *Proc. Camb. Phil. Soc.*, 30 (1934), 466–482.

the configuration of primes represented by the graph exists in Minkowskian space of $\epsilon = n+p+q+1$ dimensions. The group generated by reflexions in the primes is in this case again infinite; we may call it a generalized Coxeter group.

13. Now we have already found the Kantor group $\Gamma(\epsilon, 2)$, in ϵ dimensions, to be generated by the ϵ reflexions [123] and $[\kappa(\kappa+1)]$, $1 \leqslant \kappa \leqslant \epsilon-1$; the angles between the reflecting primes are easily found, from the intersection numbers of the virtual systems corresponding to the D-vectors; $[\kappa\lambda]$, $[\mu\nu]$ are inclined at $\frac{1}{3}\pi$ or $\frac{1}{2}\pi$ according as one of the symbols in the one bracket is or is not the same as one of the symbols in the other bracket, so that of the sequence of primes [12], [23], ..., $[(\epsilon-1)\epsilon]$, two which are consecutive are inclined at $\frac{1}{3}\pi$, and every other pair is perpendicular; while as for [123], it is evidently perpendicular to all the other primes except [34], to which it is inclined at $\frac{1}{3}\pi$. We thus construct the graph



(13.1)

and have the result:

The Kantor group $\Gamma(\epsilon, 2)$ *of a set of* ϵ *points in a plane is the group* $[3^{\epsilon-4, 2, 1}]$.

This we note is a finite Coxeter group if $\epsilon < 9$, an infinite Coxeter group if $\epsilon = 9$, and a generalized Coxeter group if $\epsilon > 9$.

14. We have already shown that Γ operates in Euclidean space of ϵ dimensions if $\epsilon < 9$, and in Minkowskian space if $\epsilon > 9$; we turn now to the case $\epsilon = 9$. Here we see that the vector $\mathbf{k}$ is neither timelike nor spacelike, but a zero-vector, *i.e.* such that it and every finite vector parallel to it have zero length. The line k through the origin, every point of which is fixed under every operation of Γ, lies in the prime $R_9{}^0$, and is parallel to every other $R_9{}^\sigma$; thus the fixed point L^σ in $R_9{}^\sigma$ is the point at infinity L^∞, except for $\sigma = 0$, when it is indeterminate, being any point of k. The metric in each prime $R_9{}^\sigma$ is such that vectors parallel to $\mathbf{k}$ all have zero length, and two vectors which differ by one parallel to $\mathbf{k}$ have the same length; so that any two figures which are in perspective from L^∞ are metrically identical. All the spheres $S_8^{0;\nu}$ in $R_9{}^0$ are cylinders, with axis

k, while the spheres $S_8^{\sigma,\nu}$ $(\sigma \neq 0)$ are paraboloids with axes parallel to $\mathbf{k}$, *i.e.* $S_8^{\sigma,\nu}$ touches the prime at infinity of $R_9{}^{\sigma}$ in L^{∞}. The line k itself is all that is real of the sphere $S_8^{0,0}$.

Consider now an eight-dimensional space $R_8^{*\sigma}$, lying in $R_9{}^{\sigma}$, and not parallel to $\mathbf{k}$. It contains an ordinary Euclidean metric (still, of course, with the trifling peculiarity that a real vector has a pure imaginary length). The mapping of $S_8^{\sigma,\nu}$ on $R_8^{*\sigma}$ by projection from L^{∞} is metrically true (except for $\sigma = 0$, when it is degenerate). The primes of reflexion of the group Γ trace on the spheres or paraboloids $S_8^{\sigma,\nu}$ systems of "spherical" simplexes, which are fundamental regions for Γ, and on $R_8^{*\sigma}$ a set of Euclidean simplexes, with the same dihedral angles as the corresponding spherical simplexes, on account of the metrical truth of the mapping; these are fundamental regions for a group of motions in $R_8^{*\sigma}$ simply isomorphic with $\Gamma(8, 2)$, which we may call $\Gamma'(8, 2)$; it is evidently this group which is strictly to be identified with the infinite Coxeter group $[3^{5,2,1}]$.

15. Since in the case under consideration $\mathbf{k}\cdot\mathbf{k} = 0$, it is easily seen that, if $\mathbf{c}$ has the invariants of a D-vector, so has every vector $\mathbf{d} = \mathbf{c}+\mu\mathbf{k}$; thus $(1+\mathbf{dd}\cdot)$ is an operation of the Kantor group Γ. The primes in $R_9{}^{\sigma}$ corresponding to these reflexions, for different values of μ, form a parallel pencil; they trace on $R_8^{*\sigma}$ a pencil of parallel primes, which are genuine reflecting primes in Γ'. In the proper group Γ, regarded as operating in $R_9{}^{\sigma}$, things are not so simple, since in this degenerate metric a reflexion is not completely determined by its prime; still, they are loci of fixed points of involutory operations.

In ordinary space, the product of reflexions in two parallel primes is a translation. Suppose that of the system of vectors $\mathbf{c}+\mu\mathbf{k}$, the prime $R_8^{*\sigma}$ has been taken parallel to one, say $\mathbf{c}$. We have, if $\mathbf{d} = \mathbf{c}+\mu\mathbf{k}$,

$$(1+\mathbf{cc}\cdot)(1+\mathbf{dd}\cdot) = 1-\mu\mathbf{ck}\cdot+\mu\mathbf{kd}\cdot.$$

This operation, acting in $R_9{}^{\sigma}$, is not a translation, but more of the nature of a shear, since the last term involves the addition of a multiple of $\mathbf{k}$ which depends on the system operated on; but in the corresponding operation of Γ' this last term is to be dropped out, since $\mathbf{k}$ is disregarded, and the operation is a translation, by the vector $-\mu\sigma\mathbf{c}$. Since the vector of the translation is twice the perpendicular distance between the two primes, we see that the primes of each parallel pencil in $R_8^{*\sigma}$ occur at intervals of $\frac{1}{2}\sigma\sqrt{(-2)}$ along a perpendicular; and this is also true in $R_8{}^{\sigma}$, though the perpendicular direction is not determinate. In $R_9{}^{0}$ all the primes of a system coincide, as we expect since we know that they all contain the line k. This does not mean, of course, that the operations $(1+\mathbf{cc}^{\bullet})$, $(1+\mathbf{dd}^{\bullet})$ have the same effect on $R_9{}^0$, since, as we have seen, a reflexion is not completely determined by its prime; but in R_8^{*0} the operation of Γ' is really degenerate, the reflexions corresponding to $(1+\mathbf{cc}^{\bullet})$, $(1+\mathbf{dd}^{\bullet})$ being identical.

16. We recall further results obtained by Coxeter†. If we call N, P, Q, respectively, the reflecting primes represented by the end dots (other than their common end) of the three chains, consider a point lying in all the ϵ primes except N. It and all its images under the group $[3^{n,p,q}]$ are the vertices of a polytope which is called $n_{p,q}$‡. Similarly a point lying in all the primes except P, with its images, and one lying in all except Q, with its images, are the vertices of two other polytopes $p_{q,n}$, $q_{n,p}$. The faces (*i.e.* elements of highest dimension) of the polytope $n_{p,q}$ are of two kinds, namely $n_{p-1,q}$ and $n_{p,q-1}$. The subgroup of $[3^{n,p,q}]$ generated by all the ϵ reflexions except that in N, leaves fixed a vertex of $n_{p,q}$, and transforms into themselves a face $p_{q,n-1}$ of $p_{q,n}$, and a face $q_{n-1,p}$ of $q_{n,p}$. Thus the unique line through the origin which is fixed under this subgroup (unique because the subgroup, being $[3^{n-1,p,q}]$, operates in $\epsilon-1$ dimensions) is at once a circumradius of $n_{p,q}$, *i.e.* the line joining the centre to a vertex, and an inradius of $p_{q,n}$ and of $q_{n,p}$, *i.e.* the perpendicular from the centre on to a face. The three polytopes are, in fact, so related that the two sets of inradii of each are the circumradii of the other two. Polytopes so related are called semi-reciprocal, by analogy with the relation between two reciprocal polytopes, where the circumradii of each are the inradii of the other.

When $q=0$ these statements need some modification. The polytope $n_{p,0}$ is a regular simplex, and $p_{0,n}$ the simplex reciprocal to it; these depend not actually on n, p, but only on $n+p$, since, as we remarked, $[3^{n,p,0}]$ is the same thing as $[3^{n+p}]$. $0_{n,p}$, on the other hand, is the n-th truncation of the simplex $n_{p,0}$, *i.e.* its vertices are the centres of the n-dimensional elements of $n_{p,0}$; similarly, of course, it is the p-th truncation of $p_{0,n}$, the identity of these two truncations being an immediate consequence of the reciprocity of the two simplexes. We call the regular simplex in r dimensions α_r, and its n-th truncation $t_n\,\alpha_r$.

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† H. S. M. Coxeter, "Wythoff's construction for uniform polytopes", *Proc. London Math. Soc.* (2), 38 (1935), 327–339. Two papers on "The polytopes with regular-prismatic vertex-figures", *Phil. Trans. Royal Soc.* (A), 229 (1930), 329–425, and *Proc. London Math. Soc.* (2), 34 (1932), 126–189, contain other properties of the polytopes in question, but we do not need them here.

‡ I have inserted a comma between the suffixes, for more clarity, where they are not single symbols: I mention this, because Coxeter is precise over commas, *cf.* "Wythoff's construction", 331, foot-note ‡‡.

The polytope $n_{1,1}$ is the cross polytope, or analogue of the octahedron in $n+3$ dimensions; we indicate it by β_{n+3}. The reciprocal of this is the measure polytope, or analogue of the cube, which we call γ_{n+3}. On account of the semi-reciprocity of $n_{1,1}$, $1_{n,1}$, and $1_{1,n}$, we see that the identical but not coincident polytopes $1_{n,1}$ and $1_{1,n}$ have together just the vertices of a γ_{n+3}; we accordingly call either of them the half measure polytope, $h\gamma_{n+3}$. (The configuration is the generalization of the familiar one of two tetrahedra inscribed in a cube; the tetrahedron is, in fact, $h\gamma_3$, as well as being α_3.)

If n, p, q are all unequal, $[3^{n,p,q}]$ is the complete group of symmetry of each of the polytopes $n_{p,q}$, $p_{q,n}$, $q_{n,p}$; if $p=q\neq n$, this group is a self-conjugate subgroup of index 2 in a larger group, which we may call $2[3^{n,p,p}]$; the operations of this group (other than those of $[3^{n,p,p}]$) interchange the identical polytopes $p_{p,n}$, $p_{n,p}$, but are symmetry operations of $n_{p,p}$. The group $2[3^{n,1,1}]$, for instance, is the complete group of symmetry of β_{n+3}, or of the reciprocal γ_{n+3}. Similarly, if $n=p=q$, $[3^{n,n,n}]$ is a self-conjugate subgroup of index 6 in a group which we may call $6[3^{n,n,n}]$, whose operations permute the three identical polytopes $n_{n,n}$ in all possible ways; this group has three conjugate subgroups $2[3^{n,n,n}]$, which have in common just their subgroups $[3^{n,n,n}]$, and which are respectively the complete groups of symmetry of the three polytopes.

17. Consider now the neighbourhood $\mathbf{a}_\epsilon$ of the last base point A_ϵ; it is transformed by the operations of Γ into all the E-curves based on (A), *i.e.* these are its image set under Γ. On the other hand, its representative point on $S^{1,-1}_{\epsilon,-1}$, the terminus of the vector $\mathbf{a}_\epsilon$ from the origin O, lies in the primes of all the generating reflexions of Γ, given by (13.1), except $[(\epsilon-1)\,\epsilon]$, since $\mathbf{a}_\epsilon$ is invariant under all the operations but this one, so that its image set under Γ consists of the vertices of $(\epsilon-4)_{2,1}$. That is to say,

> *The set of all E-curves based on a set of ϵ points is represented by the vertices of the polytope $(\epsilon-4)_{2,1}$, the symmetry operations of the polytope permuting its vertices in the same manner that the Kantor group permutes the E-curves.*

It is to be remarked that, since $2\neq 1$, the Kantor group $[3^{\epsilon-4,2,1}]$ is the complete group of symmetry of the polytope.

18. If $\mathbf{f}$, $\mathbf{f}'$ are any two vectors, the squares of whose lengths are ν, ν' respectively, the distance between their end points is

$$\sqrt{(\nu+\nu'-2\mathbf{f}\cdot\mathbf{f}')}, \qquad (18.1)$$

so that, unless $\mathbf{f}\cdot\mathbf{f}' < \frac{1}{2}(\nu+\nu')$, this distance is spacelike, and is least when $\mathbf{f}\cdot\mathbf{f}'$ is least [so long as it does not fall below $\frac{1}{2}(\nu+\nu')$]. In particular, if $\nu'=\nu$, the distance is

$$\sqrt{\{-2(\mathbf{f}\cdot\mathbf{f}'-\nu)\}}, \tag{18.2}$$

so that the least spacelike distance between two points on the same sphere S_ϵ^ν is $\sqrt{(-2)}$, and all other distances between them are multiples of this by square roots of integers. Wherever we identify the representative points of an image set of systems under Γ with the vertices of a polytope inscribed in $S_{\epsilon-1}^{\sigma,\nu}$ (as we have just done for the E-curves, and as we shall shortly do for other types of system), we see that the edge length of this polytope is $\sqrt{(-2)}$; for this reason we shall call this particular length the edge unit. Points separated by this distance (*i.e.* joined by an edge of the polytope) are said to be adjacent. In the case of $(\epsilon-4)_{2,1}$,

> *The vertices representing two E-curves whose intersection number is h are at a distance of* $\sqrt{(1+h)}$ *edge units, and are adjacent if the intersection number is* 0.

19. Next consider the pencil $|\mathbf{b}_1|$ of lines $1, (1, 0^{\epsilon-1})$, with vertex at A_1. This is transformed by the Kantor group into all proper pencils of rational curves based on (A); for it can be shown, just as for a homaloidal net, that a pencil of rational curves with at least three base points can have its order lowered by a quadratic transformation, and hence by a succession of such transformations can be transformed into a pencil of lines, which is easily seen to be the only such pencil with less than three base points. We shall refer to these briefly as R-pencils.

The point of $S_{\epsilon-1}^{2,0}$, representing the pencil $|\mathbf{b}_1|$, lies in all the primes of (13.1) except [12]; we conclude that

> *The set of all R-pencils based on the* ϵ *points is represented by the vertices of the polytope* $2_{1,\epsilon-4}$.

Recalling the relation of the circumradii of $2_{1,\epsilon-4}$ to one set of inradii of $(\epsilon-4)_{2,1}$, namely those corresponding to faces $(\epsilon-4)_{1,1}$ or $\beta_{\epsilon-1}$, we expect that each R-pencil based on (A) will be related in a particular way to a set of $2(\epsilon-1)$ E-curves, corresponding to the vertices of a $\beta_{\epsilon-1}$ belonging to $(\epsilon-4)_{2,1}$. In fact, the $\beta_{\epsilon-1}$ in question will consist of those vertices of $(\epsilon-4)_{2,1}$ which are nearest to the vertex of $2_{1,\epsilon-4}$, and will accordingly represent those E-curves whose intersection number with the pencil is least.

Now the pencil $|\mathbf{b}_1|$ (and consequently every R-pencil) contains $\epsilon-1$ curves which break up into a couple of E-curves, namely the neighbourhood

$\mathbf{a}_\kappa$ and the line $A_1 A_\kappa$ $(2 \leqslant \kappa \leqslant \epsilon)$. These are clearly the only E-curves whose intersection number with the pencil is as low as 0; they form $\epsilon-1$ pairs such that the two members of a pair have intersection number 1, and all other intersection numbers amongst them are 0; hence they are represented by $\epsilon-1$ pairs of vertices of $(\epsilon-4)_{2,1}$, such that the two members of a pair are at distance $\sqrt{2}$ edge units, and all other distances amongst them are 1 edge unit; these are clearly the vertices of a $\beta_{\epsilon-1}$.

20. In the same way, since the net of lines $|\mathbf{l}|$ is invariant under all the reflexions (13.1) except [123], we see that

> *The representative points on $S^{3,1}_{\epsilon-1}$ of the H-nets based on the ϵ points are the vertices of the polytope $1_{\epsilon-4,2}$.*

In virtue of the semi-reciprocity, we expect that the E-curves whose intersection number with a particular H-net is least will be represented by the vertices of a face of $(\epsilon-4)_{2,1}$ of the other kind, namely $(\epsilon-4)_{2,0}$, or $\alpha_{\epsilon-1}$. This is, in fact, the case; for there are just ϵ E-curves whose intersection number with $|\mathbf{l}|$ is 0, namely the neighbourhoods $\mathbf{a}_\kappa$; and since every two of these have intersection number 0, they are represented by ϵ points every two of which are separated by 1 edge unit, *i.e.* the vertices of a simplex $\alpha_{\epsilon-1}$.

21. Though of less importance than the E-curves, R-pencils, and H-nets, a certain interest attaches to what we may call Q-systems, namely systems having $\nu = 2$, $\sigma = 4$, $\pi = 0$, $\rho = 3$, so that the projective model of such a system is a quadric surface Q^2. Such a system is the sum of two R-pencils whose intersection number is 1, the typical system $2,(1, 1, 0^{\epsilon-2})$ being $|\mathbf{b}_1 + \mathbf{b}_2|$. The corresponding vector is thus the perpendicular from the origin on to an edge of $2_{1,\epsilon-4}$, since the two R-pencils correspond to adjacent vertices; so that

> *The representative points of the Q-systems are the vertices of the polytope $t_1 2_{1,\epsilon-4}$, i.e. the mid-points of the edges of $2_{1,\epsilon-4}$.*

This could also be obtained by noting that the system $2,(1, 1, 0^{\epsilon-2})$ is invariant under all the reflexions (13.1) except [23], so that its representative point lies in all the primes except this one; the image set of this point is thus what Coxeter calls $(2_1 1\, \epsilon-4)$, which he shows to be the same as $t_1 2_{1,\epsilon-4}$.

The two $\beta_{\epsilon-1}$'s of $(\epsilon-4)_{2,1}$, corresponding to the pencils $|\mathbf{b}_1|$, $|\mathbf{b}_2|$, have an $\alpha_{\epsilon-2}$ in common, which is a face of both of them, and corresponds to the

$\epsilon-1$ E-curves $A_1 A_2$ and $\mathbf{a}_\kappa$ $(3 \leqslant \kappa \leqslant \epsilon)$. The circumradius of $t_1 2_{1,\epsilon-4}$ is also the perpendicular from the centre of $(\epsilon-4)_{2,1}$ on to this $\alpha_{\epsilon-2}$. These $\epsilon-1$ E-curves have intersection number 0 with the Q-system $|\mathbf{b}_1+\mathbf{b}_2|$, and represent the neighbourhoods of a set of points (A^*) on its projective model Q^2; every linear system in the plane based on (A) corresponds to a linear system on Q^2 based on (A^*).

22. There is, of course, another kind of $\alpha_{\epsilon-2}$ in $(\epsilon-4)_{2,1}$, namely the face of an $\alpha_{\epsilon-1}$. The vertices of a typical simplex of this kind correspond to the neighbourhoods of all but one of the base points (A), say of all but $\mathbf{a}_1$. It is clear that these E-curves are also among those which have zero intersection number with the pencil $|\mathbf{b}_1|$, so that the $\alpha_{\epsilon-2}$ in question is a face of the $\beta_{\epsilon-1}$ corresponding to this pencil.

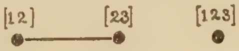
Since the subgroup of $[3^{\epsilon-4,2,1}]$ that leaves a $\beta_{\epsilon-1}$ fixed is $[3^{\epsilon-4,1,1}]$, the faces of each of these $\beta_{\epsilon-1}$'s fall into two sets, corresponding to the two $h\gamma_{\epsilon-1}$'s semi-reciprocal to it, the members of each set being equivalent under Γ; they are thus of the two kinds just described. Clearly, all the faces of an $\alpha_{\epsilon-1}$ are of the second kind.

In Coxeter's nomenclature these two types of $\alpha_{\epsilon-2}$ are readily distinguished as $(\epsilon-4)_{0,1}$ and $(\epsilon-4)_{1,0}$, the latter belonging to both $(\epsilon-4)_{2,0}$ and $(\epsilon-4)_{1,1}$, the former only to $(\epsilon-4)_{1,1}$.

23. The remarks in the preceding paragraphs require modification for small values of ϵ, since the polytopes $n_{p,q}$ are not defined for negative values of n, p, q.

In the case $\epsilon = 1$ there is only one E-curve $\mathbf{a}_1$, only one R-pencil $|\mathbf{b}_1|$, only one H-net $|\mathbf{l}|$, and no D-vector nor Q-system. For $\epsilon = 2$ there are three E-curves $\mathbf{a}_1$, $\mathbf{a}_2$, and $A_1 A_2$, which form (this case is unique in this respect) an unsymmetrical system, since $A_1 A_2$ alone of the three meets both the others; there are two R-pencils $|\mathbf{b}_1|$, $|\mathbf{b}_2|$, still only one H-net $|\mathbf{l}|$, one D-vector corresponding to the operation [12], and one Q-system $|\mathbf{b}_1+\mathbf{b}_2|$. There is no Cremona transformation based on (A_1, A_2) only, other than identity, and the group Γ is of order 2.

When $\epsilon = 3$, Γ operates in ordinary space; its graph is



so that (in this case only) Γ is reducible, being the direct product of the group of order 2 generated by [123] and the symmetric group of order 6 on the symbols 1, 2, 3 generated by [12], [23]. This is what is known in crystallography as the trigonal holohedral group, and is the complete group of symmetry of a triangular prism. This figure, in fact (of height equal to its base edge), is the polytope representing the E-curves, of which there are six, the neighbourhoods of the three base points being represented by the vertices of the upper triangle say, and the lines $A_\kappa A_\lambda$ by the lower. We may call this prism, if we like, $(-1)_{2,1}$. Its equatorial plane of symmetry is [123], and the three meridional planes of symmetry are the three planes $[\kappa\lambda]$. The two triangular faces α_2 correspond to the two H-nets $|1|$ and $2,(1,1,1)$; the three square faces β_2 to the three R-pencils $|b_\kappa|$; and the three vertical edges α_1 to the three Q-systems $|b_\kappa+b_\lambda|$. Thus the polytopes which we should like to call $2_{1,-1}$, $1_{-1,2}$ reduce respectively to a triangle inscribed in a great circle of $S_2^{2,0}$ and to a pair of antipodal points of $S_2^{3,1}$.

24. We return for a while to the general theory of polytopes, to develop the idea of a vertex figure. The vertex figure of a polytope is perhaps best defined as the base of the pyramid formed by the faces which meet at a vertex; or as the section by a prime perpendicular to the circumradius, and near the vertex. (For polytopes which have no circumsphere the vertex figure cannot be defined.) Evidently the definition applies explicitly to a particular vertex; but if the polytope is isogonal (*i.e.* if its group of symmetry is transitive on its vertices, as is by definition the case for all uniform polytopes), the vertex figure is the same whatever vertex we consider, and can therefore be considered a property of the polytope.

The vertex figure of a polytope in n dimensions is clearly a polytope in $n-1$ dimensions; every element of it is the vertex figure of a corresponding element of the original polytope; in particular, a face of the vertex figure is the vertex figure of a face of the original.

When (as in the case of a uniform polytope) the edges are all of the same length, and hence equally inclined to the circumradii at their ends, we can give the vertex figure an absolute size by the convention that the secant prime shall be perpendicular to the circumradius at such a point of it as to meet all the edges radiating from the given vertex at unit distance from that vertex; *i.e.* the pyramid of which the vertex figure is the base is to have unit slant height. If the original polytope is taken to have unit edge length, the vertices of the vertex figure are just those vertices of the original which are adjacent to the given one. With this convention, an m-gonal plane element of the original polytope corresponds to an edge of length $2\cos(\pi/m)$ in the vertex figure; and if all plane elements of the original are triangles, the vertex figure is of unit edge length.

25. If the original polytope Π is defined by the simplest type of Wythoff construction of the kind considered by Coxeter; namely, if one vertex lies

in all but one of the primes which generate the group by their reflexions, *i.e.*, if the polytope is represented in Coxeter's notation by a graph with only one dot ringed†, then the vertex figure is uniform, and has a Wythoff construction of its own, which I shall give here (though it is not very relevant to our purpose in all its generality) because it does not seem to have been published elsewhere. (It is stated without proof by Coxeter.)

Let a vertex X of Π lie in all the generating primes except R, say. The group of symmetry of the vertex figure Π' is that subgroup of the group of symmetry of Π which leaves fixed a particular vertex, which we may take to be X. This subgroup is, of course, generated by the reflexions in all the generating primes of the original group, except R. Further, taking Π to be of unit edge length, a typical vertex of Π' is the image of X in R, and this lies in all the generating primes of the subgroup which are perpendicular to R.

> *Thus we obtain the symbol for the vertex figure* Π' *by first omitting from the symbol for* Π *the ringed dot, which represents* R, *and then placing rings on those of the dots that are left which were linked to this dot, since these represent generating primes of the subgroup which are not perpendicular to* R.

(In the case, which we have not considered here, but which is considered by Coxeter, where other angles than $\frac{1}{2}\pi$, $\frac{1}{3}\pi$ exist between the primes, the same thing is true, except that the general vertex of Π', instead of being equidistant from the primes represented by the ringed dots in its symbol, will be at distances from them proportional to the cosines of their inclinations to R. Π' will not be strictly uniform if these angles are unequal, since it will have edges of various lengths. The only case of this that can occur in Euclidean space is where Π is a truncation of a regular polytope, when Π' is a prism whose constituents are regular, but of different edge lengths.)

In particular, we note that the vertex figure of $n_{p,q}$ is $(n-1)_{p,q}$ of unit edge length, for $n \geqslant 1$; and that of $0_{p,q}$ is the prism $[\alpha_p, \alpha_q]$, likewise of unit edge length. This latter we can, if we like, call $(-1)_{p,q}$; its two sets of inradii are parallel to the circumradii of the α_p and α_q reciprocal to its constituents, which we may accordingly call $p_{q,-1}$, $q_{-1,p}$, though the semi-reciprocation is not complete, since these two simplexes have no inradii parallel to the circumradii of the prism.

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† In Coxeter's notation, the graph representing a group, with rings on some of the dots, represents the polytope whose typical vertex is equidistant from the primes represented by ringed dots, and lies in the remaining generating primes of the group.

In particular,

The vertex figure of $(\epsilon-4)_{2,1}$ *is* $(\epsilon-5)_{2,1}$.

This evidently corresponds to the fact that the E-curves based on ϵ points that have zero intersection number with a particular one, which we take to be the neighbourhood of one of the points, are precisely those based on the remaining $\epsilon-1$ points.

26. Another important property of the polytopes $n_{p,q}$, which is not perhaps directly to our present purpose, but which I shall prove here because it has not been proved before, and I may need to refer to it in the further developments of this topic which I contemplate, is that they are vertex-determinate; by which I mean, that any isogonal polytope having in common with $n_{p,q}$ a particular vertex and all those adjacent to it, the vertex figures being the same, coincides with it entirely. This will be proved by induction, by showing first that the prism $[\alpha_p, \alpha_q]$ is vertex determinate, and secondly that, if the vertex figure Π' of an isogonal polytope Π of unique edge-length is isogonal and vertex determinate, then Π itself is also vertex determinate.

The first step is easy; for it is at once seen that, if X is a particular vertex of $[\alpha_p, \alpha_q]$, every other vertex is either adjacent to X, or is the fourth vertex of a square, of which one vertex is X and two others are adjacent to X. Thus the vertices not adjacent to X are directly determined by those which are.

Now let Π^* be an isogonal polytope having in common with Π a vertex X and all those adjacent to it, and having Π' as its vertex figure. The vertices adjacent to X form, of course, a polytope Π', which we distinguish as Π'_X. Let Y be one of these; the vertices adjacent to both X and Y form the vertex figure of Π', and will be distinguished as Π''_{XY}. Let the vertices of Π, Π^*, respectively adjacent to Y, be Π'_Y, Π'^*_Y, both, of course, polytopes Π'. They have in common the vertex X and those adjacent to it, namely those of Π''_{XY}; hence, since Π' is vertex determinate, they coincide entirely; *i.e.* every vertex of Π or Π^* adjacent to Y is also a vertex of the other. Similarly, of course, if Z is one of these, every vertex of either polytope adjacent to Z is also a vertex of the other. Now since a polytope is, by definition, a topological manifold, it is edge connected, that is every vertex can be joined to X by a sequence in which consecutive members are adjacent, so that by inducing from next to next we see that Π, Π' coincide entirely, which proves the theorem.

27. We may put the argument by which we showed Γ to be $[3^{\epsilon-4,2,1}]$ in a form which shows more clearly, from the point of view of the vector space R_ϵ, the significance of Nöther's inequality, on which we rely to prove that every Cremona transformation is a product of quadratic transformations. This inequality is

$$\left.\begin{aligned} i_1+i_2+i_3 &> m, \\ \text{where} \qquad i_1 \geqslant i_2 \geqslant i_3 \geqslant \dots &\geqslant i_\epsilon, \end{aligned}\right\} \tag{27.1}$$

i.e. the multiplicities are in descending order. This can be proved if $m,(i)$ is the characteristic of an H-net (except $|\mathbf{l}|$), an R-pencil (except $|\mathbf{b}_1|$), an E-curve (except $\mathbf{a}_\epsilon$), or a D-vector (except $[1\epsilon]$), indeed, any system for which $\pi=0$, $\nu \geqslant -3$, and which has at least three base points. (It can also be proved for certain classes of system having $\pi \neq 0$, but this is not now to our purpose.)

Now, since the reflexions (13.1) certainly belong to Γ, the group $[3^{\epsilon-4,2,1}]$ which they generate is a subgroup of Γ, if not the whole of it. If it is a proper subgroup, then the figure inscribed in $S^{1,-1}_{\epsilon-1}$, representing the E-curves, will consist of the vertices of $(\epsilon-4)_{2,1}$, together with some other points, transforms of the former set under operations of Γ not belonging to the subgroup; it cannot consist merely of the vertices of $(\epsilon-4)_{2,1}$, since of this configuration $[3^{\epsilon-4,2,1}]$ is the complete group of symmetry, and the figure in question must have the larger group of symmetry Γ. These extra points must lie in or upon the boundaries of the various fundamental regions into which $S^{1,-1}_{\epsilon-1}$ is divided by $[3^{\epsilon-4,2\,1}]$, and must be similarly distributed in each of these regions; hence at least one of them must be either interior to or upon the boundary of the particular fundamental region enclosed by the primes (13.1). (There are, in fact, two antipodal regions enclosed by these primes, of which either will do for us.) Now the defining inequalities of this region (strictly, of one of them) are

$$\left.\begin{aligned} i_1 \geqslant i_2 \geqslant i_3 \geqslant \dots &\geqslant i_\epsilon, \\ m &\geqslant i_1+i_2+i_3. \end{aligned}\right\} \tag{27.2}$$

Consequently, to assure ourselves that $[3^{\epsilon-4,2,1}]$ is the whole of Γ, we have to show that there is no E-curve except $\mathbf{a}_\epsilon$, or $0,(0^{\epsilon-1},-1)$, which satisfies (27.2). This, however, is just what the Nöther inequality (27.1) tells us.

28. In the case $\epsilon < 9$, the E-curves and other systems considered can be exhibited in a very convenient form upon the projective model of the anti-canonical system $|\mathbf{k}|$ or a multiple of it. For $\epsilon \leqslant 6$, $|\mathbf{k}|$ is simple, and

its projective model is the del Pezzo surface of order $9-\epsilon$ in $9-\epsilon$ dimensions; for $\epsilon=7$, $|\mathbf{k}|$ is compounded with an involution, that of Geiser, and its projective model is a double plane, branching along a general quartic curve, of genus 3; but $|2\mathbf{k}|$ is simple, its projective model being a surface considered by Castelnuovo[†], the general quadric section of the cone in six dimensions projecting a Veronese surface from a point; its prime sections are of genus 3. For $\epsilon=3$, $|\mathbf{k}|$ is a pencil, and $|2\mathbf{k}|$ is compounded with an involution, that of Bertini, its projective model being a double quadric cone, branching at its vertex and along a general sextic curve (cubic section of the cone) of genus 4; $|3\mathbf{k}|$, however, is simple, and its projective model is a surface of order 9 in six dimensions which is easily seen to be the section by a cubic primal (residual to three generating planes) of the cone projecting from a point a rational ruled quartic with linear directrix; the surface has on it a pencil of plane cubics (in the generating planes of the cone) which is the image of $|\mathbf{k}|$.

On the del Pezzo surfaces an E-curve appears as a line, an R-pencil as a pencil of conics, an H-net as a net of twisted cubics, and a Q-system as a system of rational quartics. The intersection relations between the lines are represented by the distances between the vertices of $(\epsilon-4)_{2,1}$, and the Kantor group Γ thus presents itself as the group of all permutations of the lines on the del Pezzo surface which leave invariant all their intersection relations. From this point of view the relation between the vertices of $2_{2,1}$ and the lines on the cubic surface was noted long ago as an isolated phenomenon, and was in fact the starting point of the whole of this investigation[‡].

29. By substituting in (8.2) the invariants of an E-curve, an R-pencil, and an H-net, and converting into terms of the edge unit, we obtain easily

Twice the squares of the circumradii of $(\epsilon-4)_{2,1}$, $2_{1,\epsilon-4}$, *and* $1_{\epsilon-4,2}$, *are, respectively,*

$$\frac{10-\epsilon}{9-\epsilon}, \quad \frac{4}{9-\epsilon}, \quad \frac{\epsilon}{9-\epsilon}; \tag{29.1}$$

[†] G. Castelnuovo, *Atti Acc. Torino*, 25 (1890), 695.

[‡] P. H. Schoute, *Proc. Royal Acad. Sci. Amsterdam*, 19 (1910), 375–383. He used Elte's results and discovery of the finite polytopes $n_{2,1}$, later published by the latter as his doctorate thesis, *The semiregular polytopes of the hyperspaces*, Groningen (1912). These polytopes had previously been described by Gosset in the *Messenger of Math.*, 29 (1899), 45–48, but his work remained unknown till quite lately, by which time it had all been repeated by Elte and again by Coxeter, and by G. S. Hardy, independently; but only Coxeter gave any systematic account of the groups of symmetry involved.

this, it may be noted, is in accordance with the general formula for the circumradius of $n_{p,q}$:

$$2r^2 = 1 + \frac{pq-1}{[n,\, p,\, q]} = \frac{[n-1,\, p,\, q]}{[n,\, p,\, q]}, \tag{29.2}$$

where $[n,\, p,\, q]$ stands for the discriminant $2+n+p+q-npq$. This formula (long ago noted by Coxeter) does not seem to have been published; it can easily be derived from the circumradii of the simplexes α_p, α_q by considering that the square of the circumradius of a prism is the sum of the squares of the circumradii of its constituents, and that the circumradii r, r' of a polytope and its vertex figure (both of unit edge length) are related by the formula

$$r^2 = \frac{1}{4(1-r'^2)}. \tag{29.3}$$

We observe in (29.1) that the circumradii of $2_{1,\epsilon-4}$ and $1_{\epsilon-4,2}$ are spacelike only if $\epsilon < 9$, and timelike if $\epsilon > 9$. This is what we should expect, since they are perpendicular to the faces of $(\epsilon-4)_{21}$, which are all Euclidean polytopes, $\alpha_{\epsilon-1}$, and $\beta_{\epsilon-1}$. $2_{1,\epsilon-4}$ and $1_{\epsilon-4,2}$ are accordingly (for $\epsilon > 9$) inscribed in two-sheeted spheres; each is, in fact, inscribed in a single sheet of its sphere. It would, of course, be possible to have a figure inscribed in both sheets of a two-sheeted sphere, and having symmetry operations which interchange the two sheets, but such a figure could not be a single polytope, since the parts in the two sheets would be entirely disconnected, unless the edges of the figure were timelike, and each had an end in each sheet, which is clearly not the case here.

The circumradius of $(\epsilon-4)_{2,1}$, on the other hand, is always spacelike, except in the cases $\epsilon = 9$ when it is infinite, and $\epsilon = 10$ when it is zero. This means that $6_{2,1}$ is inscribed in a light cone, or sphere of zero radius, while, for $n \geqslant 7$, $n_{2,1}$ is inscribed in a one-sheeted sphere. That $S_9^{1,-1}$ is a cone means, of course, that L_{10}^1 is a tangent prime of S_{10}^{-1}.

30. These Minkowskian polytopes have a number of curious properties which one does not at first expect. The centre of $(\epsilon-4)_{2,1}$ is the foot of the perpendicular from O or $0,(0^\epsilon)$ on to $L_\epsilon{}^1$, namely the point whose radius vector is $\mathbf{k}/(9-\epsilon)$; and it is easily verified that (for $\epsilon > 9$) this point is on the opposite side of any face of the polytope to all its vertices. For, if we take (i) as non-rectangular coordinates in $L_\epsilon{}^1$, the centre is the point $\{-1^\epsilon/(\epsilon-9)\}$; the prime containing the $\alpha_{\epsilon-1}$ corresponding to $|\mathbf{1}|$ is $\Sigma(i) = -1$, and whereas for the centre $\Sigma(i) = -\epsilon/(\epsilon-9) < -1$, for every E-curve

$\Sigma(i) \geqslant -1$; similarly the prime containing the $\beta_{\epsilon-1}$ corresponding to $|\mathbf{b}_1|$ is $\Sigma(i)-3i_1 = -1$, and whereas for the centre

$$\Sigma(i)-3i_1 = -(\epsilon-3)/(\epsilon-9) < -1,$$

for every *E*-curve $m \geqslant i_1$, *i.e.* $\Sigma(i)-3i_1 \geqslant -1$.

This, of course, is precisely contrary to what happens in Euclidean space. There, every face of a convex polytope, when produced, divides the circumsphere into two unequal parts, of which one, the greater, contains all the vertices of the polytope except those of the face itself; and the centre of the sphere is on the same side of the face as this larger portion. In the case before us, on the other hand, of the two portions into which the sphere is divided, it is the smaller (*i.e.*, that which is superposable on a proper part of the other) which contains all the vertices except those of the face, whereas the centre is still on the same side of the face as the larger portion of the sphere.

A rather similar phenomenon is the relation of the vertices to the cone projecting from the centre the circumsphere of a face; in the Euclidean case, of course, all the vertices except those of the face are exterior to this cone; in the present case, however, they are all interior to it.

31. We must guard ourselves, however, from supposing that we have here a star polytope, or anything of that kind; it is a perfectly proper convex polytope, since all the vertices are on the same side of any face, and hence no two faces can have an interior point in common. The region interior to the polytope is infinite but convex, and approximates roughly in shape to that interior to one sheet of a two-sheeted sphere; the centre, however, is not inside but outside the polytope. The cones projecting from the centre the circumspheres of the faces, being an enumerable infinity, may be regarded as enveloping† a cone to which they are all interior; the light cone at the centre is interior to them all. The former envelope, however, may be called the enveloping cone of the whole polytope; it has the property that every line through the centre interior to it meets just one element of the polytope, and penetrates into the interior of the latter; whereas every line through the centre exterior to the cone, is also wholely exterior to the polytope. The whole polytope lies within one sheet of the cone, in what we may call the positive region interior to the cone.

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† It must be borne in mind that the envelope of an enumerable infinity of loci is not really defined; there is certainly not a determinate quadric cone having the properties here outlined, since no one cone is invariant under Γ; but the region is most easily visualized as interior to a cone, and so we shall speak of it.

We can perhaps visualize all this a little more clearly by projecting the polytope from its centre on to the positive sheet (*i.e.*, that which is interior to the positive region) of any two-sheeted sphere concentric with it. Since all the inradii are timelike, the central portions of all the faces appear in the projection, and form a uniform partition of the sphere sheet; but the cells are so large that all their vertices are inaccessible; if we regard the sphere sheet as a Lobatschewskian space of $\epsilon-1$ dimensions, with which it is metrically identical, we may say that the whole of space is divided into $\alpha_{\epsilon-1}$'s and $\beta_{\epsilon-1}$'s so large that their vertices are all in the ideal field.

Indeed, as ϵ increases, not only the vertices but the edges, plane elements, etc., in turn disappear into the ideal; for it is clear that the sum of the vectors corresponding to s adjacent vertices of $(\epsilon-4)_{2,1}$ is the perpendicular from O on to an $(s-1)$-dimensional element, and that the projection of this vector on to a space L_ϵ^σ is parallel to a circumradius of $t_{s-1}(\epsilon-4)_{2,1}$. Such a vector, however, clearly has $\sigma = s$, $\nu = -s$, so that the circumradius of $t_{s-1}(\epsilon-4)_{2,1}$ is (of the same kind as) $r_{s,s}$, where

$$r^2_{s,-s} = s\left(\frac{s}{\epsilon-9}-1\right),$$

and is thus only timelike for $9 < \epsilon < 9+s$; this means that for $\epsilon \geqslant 11$ the edges, for $\epsilon \geqslant 12$ the plane elements, ..., and for $\epsilon \geqslant 9+s$ the $(\epsilon-1)$-dimensional elements, of the partition of Lobatschewskian space are entirely in the ideal field.

32. It is of interest to see in what cases $[3^{\epsilon-4,2,1}]$ contains the central inversion, *i.e.* the operation whereby every vector of its space (which is any R_ϵ^σ) is reversed in sense only; this operation is clearly the product of the similarly defined central inversion in $R_{\epsilon+1}$ and the reflexion in the prime R_ϵ^0, and may accordingly be expressed by the operator

$$\left(2\frac{\mathbf{kk}\cdot}{\mathbf{k}\cdot\mathbf{k}}-1\right). \tag{32.1}$$

This, of course, gives an integer operator (*i.e.* transforms every integer vector into an integer vector) only if $\mathbf{k}\cdot\mathbf{k}$ is a factor of 2, *i.e.* if

$$\epsilon = 7,\ 8,\ 9,\ \text{or}\ 11.$$

The last two cases, however, give operations that interchange the two regions interior to the enveloping cone, and thus do not belong to Γ. Thus

> *The central inversion belongs to Γ only for $\epsilon = 7, 8$, in which cases it corresponds to the involutory operation which interchanges the two points in every pair of the Geiser or Bertini involution.*

Another question of some interest is when are primes of symmetry of $[3^{\epsilon-4,2,1}]$ parallel to the faces of $(\epsilon-4)_{2,1}$; if this happens, it means that the projection of either $\mathbf{l}$ or $\mathbf{b}_1$ perpendicular to $\mathbf{k}$ is a D-vector. Now the vector of length $\sqrt{-2}$ parallel to the projection of $\mathbf{l}$ perpendicular to $\mathbf{k}$ is

$$\pm\sqrt{\left(\frac{2(9-\epsilon)}{\epsilon}\right)}\left(\mathbf{l}-\frac{3\mathbf{k}}{9-\epsilon}\right), \tag{32.2}$$

which is an integer vector only for $\epsilon=3$ (when it is $\mathbf{k}-2\mathbf{l}$) and for $\epsilon=6$ (when it is $\mathbf{k}-\mathbf{l}$). Thus we have

$(\epsilon-4)_{2,1}$ *has primes of symmetry parallel to its faces* $\alpha_{\epsilon-1}$ *only for* $\epsilon=3,\ 6$.

In just the same way we may find that

$(\epsilon-4)_{2,1}$ *has primes of symmetry parallel to its faces* $\beta_{\epsilon-1}$ *only for* $\epsilon=7$, *and perpendicular to its circumradii only for* $\epsilon=8$.

33. A similar classical problem is the discovery of symmetrical transformations, *i.e.* those determined by an H-net whose base multiplicities are all equal. Such a net must correspond to the point of $S^{3,1}_{\epsilon-1}$ antipodal to that representing $|\mathbf{l}|$; *i.e.* if a symmetrical transformation exists, the central inversion must be a symmetry operation of $1_{\epsilon-4,2}$, even if it is not of $(\epsilon-4)_{2,1}$. The effect of the operation (32.1) on $\mathbf{l}$ is to give the vector

$$\frac{6\mathbf{k}}{9-\epsilon}-\mathbf{l},$$

which evidently is only an integer vector if $9-\epsilon$ is a factor of 6, *i.e.* if

$$\epsilon=3,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,\ \text{or}\ 15.$$

The nets in these eight cases are

$$2,(1^3);\quad 5,(2^6);\quad 8,(3^7);\quad 17,(6^8);$$
$$-19,(-6^{10});\quad -10,(-3^{11});\quad -7,(-2^{12});\quad -4,(-1^{15}).$$

The last four cases clearly give nothing of value, for these virtual nets do not determine Cremona transformations, and there are no operations of Γ corresponding to them. The first two correspond to reflexions in primes parallel to faces α_2, α_5, of $(-1)_{2,1}$, $2_{2,1}$ respectively; in the first case $1_{-1,2}$ consists merely of a pair of antipodal points, and has thus the central inversion as a symmetry operation [though, of course, the prism $(-1)_{2,1}$ has not this symmetry], in the other the central inversion belongs to the group $2[3^{2,2,1}]$, and is thus a symmetry operation of $1_{2,2}$, though not of $2_{2,1}$.

In the next two cases the central inversion, as we have just seen, belongs to Γ, and is the symmetrical transformation. In the next two cases, the central inversion is an integer operation, but does not belong to Γ; while in the last two cases the virtual net does not correspond to any integer operation at all, there not being ϵ vectors with the invariants of E-curves, and having zero intersection numbers with the virtual net.

34. There may, as we have seen, be characteristics which satisfy the conditions, as far as the values of their invariants are concerned, for an H-net, R-pencil, or E-curve; but which yet are not systems of this kind, either because the general curve of the system is reducible, having a fixed part, the variable part having quite other invariants, or because the order of the general curve is negative, and the system accordingly merely virtual. These will be called improper H-nets of the first and second kind respectively. Instances of improper H-nets of the second kind appeared in the previous section; we shall now find some improper systems of the first kind.

It is easily seen that a system of virtual genus 1 and virtual freedom ρ, together with a fixed E-curve whose intersection number with it is zero, constitutes a system of virtual genus 0 and virtual freedom ρ. Systems of virtual genus 1 are most easily constructed by taking cubics through a suitable number of base points, *i.e.* by adding to the system $\mathbf{k}$ the neighbourhoods of a suitable number of the base points, $\rho+\epsilon-9$ of them if we want a system of virtual freedom ρ. The exceptional curves having intersection number zero with this system are just these neighbourhoods, so that by adding one of them again we obtain an improper system of virtual genus 0 and virtual freedom ρ. Thus

The system

$$\left|\mathbf{k}+\sum_{\kappa=\lambda}^{\epsilon-1}\mathbf{a}_\kappa+2\mathbf{a}_\epsilon\right|, \quad \text{or} \quad 3,(1^{\lambda-1},0^{\epsilon-\lambda},-1),$$

is an improper H-net, R-pencil, or E-curve of the first kind, if $\lambda=8$, 9, *or* 10, *respectively.*

Of course, this system only exists if $\epsilon \geqslant \lambda$; if $\epsilon=\lambda$ the summation term is absent, and the system is simply $|\mathbf{k}+2\mathbf{a}_\epsilon|$.

The representative point of this system evidently lies in all the primes (13.1) except $[(\lambda-1)\lambda]$ and $[(\epsilon-1)\epsilon]$, from which it is equidistant. It is thus clear that its images under the group Γ are the vertices of a polytope $t_{0,\epsilon-\lambda}(\epsilon-4)_{2,1}$ if $\epsilon>\lambda$, or $(\epsilon-4)_{2,1}$ if $\epsilon=\lambda$; in the former case the polytope is easily found to be of edge length 1 edge unit, in the latter case 2 edge units.

In the same way, since the system $3s, (s^9)$, or $\left| s\left(\mathbf{k} + \sum_{\kappa=10}^{\epsilon} \mathbf{a}_\kappa\right) \right|$, has virtual genus 1 and virtual freedom 0, we have, for $\epsilon \geqslant 10$,

The system

$$\left| s\left(\mathbf{k} + \sum_{\kappa=10}^{\epsilon-1} \mathbf{a}_\kappa\right) + (s+1)\,\mathbf{a}_\epsilon \right|, \quad \textit{or} \quad 3s, (s^9, 0^{\epsilon-10}, -1),$$

is an improper E-curve.

35. The improper systems of the first kind which we have here found are absent for low values of ϵ; we find them, in fact, only for $\epsilon \geqslant 8$. This, of course, is due to our arbitrary limitation of attention to H-nets, R-pencils, and E-curves; we could, for instance, obtain improper Q-systems by putting $\lambda = 7$ in the last paragraph, and these would appear for $\epsilon \geqslant 7$. We disregard these, however.

We have for $\epsilon = 8$ a set of 240 improper H-nets, represented by the vertices of a $4_{2,1}$ of edge length 2 edge units inscribed in $S_7^{3,1}$, as well as the proper H-nets represented by the vertices of $1_{4,2}$. Similarly, for $\epsilon = 9$ we have a set (now of course infinite) of improper R-pencils represented by the vertices of a $5_{2,1}$ of edge 2 edge units inscribed in $S_8^{2,0}$, in addition to the proper R-pencils represented by the vertices of $2_{1,5}$; and a set of improper H-nets represented by the vertices of a $t_{0,1}5_{2,1}$ inscribed in $S_8^{3,1}$, in addition to the proper H-nets represented by the vertices of $1_{5,2}$. In my previous paper referred to above, I gave calculations which I shall not repeat here, showing that these polytopes exhaust the integer points of $S_8^{2,0}$ and $S_8^{3,1}$, while $5_{2,1}$ itself exhausts the integer points of $S_8^{1,-1}$. It follows incidentally from this that there are no improper H-nets, R-pencils, nor E-curves, other than those we have found, for $\epsilon \leqslant 9$.

For $\epsilon = 10$ the situation is more complicated. We have inscribed in $S_9^{3,1}$ a $t_{0,2}6_{2,1}$ representing improper H-nets, and in $S_9^{2,0}$ a $t_{0,1}6_{2,1}$ representing improper R-pencils. In $S_9^{1,-1}$, on the other hand, we have ∞^1 $6_{2,1}$'s inscribed, one of edge length $s+1$ edge units representing improper E-curves of the form

$$|s\mathbf{k} + (s+1)\,\mathbf{a}|$$

(where $\mathbf{a}$ is a proper E-curve) for each value of s, positive and negative; except $s = 0$, which corresponds to the proper E-curves, and $s = -1$, for which the edge length is 0, and which gives only the single improper curve $|-\mathbf{k}|$. Of course, all the improper E-curves corresponding to negative values of s are of the second kind.

36. We shall now prove that these $6_{2,1}$'s exhaust the integer points of $S_9^{1,-1}$. In the first place, every improper E-curve contains some multiple of a proper E-curve as a part, having zero intersection number with the residual part; for the derivation of the Nöther inequality depends only on the invariants of the system, and the assumption that all its base multiplicities are positive, and consequently, if an improper E-curve has all its base multiplicities positive, it can be transformed by a succession of quadratic transformations into systems of successively lower order till a stage is reached at which some of the multiplicities are negative; this stage will certainly be reached, since it is impossible to transform the system into one of order less than or equal to 1, or it would be a proper E-curve.

For the system having some base multiplicities negative let $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_\epsilon$, so that i_ϵ is certainly negative, and equal to $-h$, say. We have now

$$\sum_{\kappa=1}^{\epsilon-1} (i_\kappa^2) = m^2+1-h^2 \leqslant m^2,$$

$$\sum_{\kappa=1}^{\epsilon-1} (i_\kappa) = 3m-1+h \geqslant 3m,$$

whence

$$\frac{m}{3} \sum_{\kappa=1}^{\epsilon-1} (i_\kappa) \geqslant \sum_{\kappa=1}^{\epsilon-1} (i_\kappa^2), \tag{36.1}$$

equality holding only if $h=1$. On the other hand, since the arithmetic mean of the squares of the quantities (i) other than i_ϵ is not less than the square of their arithmetic mean, equality holding only when they are themselves all equal, we have

$$(\epsilon-1) \sum_{\kappa=1}^{\epsilon-1} (i_\kappa^2) \geqslant \left[\sum_{\kappa=1}^{\epsilon-1} (i_\kappa) \right]^2, \tag{36.2}$$

and (36.1), (36.2) together give

$$(\epsilon-1)\frac{m}{3} \geqslant \sum_{\kappa=1}^{\epsilon-1} (i_\kappa) \geqslant 3m,$$

equality holding only when $h=1$ and all the base multiplicities (i) except i_ϵ are equal. This clearly requires

$$\epsilon-1 \geqslant 9,$$

so that

$$\epsilon \geqslant 10,$$

and if $\epsilon=10$ it requires $h=1$ and $i_1=i_2=\ldots=i_9=m/3$, which is precisely the case we have considered.

37. We may now consider the group $\Gamma^\dagger$ of all symmetry operations of the lattice of integer points in each R_ϵ^σ; *i.e.* of all homogeneous linear

transformations acting on the characteristics $m,(i)$, whose coefficients are all integers, and which leave invariant the quadratic form (3.2) and the linear form (5.2). Of this group, as we have seen, Γ is a subgroup, and may be the whole.

It is to be observed that, since $\Gamma(\epsilon, 2)$ is the whole group of symmetry of $(\epsilon-4)_{2,1}$, if Γ is a proper subgroup of $\Gamma^{\dagger}$, then the configuration of lattice points inscribed in $S^{1,-1}_{\epsilon-1}$ does not merely consist of the vertices of $(\epsilon-4)_{2,1}$, *i.e.* there are some improper E-curves. As we have observed that an improper E-curve has at least ten base points,

> *For* $\epsilon \leqslant 9$, $\Gamma^{\dagger}$ *coincides with* Γ.

Turning to higher values of ϵ, we have to distinguish between positive and negative operations of $\Gamma^{\dagger}$, *i.e.* those which leave invariant and those which interchange the two regions interior to the isotropic cone with vertex at L^{σ}. The operations of Γ, as we have seen, are all positive; there will always, however, be negative operations in $\Gamma^{\dagger}$ for $\epsilon \geqslant 11$, since the reflexion $(1-\mathbf{cc}\cdot)$, where $\mathbf{c}\cdot\mathbf{c}=2$, $\mathbf{k}\cdot\mathbf{c}=0$, is a negative operation, and an instance of a vector $\mathbf{c}$ having these invariants is given by $4,(2, 1^{10})$, which has eleven base points. We can go further, however, and assert the existence of negative operations also for $\epsilon = 10$, since we saw in sections 32, 33 that, for $\epsilon = 10$, the central inversion belongs to $\Gamma^{\dagger}$ though not to Γ, and in Minkowskian space this is a negative operation. Thus

> *For* $\epsilon \geqslant 10$, $\Gamma^{\dagger}$ *contains both positive and negative operations, whereas* Γ *consists entirely of positive operations.*

Since the product of two positive or two negative operations is positive, and that of a positive and a negative operation negative, it is clear that

> *For* $\epsilon \geqslant 10$ *the positive operations in* $\Gamma^{\dagger}$ *form a self-conjugate subgroup of index* 2.

This we shall call the positive subgroup; $\Gamma^{\dagger}$ is clearly generated by adjoining to the positive subgroup any negative operation. In particular,

> *When* (*i.e. for* $\epsilon = 10, 11$) *the central inversion belongs to* $\Gamma^{\dagger}$, $\Gamma^{\dagger}$ *is the direct product of its positive subgroup with the group of order* 2 *generated by the central inversion.*

38. In the case $\epsilon = 10$ things are fairly simple. We know what the configuration of integer points inscribed in $S^{1,-1}_{9}$ is like; those that have no other points of the configuration directly in line between themselves and the centre L^{1} (which, being thus descriptively separated out from the rest, form a configuration invariant under $\Gamma^{\dagger}$) are the vertices of two $6_{2,1}$'s of edge

length 1 edge unit, one inscribed in the positive and the other in the negative sheet of the cone $S_9^{1,-1}$, corresponding to $s=0, -2$ respectively in section 34. Each of these must be transformed into itself by the positive operations of Γ, so that, since $\Gamma=[3^{6,2,1}]$ is the whole group of symmetry of $6_{2,1}$, we have

> *For* $\epsilon=10$ *the positive subgroup of* $\Gamma^\dagger$ *coincides with* Γ, *and* $\Gamma^\dagger$ *is the direct product of* Γ *with the group of order* 2 *generated by the central inversion.*

On the other hand, for $\epsilon=11$ we have the improper D-vector $3,(1^{10},-1)$, whose reflexion transforms every neighbourhood $\mathbf{a}_\kappa$ (except $\mathbf{a}_{11}$) into an improper E-curve of the first kind, $\mathbf{a}_{10}$ for instance becoming $3,(1^9, 0, -1)$; so that here we have a positive operation belonging to $\Gamma^\dagger$ but not to Γ; and we conclude that

> *For* $\epsilon \geqslant 11$ *the positive subgroup of* $\Gamma^\dagger$ *is larger than* Γ, *and has* Γ *as a proper non-self-conjugate subgroup; the operations of the positive subgroup permute amongst themselves transitively the proper E-curves and certain improper E-curves of the first kind, while the negative operations of* $\Gamma^\dagger$ *interchange this set with a certain set of improper E-curves of the second kind.*

Further consideration of the properties of $\Gamma^\dagger$, particularly as to its generation, and whether or not it is, as I suspect, a generalized Coxeter group, we leave on one side, noting only that the problem is not without interest, and does not appear to be perfectly trivial.

39. The preceding investigation has an analogue with regard to Cremona transformations in any number of dimensions, upon which I may briefly touch here. In projective space of k dimensions a general set of $k+1$ points determines a homaloidal system of k-ic primals, having each of the points as $(k-1)$-ple, each of the lines joining them by pairs as $(k-2)$-ple, and generally the joins of r of them as $(k-r)$-ple, base elements, all these base multiplicities being necessary consequences of those at the points only. For $k=2$ this is the H-net corresponding to the ordinary quadratic transformation. We shall call the Cremona transformation defined by this system a primitive transformation; the exceptional primals, which become neighbourhoods of points, are the joins of all but one of the base points; to the point arising from each of these we give the name of the remaining point, thus completely determining an operation analogous to those of the group Γ in the case $k=2$. We indicate the primitive transformation based on points $(A_{\kappa_1} \dots A_{\kappa_{k+1}})$ by the symbol $[\kappa_1 \dots \kappa_{k+1}]$.

Now, given any set of points $(A_1 \dots A_\epsilon)$, we can base on them certain primitive transformations [one on every set of $k+1$ out of the points (A)]

and these generate a group, each of whose operations transforms into the neighbourhoods of points ϵ primals belonging to a certain set based on (A), which we shall call the E-primals; and every such operation is a Cremona transformation having the property that it is completely determined by the assigned multiplicities of its homaloids at the points of the set, their multiplicities at base elements of higher dimensions being necessary consequences of these. This group we shall call $\Gamma(\epsilon, k)$.

40. The primitive transformation $[12\ldots(k+1)]$ transforms the linear system of m-ic primals with (i)-ple assigned base points at the points (A) into one of m'-ics with (i')-ple assigned base points, where

$$\left.\begin{aligned} m' &= km - \sum_{\kappa=1}^{k+1} (i_\kappa), \\ i_\kappa' &= (k-1)m - \sum_{\lambda(\neq\kappa)=1}^{k+1} (i_\lambda), \end{aligned}\right\} \tag{40.1}$$

which, as is easily seen, leaves invariant the linear form

$$(k+1)m - \sum_{\kappa=1}^{\epsilon} (i_\kappa), \tag{40.2}$$

and the bilinear form

$$(k-1)mn - \sum_{\kappa=1}^{\epsilon} (i_\kappa j_\kappa) \tag{40.3}$$

relating to two linear systems $m, (i)$ and $n, (j)$. Every operation of the group Γ accordingly leaves these forms invariant, since it is a product of primitive operations; and the whole representation by vectors in a Minkowskian space can be carried through exactly as we have done in the case $k = 2$. The vector whose product with a given vector is the linear form (40.2) [products being defined by (40.3)] is

$$\mathbf{K} = \frac{k+1}{k-1}, (1^\epsilon), \tag{40.4}$$

which, of course, is only an integer vector for $k = 2, 3$.

It is evident that the whole group Γ is generated by the primitive transformation $[12\ldots(k+1)]$ and the interchanges $[(\kappa-1)\kappa]$ of consecutive neighbourhoods; and that the diagram corresponding to (13.1) will have $[12\ldots(k+1)]$ linked only to $[(k+1)(k+2)]$; whence

The group $\Gamma(\epsilon, k)$ *is the Coxeter group* $[3^{\epsilon-k-2, k, 1}]$. *There is a polytope* $(\epsilon-k-2)_{k,1}$ *whose vertices represent the E-primals, and* Γ *is the whole group of symmetry of this.*

Either from Coxeter's discriminant (12.1), or from the condition for the vector **K** to be timelike or spacelike, we find

$\Gamma(\epsilon, k)$ is a finite Coxeter group, an infinite Coxeter group, or a generalized Coxeter group, according as

$$\epsilon \lesseqgtr \frac{(k+1)^2}{k-1}. \tag{40.5}$$

Owing to the identity of $[3^{k, k', 1}]$, $[3^{k', k, 1}]$, we see that

$\Gamma(\epsilon, k)$ is simply isomorphic with $\Gamma(\epsilon, k')$, where

$$k+k' = \epsilon - 2.$$

We may note that the condition of equality in (40.5) is satisfied in only three cases, only two of which are essentially distinct, namely $k = 2$ or 5, $\epsilon = 9$, when the group is $[3^{5, 2, 1}]$, and $k = 3$, $\epsilon = 8$, when it is $[3^{3, 3, 1}]$. In these cases the group has the distinctive features of the group of symmetry of a lattice, such as the possession of a self-conjugate Abelian subgroup of finite index, consisting of all the translations contained in the group; and (for every positive integer n) the possession of $n^{\epsilon-1}$ conjugate subgroups of index $n^{\epsilon-1}$, each consisting of those operations which leave invariant one of the $n^{\epsilon-1}$ lattices similar to the given one, inscribed in it, and of n times its linear dimensions.

It may be hoped that it will ultimately be possible to define Γ not as being generated by primitive transformations, but by some descriptive property of its operations, such as that of having assigned base points only (other base elements being consequences of these, so that they need not be assigned); and to give descriptive definitions of the invariants (40.2), (40.3), which will make their invariance evident, as it is for $k = 2$, and not merely the fortuitous result of calculation. We may, in fact, note that the vector $(k-1)$ **K** corresponds to the anticanonical system, so that only (40.3) needs to be explained. This I am at present able to do only for the case $k = 3$, to which I hope to devote a paper soon; but it seems reasonable to suppose that an occurrence which is rational and explicable in its first two cases will not continue to repeat itself endlessly as a mere freak of nature.

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THEOREMS ON FOURIER SERIES AND POWER SERIES (II)†

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[Received 18 June, 1936.—Read 18 June, 1936.]

Introduction.

1. Suppose that $f(\theta) \epsilon L$, that is to say that f is integrable in the sense of Lebesgue in $(-\pi, \pi)$; and that

$$\tfrac{1}{2}a_0+\sum_1^\infty (a_n \cos n\theta+b_n \sin n\theta), \qquad \sum_{-\infty}^{\infty} c_n e^{n\theta i}$$

are the ordinary and the "complex" Fourier series of f. We suppose, to avoid trivial complications in stating our results, that $c_0=0$. Let

$$u(\rho, \theta)=\sum_1^\infty (a_n \cos n\theta+b_n \sin n\theta)\rho^n, \quad v(\rho, \theta)=\sum_1^\infty(-b_n \cos n\theta+a_n \sin n\theta)\rho^n;$$

$$\phi(z)=\phi(z, f)=u+iv=2\sum_1^\infty c_n z^n \quad (z=\rho e^{i\theta});$$

$$g(\theta)=g(\theta, f)=\left(\int_0^1 (1-\rho)\,|\phi'(\rho e^{i\theta})|^2 d\rho\right)^{\frac{1}{2}}.$$

Let α and $\beta \geqslant \alpha$ be positive constants, and let $\nu_1, \nu_2, \ldots$ be any sequence of positive integers satisfying

$$(1+\alpha)\,\nu_n \leqslant \nu_{n+1} \leqslant (1+\beta)\,\nu_n,$$

and $\nu_1', \nu_2', \ldots$ any other such sequence (with the same α, β). Let

$$\Delta_0=\sum_1^{\nu_1} c_m e^{m\theta i}, \quad \Delta_n=\sum_{\nu_n<m\leqslant\nu_{n+1}} c_m e^{m\theta i} \quad (n>0).$$

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† We are greatly indebted to Prof. G. H. Hardy, who read a rough draft of the paper, and detected and supplied some serious omissions.

These Δ_n we may call "long pieces" of the "positive" part of the complex Fourier series of f (each is of a length comparable with the whole of the earlier part of the series: the typical case is $\nu_n = 2^n$). The negative part $\sum\limits_{-\infty}^{-1}$ of the Fourier series is similarly divided into "long pieces" $\Delta_{-1}, \ldots, \Delta_{-n}, \ldots$ derived from the sequence $(-\nu_1', -\nu_2' \ldots)$.

Let p, q, r denote indices satisfying respectively

$$r > 1, \quad 1 < p \leqslant 2 \leqslant q,$$

and let $A(x, y, \ldots)$, or $A_{x, y, \ldots}$, denote a positive number, not always the same from one occurrence to another, depending only on the parameters $x, y, \ldots$ shown explicitly. Then we have the following theorems, in which the range of integration of θ is always $(-\pi, \pi)$.

THEOREM 1.

$$\sum_{-\infty}^{\infty} \int |\Delta_n|^q \, d\theta \leqslant A(q, \alpha, \beta) \int |f|^q \, d\theta.$$

THEOREM 2.

$$\int |f|^p \, d\theta \leqslant A(p, \alpha, \beta) \sum_{-\infty}^{\infty} \int |\Delta_n|^p \, d\theta.$$

THEOREM 3. *The ratio*

$$\int \left(\sum_{-\infty}^{\infty} |\Delta_n|^2 \right)^{\frac{1}{2}r} d\theta \Big/ \int |f|^r \, d\theta$$

lies between two numbers $A(r, \alpha, \beta)$†.

THEOREM 4. *Suppose that* (η_n) $(-\infty < n < +\infty)$ *is any set of numbers each of which has one of the two values* $+1$ *and* -1. *Suppose that* $f \in L^r$. *Then*

$$\sum_{-\infty}^{\infty} \eta_m \Delta_m$$

is (when written at length as a "complex" trigonometrical series) the Fourier series of a function F *of* L^r, *and*

$$\int |F|^r \, d\theta \leqslant A(r, \alpha, \beta) \int |f|^r \, d\theta.$$

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† More strictly: if either the numerator or the denominator is finite, then so is the other, and the ratio lies between $A(r, \alpha, \beta)$'s. A similar gloss is needed in Theorem 7.

THEOREM 5.

$$\int_0^1 d\rho \int_{-\pi}^{\pi} (1-\rho)^{q-1} |\phi'(z)|^q \, d\theta \leqslant A(q) \int_{-\pi}^{\pi} |f|^q \, d\theta.$$

THEOREM 6.

$$\int_{-\pi}^{\pi} |f|^p \, d\theta \leqslant A(p) \int_0^1 d\rho \int_{-\pi}^{\pi} (1-\rho)^{p-1} |\phi'(z)|^p \, d\theta.$$

THEOREM 7. *The ratio*

$$\int_{-\pi}^{\pi} |g|^r \, d\theta \Big/ \int_{-\pi}^{\pi} |f|^r \, d\theta$$

lies between two numbers $A(r)$.

We add finally, for completeness, a strengthened form of Theorem 7 for the special case when q is an even integer $2k$. This is essential for the deduction of Theorems 3 and 4, but we have not attempted to develop it further for its own sake.

THEOREM 7*. *Let*

$$g^*(\theta) = \left(\int_0^1 (1-\rho)\, \chi^2(\rho, \theta) \, d\rho \right)^{\frac{1}{2}},$$

where†

$$\chi^2(\rho, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\phi'(\rho e^{i(\theta+t)})|^2 P(\rho, t) \, dt.$$

Then

$$\int g^{*2k} \, d\theta \leqslant A(k) \int |f|^{2k} \, d\theta.$$

2. Theorems 1 to 7 (with some minor differences of detail, and along with some other theorems) were stated without proof in a former communication with the same general title‡. We explained how there is a heuristic connection, which we will call "*H*", between the "Fourier series group" of Theorems 1, 2, 3, and the "power series group" of Theorems 5, 6, 7§.

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† $P(\rho, \theta) = (1-\rho^2)/(1-2\rho \cos \theta + \rho^2)$ is the "Poisson kernel" (defined again below).

‡ Littlewood and Paley, **4**.

§ 1 corresponds with 5, 2 with 6, 3 with 7. 4 has no interesting analogue. It is derived from 3, and is best ignored in the explanation above, as is the difference between 7 and 7*.

The deepest theorem is 3 (with 4). 3 is deduced from 7 (actually 7*) by a (difficult) argument A founded on H. 1 and 2 are trivial corollaries of 3. They can also be derived from 5 and 6 respectively by arguments A' and A'' founded on H, and like A, but far simpler. We should then expect, by analogy, that 5 and 6 would be easy consequences of 7; the deductions are, in fact, fairly easy, though not entirely trivial like those of 1 and 2 from 3.

There are also *ab initio* proofs of 5 and 6, not depending on the deep Theorem 7. But there are no *ab initio* proofs of 1 and 2, which come at best by A' and A'' (from 5 and 6). No Fourier theorem is proved except *via* a power series theorem.

We give a diagram at the end of the paper, showing in more detail the various possible chains of deductions.

3. It will be convenient to summarize for reference a number of well-known results in the theory of power series and Fourier series. We first summarize our notation and fundamental hypotheses (repeating for completeness some of § 1).

$$r > 1, \quad 1 < p \leqslant 2 \leqslant q.$$

$$f \in L, \quad f \sim \sum_{-\infty}^{\infty} c_n e^{n\theta i}, \quad c_0 = 0.$$

$$P(\rho, \theta) = \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}.$$

u is the Poisson integral of f,

$$u(\rho, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+t)\,P(\rho, t)\,dt;$$

v is the conjugate of u, with $v(0) = 0$;

$$\phi(z) = \phi(z, f) = u + iv;$$

also

$$\phi(z) = 2\sum_{1}^{\infty} c_n z^n.$$

$$g(\theta) = g(\theta, f) = \left(\int_0^1 (1-\rho)\,|\phi'(\rho e^{i\theta})|^2\,d\rho\right)^{\frac{1}{2}}.$$

$$g^*(\theta) = \left(\int_0^1 (1-\rho)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi'(\rho e^{i(\theta+t)})|^2\,P(\rho, t)\,dt\right)d\rho\right)^{\frac{1}{2}}.$$

The sequences (ν_n), (ν_n') $(n=1, 2, \ldots)$ satisfy

$$(1+\alpha)\,\nu_n \leqslant \nu_{n+1} \leqslant (1+\beta)\,\nu_n,$$

$$\Delta_0 = \sum_{0<m\leqslant\nu_1} c_m e^{m\theta i}, \quad \Delta_n = \sum_{\nu_n<m\leqslant\nu_{n+1}} c_m e^{m\theta i} \qquad (n>0),$$

$$\Delta_{-1} = \sum_{0<m\leqslant\nu_1'} c_{-m} e^{-m\theta i}, \quad \Delta_{-n} = \sum_{\nu_n'<m\leqslant\nu_{n+1}'} c_{-m} e^{-m\theta i} \quad (n>1).$$

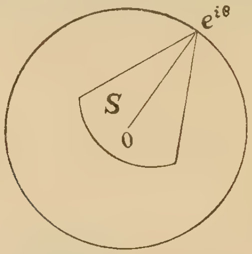
$$M_r(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^r\,d\theta\right)^{1/r}, \quad M_r(\phi) = M_r(\phi, \rho) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(\rho e^{i\theta})|^r\,d\theta\right)^{1/r}$$

(with a similar notation for any function which, like ϕ, is regular in $|z|<1$);

$$J_r(\phi) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 |\phi'(\rho e^{i\theta})|^r\,(1-\rho)^{r-1}\,d\rho\,d\theta\right)^{1/r}.$$

Let $S(\theta)$ be the z-region defined by

$$|z|<1, \quad |e^{i\theta}-z| \leqslant \tfrac{5}{4}, \quad |\arg(e^{i\theta}-z)-\theta| \leqslant \tfrac{1}{4}\pi.$$



(It is a kite-shaped region pointing into the unit circle from the point $e^{i\theta}$; its straight edges make angles $\frac{1}{4}\pi$ with the radius vector to this point; it contains the circle $|z| \leqslant \frac{1}{4}$.) Let

$$\Phi(\theta) = \max|\phi(z)| \quad \big(z \in S(\theta)\big),$$

with a similar notation for other functions regular in $|z|<1$†.

Next, for any periodic function $\psi(\theta)$ of L, we define

$$(\psi)_A,$$

the "maximum average of ψ at the point θ", to be

$$\max\left|\frac{1}{t}\int_0^t |\psi(\theta+u)|\,du\right|,$$

the upper bound being taken for t varying in the range $|t| \leqslant \pi$.

---

† The capitals F, U, V are not of this type. But U, V have traditional meanings; further, U, V, F are used only in the Fourier series section (and F does not even occur in the present summary); things like Φ are used only in the power series section.

So far we have been concerned chiefly with definitions; we proceed to set out the results we shall have to use most frequently. Proofs can be found in Zygmund's book on trigonometrical series, and full references: here we refer only to recent work.

(A) As $\rho \to 1$, $u(\rho, \theta)$ [the Poisson integral of $U(\theta) = f(\theta)$†] tends to $u(e^{\theta i}) = U(\theta) = f(\theta)$, p.p. in θ; and $v(\rho, \theta)$ tends, p.p. in θ, to a limit function $V(\theta)$, given by

$$V = -\frac{1}{\pi} \int \frac{U(\theta+t) - U(\theta-t)}{2 \tan \frac{1}{2}t}\, dt,$$

the integral being a principal value one. V is the (real variable) conjugate of U. Clearly, also,

$$\lim_{\rho \to 1} \phi(z) = \phi(e^{\theta i}) = U + iV = f + iV, \quad \text{p.p. in } \theta.$$

(B) If $f = U \in L^r$, then $M_r(u) \leqslant M_r(U)$,

$$\lim_{\rho \to 1} M_r(u) = M_r(U), \quad \text{and} \quad M_r(u-U) \to 0;$$

u converges strongly to its limit function U, with index r. Conversely, if $M_r(u) \leqslant C$, then u is the Poisson integral of a U with $M_r(U) \leqslant C$.

(C)‡ If $f = U \in L^r$, then $M_r(V) \leqslant A_r M_r(U)$, $M_r(v) \leqslant A_r M_r(u)$, and $M_r(\phi) \leqslant A_r M_r(u)$, with consequences arising out of (B); *e.g.* ϕ converges strongly to its limit function $\phi(e^{\theta i})$, $M_r(\phi) \to M_r\big(\phi(e^{\theta i})\big)$, and $\phi(z)$ is the Poisson integral and the Cauchy integral of $\phi(e^{\theta i})$.

(D) $M_r(\rho, \phi)$ increases with r, and increases with ρ.

(E)§ ("Real max")

$$\int_{-\pi}^{\pi} \big(\psi(\theta)\big)_{\mathcal{A}}^{r}\, d\theta \leqslant A_r \int_{-\pi}^{\pi} |\psi(\theta)|^r\, d\theta.$$

(F)§ ("Complex max")

$$\int_{-\pi}^{\pi} \Phi^r\, d\theta \leqslant A_r \int_{-\pi}^{\pi} |\phi(e^{\theta i})|^r\, d\theta \leqslant A_r \int_{-\pi}^{\pi} |f|^r\, d\theta.$$

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† It is convenient sometimes (for symmetry or analogy) to give $f(\theta)$ the alternative name $U(\theta)$.

‡ M. Riesz, 7.

§ Hardy and Littlewood, 2. Note that $(\psi)_{\mathcal{A}}^{r}$ is to mean $\big((\psi)_{\mathcal{A}}\big)^r$, not $(\psi^r)_{\mathcal{A}}$.

(G) $$\left|\int_{-\pi}^{\pi} \psi(\theta+t)\, P(\rho, t)\, dt\right| \leqslant A\Big(\psi(\theta)\Big)_{\mathcal{A}} \quad \text{(independent of } \rho\text{).}$$

(H)† $$|\phi(z)| \leqslant A\rho\,\Phi(\theta), \quad |\phi'| \leqslant \frac{A\Phi}{1-\rho}, \quad |\phi''| \leqslant \frac{A\Phi}{(1-\rho)^2}.$$

(K) If $\psi(z)$ is regular in $|z|<1$, and

$$M_r(\psi) \leqslant C = M_r\Big(\psi(e^{\theta i})\Big) < \infty,$$

then (i) $\psi=\omega\chi$, where either ω is identically 1 or else $|\omega|<1$, χ is never zero in $|z|<1$, and $M_r\Big(\chi(e^{\theta i})\Big)=C$; (ii) $\psi=\psi_1+\psi_2$, where ψ_1, ψ_2 are without zeros in $|z|<1$, and $M_r(\psi_1) \leqslant C$, $M_r(\psi_2) \leqslant 2C$.

(L)‡ Suppose that $f \in L^r$, and that $s_n(f)$ denotes the sum to n terms of the Fourier series of f§,

$$s_n(f) = \tfrac{1}{2}a_0 + \sum_{1}^{n}(a_m \cos m\theta + b_m \sin m\theta) = \sum_{-n}^{n} c_m e^{m\theta i}.$$

Then (i) if $S = \sum_{\mu}^{\nu} c_m e^{m\theta i}$ is any connected piece (finite or not) of the Fourier series of f, $M_r(S) \leqslant A_r M_r(f)$. In particular this is true for $S = \sum_{0}^{\infty}$ (or $\sum_{0}^{-\infty}$), for $S = \sum_{0}^{n}$, and for $S = s_n = \sum_{-n}^{n}$.

(ii) $M_r(s_n - f) \to 0$ as $n \to \infty$; s_n converges strongly to f with index r.

Conversely, (iii) if $s_n = \sum_{-n}^{n} c_m e^{m\theta i}$ and $M_r(s_n) \leqslant K$ for all n, then there exists an $f \sim \sum_{-\infty}^{\infty} c_m e^{m\theta i}$ with $M_r(f) \leqslant A_r K$.

(M) "Fatou's lemma". If $\psi_n(\theta) \geqslant 0$ and $\lim \psi_n = \psi$, *p.p.* in θ, then

$$\int \psi\, d\theta \leqslant \varliminf \int \psi_n\, d\theta.$$

---

† (i) $|\phi(z)/z|$ does not attain its maximum in $S(\theta)$ where $|z|<\frac{1}{4}$, and where $|z| \geqslant \frac{1}{4}$ (in S) it is bounded by 4Φ. (ii) $2\pi i\phi'$ is the integral of $\phi(\zeta)/(\zeta-z)^2$ taken round a circle $|\zeta-z| = \frac{1}{4}(1-\rho)$; since this is contained in S, $2\pi|\phi'|$ is bounded by $2\pi\Phi/\frac{1}{4}(1-\rho)$; similarly for ϕ''.

‡ M. Riesz, 7.

§ The reader may be surprised to learn that this paper of 37 pages does not mention the Cesáro mean σ_n.

(N) "Hölder's inequality"[†].

$$\left|\int a(x)\,b(x)\,dx\right| \leqslant \left(\int |a(x)|^r\,dx\right)^{1/r}\left(\int |b(x)|^{r'}\,dx\right)^{1/r'},$$

where r and r' are conjugate indices, satisfying

$$\frac{1}{r}+\frac{1}{r'}=1, \quad r'=\frac{r}{r-1}.$$

(Note that p' is a q, q' a p.)

Two final points, not worth the dignity of a letter, will, however, continually recur. The first is that

$$\left|\frac{d}{dx}\,|\psi(x)|\right| \leqslant \left|\frac{d}{dx}\,\psi(x)\right|$$

(and the left-hand side certainly exists if the right-hand one does).

The second is that

$$\int_a^1 (1-\rho)\,\psi(\rho)\,d\rho$$

is roughly a repeated integral of ψ from a to 1; more precisely, ψ being continuous except at $\rho=1$,

$$\int_a^1 (1-\rho)\,\psi(\rho)\,d\rho = \int_a^1 d\rho \int_a^\rho \psi(\sigma)\,d\sigma$$

is valid in general subject to convergence conditions, and is *certainly* valid if ψ is non-negative (the case when one and so both sides are $+\infty$ being included).

Power series theorems.

4. We begin now our attack on Theorem 7. We need first

LEMMA 1. *Supposing that* $f \in L^r$, *let* $R<1$ *and*

$$f_R = u(R,\theta), \quad g_R = g(f_R).$$

[†] We use also the "converse of Hölder's inequality", and Minkowski's inequality; the last seldom, the first only once.

Then

(i) $$g_R \leqslant g,$$

(ii) $$M_r(g) = \lim_{R\to 1} M_r(g_R)$$

(*both sides possibly being* $+\infty$).

Evidently
$$\phi(z, f_R) = \phi(Rz),$$

$$g_R^2 = \int_0^1 (1-\rho)\,|R\phi'(R\rho e^{\theta i})|^2\,d\rho \leqslant \int_0^1 (1-R\rho)\,|\phi'(R\rho e^{\theta i})|^2\,R\,d\rho \leqslant \int_0^{1/R} = g^2,$$

which is (i). Clearly also $g_R \to g$ as $R \to 1$, and (ii) follows from (i) together with Fatou's lemma (or by Dini's theorem).

5. We begin with the case $r = 2k$, and have to prove

$$M_{2k}(g) \leqslant BM_{2k}(f),$$

where B stands for $A(k)$. [Generally, in the sequel, we use B for the $A(r)$, $A(r, \alpha, \beta)$, etc., which occur in the theorem being considered.]

First, we may suppose ϕ regular on $|z| = 1$. For suppose the result known for this case, and consider the general one. Since $\phi(z, f_R)$ is regular on $|z| = 1$, we have

$$M_{2k}(g_R) \leqslant BM_{2k}(f_R) = BM_{2k}\big(u(R,\ \theta)\big) \leqslant BM_{2k}(f),$$

by (B). By Lemma 1, $M_{2k}(g) \leqslant BM_{2k}(f)$, the general result.

For clarity of exposition we set out the argument for the special case $2k = 4$, explaining later the modifications needed in the general one[†]. We have

$$M_4^4(g) = \frac{1}{2\pi}\int g^4\,d\theta = \frac{1}{2\pi}\iiint (1-\rho_1)(1-\rho_2)\,|\phi_1'|^2\,|\phi_2'|^2\,d\theta\,d\rho_1\,d\rho_2,$$

where we write ϕ_1, ϕ_2 for $\phi(\rho_1 e^{\theta i})$, $\phi(\rho_2 e^{\theta i})$ respectively. The variables ρ_1, ρ_2 range from 0 to 1. The (treble) integral is, however, equal to a multiple A of the same integral with the variables ρ_1, ρ_2 restricted by

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[†] There are forms of the proof when $2k = 4$ that do not extend (in particular there is one which does not use "max"): we set out, of course, the one that does.

$0 \leqslant \rho_2 \leqslant \rho_1 \leqslant 1$ (in general, $0 \leqslant \rho_k \leqslant \ldots \leqslant \rho_2 \leqslant \rho_1 \leqslant 1$);

$$M_4^4(g) \leqslant A \iiint\limits_{0 \leqslant \rho_2 \leqslant \rho_1 \leqslant 1} (1-\rho_1)(1-\rho_2)\,|\phi_1'|^2\,|\phi_2'|^2\,d\theta\,d\rho_1\,d\rho_2$$

(and we suppose the range of the ρ's to be this in future except when the contrary is stated). We now substitute for $|\phi_1'|^2$ (but *not* for $|\phi_2'|^2$) from the identity†

$$|\phi'|^2 = \tfrac{1}{4}\,\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{d}{d\rho}\,(|\phi|^2)\right) + \tfrac{1}{4}\,\frac{1}{\rho^2}\left(\frac{d}{d\theta}\right)^2 (|\phi|^2). \tag{1}$$

So far as the first term [on the right-hand side of (1)] is concerned, we perform the integration with respect to ρ_1 (the range being ρ_2 to 1). We have (ϕ being regular on $\rho = 1$)

$$\int_{\rho_2}^{1} (1-\rho)\,\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{d}{d\rho}\,|\phi|^2\right) d\rho$$

$$= \left[\frac{1-\rho}{\rho}\,|\phi|^2\right]_{\rho_2}^{1} + \int_{\rho_2}^{1} \left|\frac{\phi}{\rho}\right|^2 d\rho + \Big[|\phi|^2\Big]_{\rho_2}^{1} + (1-\rho_2)\left[\frac{d}{d\rho}\,|\phi|^2\right]_{\rho=\rho_2},$$

and so, rejecting negative terms,

$$\leqslant |\phi(e^{\theta i})|^2 + A(1-\rho_2)\,|\phi_2\phi_2'| + \int_{\rho_2}^{1} \left|\frac{\phi}{\rho}\right|^2 d\rho$$

$$\leqslant A\,\Phi^2,$$

by (H). Hence

$$M_4^4(g) \leqslant A \iint \Phi^2 (1-\rho_2)\,|\phi_2'|^2\,d\rho_2\,d\theta$$
$$+ A \iiint (1-\rho_1)(1-\rho_2)\,|\phi_2'|^2\,\frac{1}{\rho_1^2}\left(\frac{d}{d\theta}\right)^2 |\phi_1|^2\,d\theta\,d\rho_1\,d\rho_2. \tag{2}$$

---

† P. Stein (9, 1933) uses the identities

$$\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{d}{d\rho}\int|\phi|^r\,d\theta\right) = \int \Delta(|\phi^r|)\,d\vartheta = r^2 \int |\phi|^{r-2}\,|\phi'|^2\,d\theta;$$

where Δ is Laplace's operator

$$\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{d}{d\rho}\right) + \frac{1}{\rho^2}\left(\frac{d}{d\theta}\right)^2.$$

Similar ideas occur much earlier in G. H. Hardy (1, 1914); but it was actually Dr. Stein's rediscovery, then recent (and unpublished), that drew our attention to the possibilities of the identities.

What is vital is that $\Delta(|\phi|^2)$ [and more generally $\Delta(|\phi|^r)$] does not contain ϕ''; the effect of this here is that we can replace $|\phi_1'|^2$ by $|\phi_1\phi_1'|$, at the expense of introducing second derivatives of $\phi_2, \phi_3, \ldots$ (which can be got rid of by the use of "max"): some idea of the kind seems indispensable.

The first term on the right-hand side is [identically, the range of ρ_2 being (0, 1)]

$$A\int \Phi^2 g^2\,d\theta. \tag{3}$$

In the second term we integrate (once) by parts with respect to θ, observing that the integrated part vanishes when taken between $-\pi$ and π. We obtain

$$\text{(4)}\quad -A\iiint (1-\rho_1)(1-\rho_2)\frac{1}{\rho_1^{\,2}}\frac{d}{d\theta}(|\phi_2'|^2)\frac{d}{d\theta}(|\phi_1|^2)\,d\theta\,d\rho_1\,d\rho_2$$

$$\leqslant A\iiint (1-\rho_1)(1-\rho_2)\frac{1}{\rho_1^{\,2}}|\rho_2\phi_2'\phi_2''|\,|\rho_1\phi_1\phi_1'|\,d\theta\,d\rho_1\,d\rho_2$$

$$\leqslant A\iiint (1-\rho_1)(1-\rho_2)|\phi_2'|\frac{\Phi}{(1-\rho_2)^2}\Phi|\phi_1'|\,d\theta\,d\rho_1\,d\rho_2,$$

by (H),

$$=A\int \Phi^2\left(\iint \frac{1-\rho_1}{1-\rho_2}|\phi_1'\phi_2'|\,d\rho_1\,d\rho_2\right)d\theta$$

$$\leqslant A\int \Phi^2\left(\iint\left(|\phi_2'|^2+\left(\frac{1-\rho_1}{1-\rho_2}\right)^2|\phi_1'|^2\right)d\rho_1\,d\rho_2\right)d\theta$$

$$=A\int \Phi^2\left(\int_0^1\left(|\phi_2'|^2\int_{\rho_2}^1 d\rho_1\right)d\rho_2+\int_0^1(1-\rho_1)^2|\phi_1'|^2\left(\int_0^{\rho_1}\frac{d\rho_2}{(1-\rho_2)^2}\right)d\rho_1\right)d\theta$$

$$=A\int \Phi^2(g^2+g^2)\,d\theta=A\int \Phi^2 g^2\,d\theta.$$

From (2), (3), and (4),

$$M_4^4(g)\leqslant A\int \Phi^2 g^2\,d\theta \leqslant AM_4^2(\Phi)\,M_4^2(g).$$

Since (ϕ being regular on $|z|=1$) $M_4(g)$ is finite,

$$M_4(g)\leqslant AM_4(\Phi)$$

$$\leqslant AM_4(f),$$

by (E). This completes the proof.

In the general case,

$$M_{2k}^{2k}(g)=B\int\dots\int \prod_1^k\{(1-\rho_n)|\phi_n'|^2\}\,d\theta\,\Pi\,d\rho_n,$$

subject to $0 \leqslant \rho_k \leqslant \dots \leqslant \rho_2 \leqslant \rho_1 \leqslant 1$. We substitute for $|\phi_1'|^2$ as before; the first term contributes at most $B \int \Phi^2 g^{2k-2}\, d\theta$, just as before. After the integration by parts for the second term, the (second) operation $d/d\theta$ bears on the product $\prod_2^k |\phi_n'|^2$, and is majorized by the sum of $k-1$ things like $|\phi_\nu' \phi_\nu''|\, \Pi' |\phi_n'|^2$, where Π' is taken over $n \neq 1, \nu$. When we treat the suffixes 1 and ν as we did 1 and 2 in the special case, the typical term contributes to $M_{2k}^{2k}(g)$ at most

$$B \int \dots \int \frac{1-\rho_1}{1-\rho_\nu} |\phi_1 \phi_\nu'|\, \Pi' \{(1-\rho_n) |\phi_n'|^2\}\, d\theta\, d\rho_1\, d\rho_\nu\, \Pi' \, d\rho_n.$$

This is increased by performing the integration over the extended range $0 \leqslant \rho_\nu \leqslant \rho_1 \leqslant 1$, $0 \leqslant \rho_n \leqslant 1$ $(n \neq 1, \nu)$, when the expression becomes (we integrate first with respect to the ρ_n with $n \neq 1, \nu$)

$$B \iiint \left(\frac{1-\rho_1}{1-\rho_\nu}\right) |\phi_1 \phi_\nu'|\, g^{2k-2}\, d\theta\, d\rho_1\, d\rho_\nu,$$

the range of the ρ's being $0 \leqslant \rho_\nu \leqslant \rho_1 \leqslant 1$. We treat this just as before, getting at most

$$B \int \Phi^2 g^{2k-2}\, d\theta.$$

In all,
$$M_{2k}^{2k}(g) \leqslant B \int \Phi^2 g^{2k-2}\, d\theta \leqslant B M_{2k}^2(\Phi)\, M_{2k}^{2k}(g),$$

$$M_{2k}(g) \leqslant B M_{2k}(\Phi) \leqslant B M_{2k}(f).$$

6. It is best to go straight on to the difficult Theorem 7* (the extension of what we have just proved from g to g^*)†. This time the case $2k=4$ is definitely untypical, and it is easiest to deal at once with the general one. We suppose, in fact, that $k > 2$, the necessary changes for the easier cases being left to the reader.

It is enough to consider the case in which ϕ is regular on $|z|=1$, since Lemma 1 and the argument at the beginning of § 5 extend at once from g to g^*. We have then (the range of the "angle" variables $\theta, x_1, x_2, \dots, x_n$

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† $$g^2 = \int_0^1 (1-\rho^2) |\phi'(\rho^2 e^{\theta i})|^2\, 2\rho\, d\rho \leqslant 4 \int (1-\rho) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi'(\rho e^{i(\theta+t)})\, P(\rho, t)\, dt \right|^2 d\rho$$
$$\leqslant 4 \int (1-\rho) \left(\frac{1}{2\pi} \int \left| \phi'(\rho e^{i(\theta+t)} \right|^2 P(\rho, t)\, dt \right) d\rho = 4 g^{*2}.$$

Thus g^* is effectively a majorant of g.

being from $-\pi$ to π)

$$(1)\qquad \int g^{*2k}\,d\theta = \int \dots \int \prod_{n=1}^{k} \left((1-\rho_n)\,|\phi_n'|^2 P_n\right) d\theta\, \Pi\, dx_n\, \Pi\, d\rho_n,$$

where we have written, for brevity,

$$\phi_n' = \phi'(\rho_n, \theta+x_n) = \phi'(\rho_n e^{i(\theta+x_n)}), \quad P_n = P(\rho_n, x_n).$$

The integral is equal to a multiple of the same integral with the ρ's restricted by the inequalities

$$(2)\qquad 0 \leqslant \rho_k \leqslant \dots \leqslant \rho_2 \leqslant \rho_1 \leqslant 1,$$

which we suppose satisfied unless the contrary is stated.

We begin by showing that the integral (1) [subject to (2)] is increased at most by a factor B when we replace the ρ_1 in P_1 by ρ_2 (and make no other change). (We thereby avoid a crucial difficulty† at the expense of making the variable ρ_2 anomalous.) The integration with respect to the "angle" variables θ, x_1, x_2, ..., x_n may be performed with a change of variables to θ', x_1, ξ_2, ..., ξ_n, where

$$\theta' = \theta + x_1, \quad x_n = x_1 + \xi_n \quad (n > 1),$$

and the new ranges are all $(-\pi, \pi)$. The integration with respect to x_1 then involves only

$$\int_{-\pi}^{\pi} P_1(x_1)\, P_2(x_1+\xi_2) \dots P_k(x_1+\xi_k)\, dx_1 = \int_{-\pi}^{\pi} P(\rho_1, x_1)\, Q(x_1)\, dx_1,$$

say, and it is enough to prove that *this* is increased at most by a factor B when the ρ_1 in P_1 is changed to ρ_2; for we can then return to the original variables θ, x_1, x_2, ..., x_k.

We divide the range of x_1 into

$$R, \ |x_1| \leqslant 1-\rho_2; \quad R', \ 1-\rho_2 < |x_1| \leqslant \pi.$$

In R' we have‡ (since $\rho_2 \leqslant \rho_1$)

$$P(\rho_1, x_1) \leqslant A P(\rho_2, x_1),$$

---

† The presence of ρ_1 in P_1 upsets—it would seem hopelessly—the "double integration (with respect to ρ_1) of the second differential coefficient of $|\phi_1|^2$" that figures in the argument for g.

‡ In such contexts one should think of $P(\rho, x)$ as being substantially

$$\frac{\delta}{\delta^2 + x^2},$$

where $\delta = 1 - \rho$.

so that

$$\int_{R'} P(\rho_1, x_1)\, Q(x_1)\, dx_1 \leqslant A \int_{R'} P(\rho_2, x_1)\, Q(x_1)\, dx_1. \tag{3}$$

In R $Q(x_1)$ varies at most by a factor B (both ways) from the value $Q(0)$, so that

$$\int_R P(\rho_1, x_1)\, Q(x_1)\, dx_1 \leqslant BQ(0) \int_{-\pi}^{\pi} P(\rho_1, x_1)\, dx_1 = BQ(0)$$

$$\leqslant BQ(0) \int_R P(\rho_2, x_1)\, dx_1,$$

the integral last written being greater than an A,

$$\leqslant B \int_R P(\rho_2, x_1)\, Q(x_1)\, dx_1. \tag{4}$$

(3) and (4) give us what we want.

We have, then,

$$\int g^{*2k}\, d\theta \leqslant B \int \dots \int (1-\rho_1)\, |\phi_1'|^2\, P(\rho_2, x_1) \times \prod_2^k \Big((1-\rho_n)\, |\phi_n'|^2\, P_n \Big)\, d\theta\, \Pi\, dx_n\, \Pi\, d\rho_n. \tag{5}$$

We now substitute (as before)

$$|\phi_1'|^2 = \tfrac{1}{4} \frac{1}{\rho_1} \frac{d}{d\rho_1} \left(\rho_1 \frac{d}{d\rho_1} (|\phi_1|^2) \right) + \tfrac{1}{4} \frac{1}{\rho_1^2} \left(\frac{d}{d\theta} \right)^2 (|\phi_1|^2). \tag{6}$$

Since ρ_1 occurs in (5) only in $(1-\rho_1)\,|\phi_1'|^2$, we can perform the ρ_1-integration for the first term [as obtained from (6)]; the only difference as against § 5 is that the argument of ϕ_1 is $\theta + x_1$ in place of θ. For the contribution T_1 of the first term we have

$$T_1 \leqslant B \int \dots \int \Phi^2(\theta + x_1)\, P(\rho_2, x_1) \prod_2^k \Big((1-\rho_n)\, |\phi_n'|^2\, P_n \Big)\, d\theta\, dx_1 \prod_2^k (dx_n\, d\rho_n),$$

or, performing the integration with respect to x_1 (which bears only on the first two factors) and using (G),

$$T_1 \leqslant B \int \dots \int \Big(\Phi^2(\theta) \Big)_{\mathscr{A}} \prod_2^k \Big((1-\rho_n)\, |\phi_n'|^2\, P_n \Big)\, d\theta \prod_2^k (dx_n\, d\rho_n),$$

or again, restoring the complete range (0, 1) for the $\rho_2, \rho_3, \ldots, \rho_k$,†

$$T_1 \leqslant B \int (\Phi^2)_{\mathcal{A}} \, g^{*2k-2} \, d\theta. \tag{7}$$

We have now to consider the contribution of the second term on the right-hand side of (6) to (5). As before, we integrate by parts (once) with respect to θ, reject the integrated part, which is zero, and use the principle $\big| D|\psi| \big| \leqslant |D\psi|$ in the appropriate places. The new differentiation $d/d\theta$ bears on $\prod_2^k |\phi_n'|^2$, and we get a total contribution at most $\sum_2^k T_\nu$, where

$$T_\nu = B \int \ldots \int \frac{1}{\rho_1^2} |\rho_1 \phi_1 \phi_1'| |\rho_\nu \phi_\nu' \phi_\nu''| (1-\rho_1)(1-\rho_\nu) P(\rho_2, x_1) P_\nu \\ \times \Pi' \big((1-\rho_n) |\phi_n'|^2 P_n \big) \, d\theta \, \Pi (dx_n \, d\rho_n),$$

where Π' stands for $\prod_{n \neq 1, \nu} (1-\rho_n) |\phi_n'|^2 P_n$.

Replacing ϕ_1 and ϕ_ν'' by their majorants $\rho_1 \Phi(\theta + x_1)$ and

$$A\Phi(\theta + x_\nu)/(1-\rho_\nu)^2,$$

we obtain (much as in § 5)

$$T_\nu \leqslant B \int \ldots \int \Phi(\theta + x_1) \Phi(\theta + x_\nu) \frac{1-\rho_1}{1-\rho_\nu} |\phi_1' \phi_\nu'| P(\rho_2, x_1) P_\nu \\ \times \Pi' \, . \, d\theta \, \Pi (dx \, d\rho) \tag{8}$$

$$\leqslant B \int \ldots \int \left(\Phi^2(\theta + x_1) P(\rho_2, x_1) |\phi_\nu'|^2 P_\nu + \left(\frac{1-\rho_1}{1-\rho_\nu} \right)^2 \Phi^2(\theta + x_\nu) |\phi_1'|^2 P(\rho_2, x_1) P_\nu \right) \\ \times \Pi' \, . \, d\theta \, \Pi (dx \, d\rho).$$

We now perform the integrations with respect to x_1 and ρ_1 (ρ_2 to 1). In the first term ρ_1 does not occur at all, and x_1 only in the first two factors. The x_1-integration on these factors gives something majorized by $A(\Phi^2)_{\mathcal{A}}$; the ρ_1-integration introduces a further factor $1-\rho_2 \leqslant 1-\rho_\nu$. So the first term contributes to T_ν at most

$$T_\nu^{(1)} = B \int \ldots \int (\Phi^2)_{\mathcal{A}} (1-\rho_\nu) |\phi_\nu'|^2 P_\nu \, \Pi' \, . \, d\theta \prod_2^k (dx_n \, d\rho_n) \tag{9}$$

$$\leqslant B \int (\Phi^2)_{\mathcal{A}} \, g^{*2k-2} \, d\theta$$

(the missing term $n = \nu$ in Π' being supplied again).

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† (i) The ρ_2 of $P(\rho_2, x_1)$ has disappeared "into the majorant $(\Phi^2)_{\mathcal{A}}$" and the variable ρ_2 has ceased to be anomalous.

(ii) The argument of the functions of type $(\psi)_{\mathcal{A}}$ that occur in the sequel is in every case θ.

In the contribution $T_\nu^{(2)}$ of the second term in (8) to T_ν we first integrate with respect to x_ν, getting

$$(10)\qquad T_\nu^{(2)} \leqslant B\int\cdots\int \left(\frac{1-\rho_1}{1-\rho_\nu}\right)^2 (\Phi^2)_{\mathscr{A}}\, |\phi_1'|^2 P(\rho_2, x_1)\, \Pi' \,.\, d\theta \prod_{n\neq\nu} dx_\nu \prod d\rho_n.$$

We must now distinguish two cases, according as ν is not or is the anomalous suffix 2.

When $\nu \neq 2$, we perform first the integration with respect to ρ_ν, whose range (for the remaining ρ's fixed) is at most 0 to ρ_1; the integration bears only on the factor $1/(1-\rho_\nu)^2$, and turns this to $1/(1-\rho_1)$. So

$$T_\nu^{(2)} \leqslant B\int\cdots\int (1-\rho_1)(\Phi^2)_{\mathscr{A}}\, |\phi_1'|^2 P(\rho_2, x_1)\, \Pi' \,.\, d\theta \prod_{n\neq\nu} (dx_n\, d\rho_n).$$

We next perform the integration with respect to ρ_1, extending it to the full range $(0, 1)$; this converts $(1-\rho_1)|\phi_1'|^2$ into $g^2(\theta+x_1)$; and this, multiplied by $P(\rho_2, x_1)$ and integrated with respect to x_1, produces at most $A(g^2)_{\mathscr{A}}$. Thus

$$(11)\qquad T_\nu^{(2)} \leqslant B\int\cdots\int (\Phi^2)_{\mathscr{A}}\, (g^2)_{\mathscr{A}}\, \Pi' \,.\, d\theta \prod_{n\neq 1,\,\nu} (dx_n\, d\rho_n)$$

$$\leqslant B\int (\Phi^2)_{\mathscr{A}}\, (g^2)_{\mathscr{A}}\, g^{*2k-4}\, d\theta.$$

Finally, when $\nu = 2$, Π' contains only variables x and ρ with suffixes 3, 4, ...; and integrating with respect to them [over the full (0, 1) for the ρ's] gives

$$T_2^{(2)} \leqslant B\iiiint \left(\frac{1-\rho_1}{1-\rho_2}\right)^2 (\Phi^2)_{\mathscr{A}}\, |\phi_1'(\theta+x_1)|^2 P(\rho_2, x_1)\, g^{*2k-4}\, d\theta\, dx_1\, d\rho_1\, d\rho_2$$

$$= B\iiiint R(\theta)\, P(\rho_2, x_1) \left(\frac{1-\rho_1}{1-\rho_2}\right)^2 |\phi_1'(\theta+x_1)|^2\, d\theta\, dx_1\, d\rho_1\, d\rho_2,$$

where
$$R(\theta) = (\Phi^2)_{\mathscr{A}}\, g^{*2k-4}.$$

We now change the variable θ to $\theta - x_1$ (keeping the range $-\pi$, π for the new θ) and get

$$T_2^{(2)} \leqslant B\iiiint R(\theta-x_1)\, P(\rho_2, x_1) \left(\frac{1-\rho_1}{1-\rho_2}\right)^2 |\phi_1'(\theta)|^2\, dx_1\, d\theta\, d\rho_1\, d\rho_2,$$

and so, performing the x_1-integration,

$$T_2^{(2)} \leqslant B \iiint (R)_{\mathscr{A}} \left(\frac{1-\rho_1}{1-\rho_2}\right)^2 |\phi_1'(\theta)|^2 \, d\theta \, d\rho_1 \, d\rho_2.$$

The integration with respect to ρ_2 turns $(1-\rho_1)^2/(1-\rho_2)^2$ into $1-\rho_1$ and that with respect to ρ_1 turns $(1-\rho_1)|\phi_1'|^2$ into g^2. Hence

$$T_2^{(2)} \leqslant B \int (R)_{\mathscr{A}} \, g^2 \, d\theta. \tag{12}$$

Summing up from (5), (7), (9), (10), (11), (12), we have[†]

$$\int g^{*2k} \, d\theta \leqslant B\Big(\int (\Phi^2)_{\mathscr{A}} \, g^{*2k-2} \, d\theta + \int (\Phi^2)_{\mathscr{A}} \, (g^2)_{\mathscr{A}} \, g^{*2k-4} \, d\theta$$

$$+ \int \big((\Phi^2)_{\mathscr{A}} \, g^{*2k-4} \big)_{\mathscr{A}} \, g^2 \, d\theta \Big).$$

The first term on the right-hand side is not greater than

$$B\Big(\int \{(\Phi^2)_{\mathscr{A}}\}^k \, d\theta\Big)^{2/2k} \Big(\int g^{*2k} \, d\theta\Big)^{1-(2/2k)}, \tag{13}$$

the second is not greater than

$$B\Big(\int \{(\Phi^2)_{\mathscr{A}}\}^k \, d\theta\Big)^{2/2k} \Big(\int \{(g^2)_{\mathscr{A}}\}^k \, d\theta\Big)^{2/2k} \Big(\int g^{*2k} \, d\theta\Big)^{1-(4/2k)}, \tag{14}$$

and the third is not greater than

$$B\Big(\int \Big\{ \big((\Phi^2)_{\mathscr{A}} \, g^{*2k-4} \big)_{\mathscr{A}} \Big\}^{2k/(2k-2)} d\theta \Big)^{(2k-2)/2k} \Big(\int g^{2k} \, d\theta \Big)^{2/2k}. \tag{15}$$

In (13) and (14) we can apply (E), since the brackets affected by $\mathscr{A}$ occur to a power greater than unity, and the symbols $\mathscr{A}$ may be dropped. In (15) the outer $\mathscr{A}$ may be dropped for a similar reason: the integral affected then becomes

$$\int \{ (\Phi^2)_{\mathscr{A}} \, g^{*2k-4} \}^{2k/(2k-2)} \, d\theta \leqslant \Big(\int \big((\Phi^2)_{\mathscr{A}} \big)^k \, d\theta \Big)^{2/(2k-2)} \Big(\int g^{*2k} \, d\theta \Big)^{(2k-4)/(2k-2)},$$

and the remaining $\mathscr{A}$ may be dropped (at the expense of a factor B).

[†] This inequality displays the rôle of "max" in the proof (Φ is itself, of course, a "max").

Collecting results, we have

$$\int g^{*2k}\,d\theta \leqslant B\Bigl(\int \Phi^{2k}\,d\theta\Bigr)^a \Bigl(\int g^{*2k}\,d\theta\Bigr)^b + B\Bigl(\int \Phi^{2k}\,d\theta\Bigr)^c \Bigl(\int g^{2k}\,d\theta\Bigr)^d \Bigl(\int g^{*2k}\,d\theta\Bigr)^e$$
$$+B\Bigl(\int \Phi^{2k}\,d\theta\Bigr)^{c'} \Bigl(g^{2k}\,d\theta\Bigr)^{d'} \Bigl(\int g^{*2k}\,d\theta\Bigr)^{e'},$$

where the indices a, b, ..., e' are strictly positive and less than 1, and, of course, $a+b=c+d+e=c'+d'+e'=1$. Since the integrals other than that of g^{*2k} are not greater than $B\int |f|^{2k}\,d\theta$, and $\int g^{*2k}\,d\theta$, which is finite, occurs to a lower power everywhere on the right than on the left, we finally obtain

$$\int g^{*2k}\,d\theta \leqslant B\int |f|^{2k}\,d\theta,$$

the desired result.

7. Returning now to Theorem 7, we prove next that

$$M_r(g) \leqslant BM_r(f)$$

for general r. Supposing $M_r(f) < \infty$ (or there is nothing to prove), we have, by (C), $M_r(\phi) \leqslant BM_r(f)$, and so $M_r(\psi) \leqslant BM_r(f)$, where $\phi = z\psi$. By (K), $\psi = \psi_1 + \psi_2$, where ψ_1, ψ_2 are without zeros in $|z| < 1$, and $M_r(\psi_i) \leqslant BM_r(f)$. Since, evidently,

$$g(f_1+f_2) \leqslant A\bigl(g(f_1)+g(f_2)\bigr),$$

it is enough to prove the result for the case in which ϕ is of the form $z\psi$, where ψ is without zeros.

Let $\psi = \chi^s$, where $s = 2k/r$ and $2k$ is the least even integer for which $2k \geqslant 2r$ (so that $s \geqslant 2$ and depends only on r), and $\omega = z\chi$. Then χ and ω are regular, and $\phi = z^{1-s}\omega^s$. Hence

$$|\phi'| \leqslant B(\rho^{-s}|\omega|^s + \rho^{1-s}|\omega|^{s-1}|\omega'|)$$
$$\leqslant B(\Omega^s + \Omega^{s-1}|\omega'|),$$
$$|\phi'|^2 \leqslant B(\Omega^{2s} + \Omega^{2s-2}|\omega'|^2),$$
$$g^2(\phi) \leqslant B\bigl(\Omega^{2s} + \Omega^{2s-2} g^2(\omega)\bigr)\dagger,$$
$$g^r(\phi) \leqslant B\bigl(\Omega^{rs} + \Omega^{r(s-1)} g^r(\omega)\bigr),$$

(1) $$\int g^r(\phi)\,d\theta \leqslant B\Bigl(\int \Omega^{2k}\,d\theta + \int \Omega^{2k/s'} g^r(\omega)\,d\theta\Bigr).$$

---

† We write $g(\varphi)$ momentarily for the usual $g(f)$, and $g(\omega)$ means a corresponding thing.

The last term is not greater than

$$\left(\int \Omega^{2k}\,d\theta\right)^{1/s'}\left(\int g^{rs}(\omega)\,d\theta\right)^{1/s};$$

since $rs = 2k$, we can apply the result for the index $2k$ to the second factor, which is therefore not greater than

$$B\left(\int |\omega(e^{\theta i})|^{2k}\,d\theta\right)^{1/s}.$$

Thus (1) gives

$$\int g^{r}(\phi)\,d\theta \leqslant B\int \Omega^{2k}\,d\theta \leqslant B\int |\omega(e^{\theta i})|^{2k}\,d\theta = B\int |\phi(e^{\theta i})|^{r}\,d\theta \leqslant B\int |f|^{r}\,d\theta,$$

the desired result.

8. We prove next (by "conjugacy with the opposite inequality with q") that

$$M_p(f) \leqslant BM_p(g).$$

We may suppose ϕ regular on $|z| = 1$. For supposing the result proved for this case, we have, by Lemma 1,

$$M_r\big(u(R, \theta)\big) = M_r(f_R) \leqslant BM_r(g_R) \leqslant BM_r(g),$$

and so
$$M_r(f) = \lim_{R\to 1} M_r\big(u(R, \theta)\big) \leqslant M_r(g).$$

Now let

(1) $$h = |f|^{p-1}\,\overline{\mathrm{sgn}}\,f + c\,M_p^{p-1}(f),$$

the constant c being chosen to make $\int_{-\pi}^{\pi} h\,d\theta = 0$. By Hölder's inequality $|c| \leqslant 1$, and so [by Minkowski's inequality and (1)]

(2) $$M_{p'}(h) \leqslant 2M_p^{p-1}(f).$$

Let $\psi = u^* + iv^*$ be the ϕ of h, γ the g of h. Then h is continuous, u^* is continuous in $|z| \leqslant 1$, and so $(1-\rho)\,u_\rho^* \to 0$ uniformly in θ as $\rho \to 1$. Now, if $0 < R < 1$,

$$\int_0^R (1-\rho)\left(\frac{d}{d\rho}\right)^2 (uu^*)\,d\rho = [uu^*]_{\rho=R} + (1-R)\left[\frac{d}{d\rho}\,(uu^*)\right]_{\rho=R} - \left[\frac{d}{d\rho}\,(uu^*)\right]_{\rho=0}.$$

The last term on the right-hand side is zero, and the right-hand side converges, *uniformly* in θ, to $u(e^{\theta i})\,u^*(e^{\theta i}) = fh$ as $R \to 1$.

Hence

$$(3) \qquad \int |f|^p \, d\theta = \int fh \, d\theta = \lim_{R \to 1} \int_0^R \int_{-\pi}^{\pi} (1-\rho) \left(\frac{d}{d\rho}\right)^2 (uu^*) \, d\theta \, d\rho$$

$$= \lim_{R \to 1} \int_0^R \int (1-\rho)(u_{\rho\rho} u^* + uu^*_{\rho\rho} + 2u_\rho u_\rho{}^*) \, d\rho \, d\theta.$$

Now
$$u_{\rho\rho} = -\frac{1}{\rho} u_\rho - \frac{1}{\rho^2} u_{\theta\theta},$$
so that

$$(4) \qquad \int u_{\rho\rho} u^* \, d\theta = -\frac{1}{\rho} \int u_\rho u^* \, d\theta - \frac{1}{\rho^2} \int u_{\theta\theta} u^* \, d\theta$$

$$= -\frac{1}{\rho} \int u_\rho u^* \, d\theta + \frac{1}{\rho^2} \int u_\theta u_\theta{}^* \, d\theta.$$

Similarly

$$(5) \qquad \int uu^*_{\rho\rho} \, d\theta = -\frac{1}{\rho} \int uu_\rho{}^* \, d\theta + \frac{1}{\rho^2} \int u_\theta u_\theta{}^* \, d\theta.$$

In (5) [but not in (4)] we transform the first term on the right-hand side;

$$(6) \qquad -\frac{1}{\rho} \int uu_\rho{}^* \, d\theta = -\frac{1}{\rho^2} \int uv_\theta{}^* \, d\theta = \frac{1}{\rho^2} \int u_\theta v^* \, d\theta.$$

Observing now that $|u_\rho| \leqslant |\phi'|$, $|u_\theta| \leqslant \rho |\phi'|$, and similarly for u^* and ψ, we have, from (4), (5), and (6),

$$\left| \int (u_{\rho\rho} u^* + uu^*_{\rho\rho} + 2u_\rho u_\rho{}^*) \, d\theta \right| \leqslant A \int \left(|\phi' \psi'| + \frac{1}{\rho} |\phi' \psi| \right) d\theta.$$

It follows from (3) that

$$\int |f|^p \, d\theta \leqslant \overline{\lim_{R \to 1}} \int_0^R \int (1-\rho) \left(|\phi' \psi'| + \frac{1}{\rho} |\phi' \psi| \right) d\rho \, d\theta.$$

The integrand on the right-hand side being now non-negative, we can drop the limit symbol and write 1 for R.

We have therefore the following result, which, as we need to use it again, we state as

LEMMA 2. *If ϕ is regular on $|z| = 1$, h is defined by* (1), *and ψ is the ϕ of h, then*

$$\int |f|^p \, d\theta \leqslant A \int_0^1 \int_{-\pi}^{\pi} (1-\rho) \left(|\phi' \psi'| + \frac{1}{\rho} |\phi' \psi| \right) d\rho \, d\theta.$$

For our present purposes we observe that

$$\iint (1-\rho)|\phi' \psi'|\,d\rho\,d\theta \leqslant \int \left(\left(\int (1-\rho)|\phi'|^2 d\rho \right)^{\frac{1}{2}} \left(\int (1-\rho)|\psi'|\,d\rho \right)^{\frac{1}{2}} d\theta = \int g\gamma\,d\theta,$$

$$\iint (1-\rho)\,\frac{1}{\rho}\,|\phi' \psi|\,d\rho\,d\theta \leqslant \iint (1-\rho)|\phi'|\,\Psi\,d\rho\,d\theta$$

$$\leqslant \int \Psi \left(\int (1-\rho)|\phi'|^2 d\rho \right)^{\frac{1}{2}} \left(\int (1-\rho)\,d\rho \right)^{\frac{1}{2}} d\theta = A \int \Psi g\,d\theta,$$

so that
$$\int |f|^p\,d\theta \leqslant A \int (g\gamma + g\Psi)\,d\theta.$$

Now
$$\int g\gamma\,d\theta \leqslant M_p(g)\,M_{p'}(\gamma)$$

$$\leqslant M_p(g)\,BM_{p'}(h)$$

(by the opposite q-inequality)

$$\leqslant BM_p(g)\,M_p^{p-1}(f);$$

and

$$\int g\Psi\,d\theta \leqslant M_p(g)\,M_{p'}(\Psi) \leqslant BM_p(g)\,M_{p'}(h) \leqslant BM_p(g)\,M_p^{p-1}(f).$$

Hence
$$M_p{}^p(f) \leqslant BM_p(g)\,M_p^{p-1}(f),$$

and, finally, $M_p(f) \leqslant BM_p(g)$, since $M_p(f)$ is finite.

9. We now complete the proof of Theorem 7 by showing that

$$M_q(f) \leqslant M_q(g).$$

We may suppose that ϕ is regular on $|z| = 1$, since, if the result is true then, we have in the general case,

$$M_q\big(u(R,\theta)\big) = M_q(f_R) \leqslant BM_q(g_R) \leqslant BM_q(g),$$

and so
$$M_q(f) \leqslant BM_q(g) \quad \text{by (B).}$$

We have then, after Hardy-Stein,

$$\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{d}{d\rho}\int |\phi|^q\,d\theta\right) = \int \Delta(|\phi|^q)\,d\theta = q^2 \int |\phi'|^2 |\phi|^{q-2}\,d\theta$$

$$\leqslant q^2 \int |\phi'|^2\,\Phi^{q-2}\,d\theta.$$

But (ϕ being regular for $\rho=1$ and zero at $z=0$)

$$\int_0^1 (1-\rho)\,\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\,\frac{d}{d\rho}\,(|\phi|^q)\,d\rho\right)=|\phi(e^{\theta i})|^q+\int_0^1 \frac{1}{\rho^2}\,|\phi|^q\,d\rho.$$

The last term being positive, we have

$$\int|\phi(e^{\theta i})|^q\,d\theta\leqslant q^2\iint(1-\rho)\,|\phi'|^2\,\Phi^{q-2}\,d\rho\,d\theta=q^2\int g^2\,\Phi^{q-2}\,d\theta,$$

$$M_q{}^q(\Phi)=\int\Phi^q\,d\theta\leqslant B\int|\phi(e^{\theta i})|^q\,d\theta\leqslant B\int g^2\,\Phi^{q-2}\,d\theta$$

$$\leqslant BM_q{}^2(g)\,M_q^{q-2}(\Phi);$$

$$M_q(f)\leqslant M_q(\Phi)\leqslant BM_q(g),$$

since $M_q(\Phi)$ is finite.

10. There remain the two theorems about J_r. We must prove first Theorem 5, that

$$J_q\leqslant BM_q(f).$$

This follows at once from Theorem 7. For, using (H), we have

$$J_q{}^q=\iint|\phi'|^q(1-\rho)^{q-1}\,d\rho\,d\theta\leqslant B\iint|\phi'|^2(1-\rho)\,\Phi^{q-2}\,d\rho\,d\theta=B\int g^2\,\Phi^{q-2}\,d\theta$$

$$\leqslant BM_q{}^2(g)\,M_q^{q-2}(\Phi)\leqslant BM_q{}^2(f)\,M_q^{q-2}\,(f).$$

It is perhaps desirable, however, to give a proof of this theorem that does not appeal to the deep Theorem 7; and it is, in fact, possible to prove it by "interpolation between $q=2$ and $q=\infty$".

We must first alter the theorem slightly into the form

$$\left(\iint|u_\rho|^q(1-\rho)^{q-1}\,d\rho\,d\theta\right)^{1/q}\leqslant B\left(\int|f|^q\,d\theta\right)^{1/q}.$$

If proved in this form it follows in the original form, since, by (C),

$$\int|\phi'|^q\,d\theta\leqslant B\int|u_\rho|^q\,d\theta\dagger.$$

(But the new form, unlike the old, is "true for $q=\infty$".)

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† We have $e^{i\theta}\phi'(z)=u_\rho+iv_\rho$, v_ρ is, for fixed ρ, the real variable conjugate of u_ρ, and the constant term of v_ρ is zero.

Let us define, for a function ψ of ρ, θ,

$$R_q(\psi) = \int_0^1 \int_{-\pi}^{\pi} |\psi|^q \frac{d\rho}{1-\rho}\, d\theta \quad (q < \infty),$$

$$R_\infty(\psi) = (\text{effective upper bound of } |\psi|).$$

The function $\psi(\rho, \theta) = (1-\rho)\, u_\rho$ is formed from f by an operation which is linear, and for which the "norm"

$$N_q = \max \frac{R_q(\psi)}{M_q(f)}$$

(where max denotes the upper bound for varying non-null f of L^q) exists, as we shall see in a moment, when $q = 2$ and when $q = \infty$. The necessary conditions are then satisfied for the application of M. Riesz's theorem† on the convexity of bilinear forms, and the theorem tells us that $\log N_q$ is convex in $1/q$ in the range $0 \leqslant 1/q \leqslant \frac{1}{2}$. Thus N_q is bounded for $2 \leqslant q \leqslant \infty$, and this *is* the modified theorem.

We must verify that N_2 and N_∞ are finite. The inequality $N_2 < A$ is equivalent to

$$\iint (1-\rho)\, |u_\rho|^2\, d\rho\, d\theta \leqslant A \int |f|^2\, d\theta,$$

which is true, since the left side is not greater than

$$\iint (1-\rho)\, |\phi'|^2\, d\rho\, d\theta = 2\pi \int (1-\rho)\, \Sigma\, |2nc_n|^2 \rho^{2n-2}\, d\rho \leqslant A\, \Sigma\, |c_n|^2 = A \int |f|^2\, d\theta.$$

The inequality $N_\infty < A$ is equivalent to the assertion

$$\text{“}\, |(1-\rho)\, u_\rho| < A \quad \text{if} \quad |f| \leqslant 1 \,\text{”},$$

and this also is true.

11. There remains (of the power series theorems) only Theorem 6. This is the "conjugate" of Theorem 5.

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† The theorem cannot be stated shortly, and this is not the place for a reader quite unfamiliar with it to learn it. There is a full account in Zygmund's book on trigonometrical series, 192–200. The application here is a little startling, but close attention will show it to be legitimate. There is nothing to prevent the transform ψ of f being a function of a pair of variables ρ, θ, with a weighted integral operation $d\theta\, d\rho/(1-\rho)$ to be applied to $|\psi|^q$.

We may suppose ϕ regular on $|z|=1$. For, assuming the result for this case, consider the general one. Since

$$\iint |R\phi'(Rz)|^p(1-\rho)^{p-1}\,d\rho\,d\theta$$

increases with R in $0 \leqslant R \leqslant 1$ (since the integral with respect to θ does when ρ is fixed), we have

$$M_p\big(u(R,\theta)\big) = M_p(f_R) \leqslant B\,J_p(f_R) \leqslant BJ_p(f)$$

and so, by (B),

$$M_p(f) = \lim M_p\big(u(R,\theta)\big) \leqslant BJ_p,$$

the desired result.

Taking now the special case, we have from Lemma 2, with the notation of §8,

$$M_p{}^p(f) = \int |f|^p\,d\theta \leqslant A\iint (1-\rho)\Big(|\phi'\psi'| + \frac{1}{\rho}|\phi'\psi|\Big)\,d\rho\,d\theta. \tag{1}$$

Now

$$\begin{aligned}\iint (1-\rho)|\phi'\psi'|\,d\rho\,d\theta &\\ &\leqslant \Big(\iint (1-\rho)^{p-1}|\phi'|^p\,d\rho\,d\theta\Big)^{1/p}\Big(\iint (1-\rho)^{p'-1}|\psi'|^{p'}\,d\rho\,d\theta\Big)^{1/p'}\\ &= J_p(f)\,J_{p'}(h) \leqslant BJ_p(f)\,M_{p'}(h),\end{aligned}$$

by the opposite inequality for q,

$$\leqslant BJ_p M_p^{p-1}(f), \tag{2}$$

by (2) of §8. Also

$$\begin{aligned}\iint (1-\rho)\frac{1}{\rho}|\phi'\psi|\,d\rho\,d\theta &\\ &\leqslant A\iint (1-\rho)|\phi'|\Psi\,d\rho\,d\theta\\ &\leqslant A\Big(\iint (1-\rho)^{p-1}|\phi'|^p\,d\rho\,d\theta\Big)^{1/p}\Big(\iint (1-\rho)^{p'-1}\Psi^{p'}\,d\rho\,d\theta\Big)^{1/p'}\\ &= BJ_p M_{p'}(\Psi)\\ &\leqslant BJ_p M_{p'}(h) \leqslant BJ_p M_p^{p-1}(f).\end{aligned} \tag{3}$$

From (1), (2), and (3),

$$M_p^{\,p}(f) \leqslant BJ_p\, M_p^{p-1}(f),$$

$$M_p\,(f) \leqslant J_p,$$

since $M_p\,(f)$ is finite, and the proof is completed.

12. We complete this section on power series by giving a direct proof that

$$M_p(g) \leqslant BJ_p,$$

which amounts to a deduction of Theorem 6 from Theorem 7.

Let $\rho_n = 1-2^{-n}$, so that $1-\rho_n = \rho_n - \rho_{n-1}$. Then

$$g^2(\theta) = \int (1-\rho)\,|\phi'(\rho e^{\theta i})|^2\, d\rho \leqslant \sum_1^\infty (1-\rho_{n-1})\,\mu_n{}^2(\theta)\,(\rho_n-\rho_{n-1}),$$

where $\mu_n(\theta) = \max|\phi'(\rho e^{i\theta})|$ for $\rho_{n-1} \leqslant \rho \leqslant \rho_n$. Since

$$(\Sigma a)^\lambda \leqslant \Sigma\, a^\lambda$$

when $\lambda < 1$, we have

$$g^p(\theta) \leqslant \sum_1^\infty \{(1-\rho_{n-1})(\rho_n-\rho_{n-1})\}^{\frac{1}{2}p}\,\mu_n{}^p \leqslant B \sum_1^\infty (1-\rho_{n+1})^{p-1}\,(\rho_{n+1}-\rho_n)\,\mu_n{}^p,$$

$$\int g^p\, d\theta \leqslant B \sum_1^\infty (1-\rho_{n+1})^{p-1}\,(\rho_{n+1}-\rho_n) \int \mu_n{}^p\, d\theta.$$

But, by "max" applied to the circle $|z| \leqslant \rho_n$,

$$\int \mu_n{}^p\, d\theta \leqslant B \int |\phi'(\rho_n e^{\theta i})|^p\, d\theta.$$

Hence

$$\int g^p\, d\theta \leqslant B \sum_1^\infty (1-\rho_{n+1})^{p-1}(\rho_{n+1}-\rho_n) \int |\phi'(\rho_n e^{\theta i})|^p\, d\theta$$

$$\leqslant B \int_0^1 (1-\rho)^{p-1} \int |\phi'(\rho e^{i\theta})|^p\, d\theta\, d\rho,$$

since the integral last written increases with ρ. This is the desired result.

Fourier series theorems.

13. "Complex max" does not occur again, and we can use capital letters F, G, etc., without fear of confusion.

The crucial theorem (part of Theorem 3) is

$$(1) \qquad \int G^{2k}\,d\theta \leqslant B \int |f|^{2k}\,d\theta,$$

where

$$G^2 = \sum_{-\infty}^{\infty} |\Delta_n|^2.$$

When this is proved, the rest is comparatively easy.

We shall need

LEMMA 3. *Let* $U_n(\theta)$ $(n = 1, 2, \ldots, k)$ *be* k *functions of* L^{2k},

$$V_n(\theta) = -\frac{1}{2\pi}\int U_n(\theta+t)\cot\tfrac{1}{2}t\,dt,$$

the conjugate of U_n *as in* (A). *Let*

$$a_n = \frac{1}{2\pi}\int U_n\,d\theta.$$

Then

$$(1) \quad \int |V_1 V_2 \ldots V_k|^2\,d\theta \leqslant B\Big(|a_1 a_2 \ldots a_n|^2 + \Sigma\int |U_1 V_2 V_3 \ldots V_k|^2\,d\theta$$

$$+\Sigma\int |U_1 U_2 V_3 \ldots V_k|^2\,d\theta + \ldots + \int |U_1 U_2 \ldots U_k|^2\,d\theta\Big),$$

where the various sums contain the different products of one U *and* $(k-1)$ V*'s, two* U*'s and* $(k-2)$ V*'s, ...,* k U*'s, respectively*†.

We may suppose the U_n (and so the V_n) real. Let

$$f_n(z) = u_n + iv_n, \quad \mathfrak{F}(z) = \Pi f_n^2 = \Pi(u_n^2 - v_n^2 + 2iu_n v_n).$$

Then

$$\int \mathfrak{F}(z)\,d\theta = 2\pi\,\mathfrak{F}(0) = A\,\Pi a_n^2.$$

The left-hand side, when expanded, contains one term

$$\pm\int v_1^2 v_2^2 \ldots v_k^2\,d\theta,$$

and other terms of type

$$T = \pm B\int \Pi_1(u_n v_n)\,\Pi_2(u_n^2)\,\Pi_3(v_n^2),$$

---

† The theorem is a kind of k-dimensional modification of (M. Riesz's) theorem (7, 221) for the even integer case. The proof is an obvious modification of Riesz's original proof.

Π_1 and Π_2 not being null together. Now

$$|T| \leqslant T^* = B \int \Pi_1 \left(\frac{v_n^2}{K} + K u_n^2\right) \Pi_2 u_n^2 \, \Pi_3 v_n^2 \, d\theta,$$

K being a positive constant at our disposal, and we have

$$(2) \qquad \int v_1^2 v_2^2 \ldots v_k^2 \, d\theta \leqslant 2\pi \Pi |a_n|^2 + \Sigma T^*.$$

In ΣT^* the term $\int v_1^2 v_2^2 \ldots v_k^2 \, d\theta$ will occur with a total factor C depending on K and small when K is large. The remaining terms of ΣT^* are all of the type of those on the right-hand side of (1) (with small u, v for large). If we choose K so that, say, $C < \frac{1}{2}$, we obtain an inequality of the form (1) with small u, v for large. Since, by (C), each u, v converges strongly with index $2k$ to its limit function U, V when $\rho \to 1$, the inequality (1) itself follows by taking the limit.

COROLLARY. *Suppose that (with changed notation)* U_n $(n = 1, 2, \ldots)$ *is a sequence of functions of* L^{2k}†, V_n *the conjugate of* U_n, $a_n = \frac{1}{2\pi} \int U_n \, d\theta$. *Then*

$$\int \left(\sum_1^\infty |V_n|^2\right)^k d\theta \leqslant B(\Sigma |a_n|^2)^k + B \int \left(\sum_1^\infty |U_n|^2\right)^k d\theta.$$

For, if each U and V in (1) ranges through the U_n and V_n respectively, and we write

$$S = \Sigma |U_n|^2, \quad T = \Sigma |V_n|^2,$$

we get

$$\int T^k \, d\theta \leqslant B(\Sigma |a_n|^2)^k + B \int S T^{k-1} \, d\theta + B \int S^2 T^{k-2} \, d\theta + \ldots + B \int S^k \, d\theta,$$

$$M_k^k(T) \leqslant B(\Sigma |a_n|^2)^k + B M_k(S) M_k^{k-1}(T)$$
$$+ B M_k^2(S) M_k^{k-2}(T) + \ldots + B M_k^k(S),$$

from which follows the desired result

$$M_k^k(T) \leqslant B(\Sigma |a_n|^2)^k + B M_k^k(S).$$

14. We may suppose f a Fourier power series and that all the terms in Δ_0 are zero, so that

$$f \sim \sum_{n > \nu_1}^\infty c_n e^{n\theta i}.$$

---

† It is enough, of course, that each should be integrable; unless each $U_n \epsilon L^{2k}$ there is nothing to prove.

For, in the general case, the parts $\sum\limits_{\nu_1+1}^{\infty}$ $\sum\limits_{-\nu_1'-1}^{-\infty}$ of the series are, by (L), Fourier power series of f_1, f_2 with $M_r(f_i) \leqslant BM_r(f)$, so are the pieces Δ_0, Δ_{-1}, and (1) of § 13 is true for f by combination if it is true for the parts.

We shall need the abbreviations

$$N = \nu_n, \quad N' = \nu_{n+1};$$

$$\rho_n = 1-1/\nu_n = 1-1/N, \quad \delta = \delta_n = 1-\rho_n = 1/\nu_n = 1/N.$$

Following the heuristic idea referred to in the Introduction (and explained in our summary of results **3**), we represent Δ_n in terms of an integral of $\phi'(\rho_n e^{\theta i})$ with respect to a kernel. We have at once

$$\Delta_n = \int_{-\pi}^{\pi} \phi'(\rho_n, \theta+t)\,\chi(t)\,dt, \tag{1}$$

[where we have written $\phi'(\rho_n, \theta+t)$ as more convenient in this context than $\phi'(\rho_n e^{i(\theta+t)})$, and] where[†]

$$\chi(t) = \frac{1}{2\pi} \sum_{N+1}^{N'} \frac{\cos mt}{m} \rho_n^{-m}.$$

Straightforward and elementary calculations based on this representation lead to the following results.

LEMMA 4. *We have*

$$\chi = \left[\frac{1}{4\pi}\frac{\rho_n^{-m}}{m} \sin(m-\tfrac{1}{2})t \cot \tfrac{1}{2}t\right]_{N+1}^{N'+1} + O(\delta) + O\big(\delta P(\rho, t)\big)\,\ddagger, \tag{2}$$

$$\Delta_n = S'-S+O\left(\delta_n \int \Phi(\theta)\,d\theta\right)+O(T_n), \tag{3}$$

where

$$T_n^2 = T_n^2(\theta) = \frac{\delta_n^2}{2\pi} \int |\phi'(\rho_n, \theta+t)|^2 P(\rho_n, t)\,dt, \tag{4}$$

$$S' = \frac{1}{4\pi}\frac{\rho_n^{-N'-1}}{N'+1} \int_{-\pi}^{\pi} \sin(N'+\tfrac{1}{2})t\,\phi'(\rho_n, \theta+t) \cot \tfrac{1}{2}t\,dt, \tag{5}$$

and S is obtained from S' by changing N' into N where it occurs. Finally

$$|\Delta_n|^2 \leqslant BM^2_{2k}(f) + BT_n^2 + B\delta_n^2 \sum_{i=1}^{4} |V_{ni}|^2, \tag{6}$$

[†] The reader may be reminded that $\phi = 2 \sum\limits_{1}^{\infty} c_n z^n$.

[‡] The constants implied in the O's are B's.

where the U_{ni} $(i = 1, 2, 3, 4)$ *are the four functions*

$$\genfrac{}{}{0pt}{}{\cos}{\sin}\,(N+\tfrac{1}{2})\,\theta\,\phi'(\rho_n,\theta), \qquad \genfrac{}{}{0pt}{}{\cos}{\sin}\,(N'+\tfrac{1}{2})\,\theta\,\phi'(\rho_n,\theta),$$

and the V_{ni} *are their conjugates.*

(6) is what we use; the remaining results are stages on the route (and are stated to make it as easy as possible for the reader to skip the tedious but necessary calculations that follow).

We observe first that $\rho_n^{-m} < B$ for all m of $N-1 \leqslant m \leqslant N'+2$, and that $\Sigma\, 1/m < B$ for a sum taken over this range of m, so that incidentally $|\chi| < B$. We now multiply χ by $\sin\frac{1}{2}t$, replace

$$2\cos mt \sin\tfrac{1}{2}t \quad \text{by} \quad \sin(m+\tfrac{1}{2})\,t - \sin(m-\tfrac{1}{2})\,t,$$

and sum twice by parts. Since

$$|\Sigma \sin(m+\tfrac{1}{2})\,t| \leqslant A\,|\operatorname{cosec}\tfrac{1}{2}t|,$$

whatever the limits of summation, we find that

$$D = \chi \sin\tfrac{1}{2}t - \left[\frac{1}{4\pi}\,\frac{\rho_n^{-m}}{m}\sin(m-\tfrac{1}{2})\,t\right]_{N+1}^{N'+1}$$

is the difference of two expressions of the form

$$A\left(\sum_{\lambda}^{\mu}\sin(m+\tfrac{1}{2})\,t\right)\Delta\left(\frac{\rho_n^{-\nu}}{\nu}\right),$$

where λ, μ, ν lie between $N-1$ and $N'+2$, together with a sum majorised by

$$A\,|\operatorname{cosec}\tfrac{1}{2}t|\sum_{N-1}^{N'+1}\left|\Delta^2\left(\frac{\rho_n^{-m}}{m}\right)\right|.$$

It follows that $|D| < B\delta^2/|t|$, and hence that

$$(7)\quad \chi - \left[\frac{1}{4\pi}\,\frac{\rho_n^{-m}}{m}\sin(m-\tfrac{1}{2})\,t\cot\tfrac{1}{2}t\right]_{N+1}^{N'+1}$$

$$= D\operatorname{cosec}\tfrac{1}{2}t - (1-\cos\tfrac{1}{2}t)\operatorname{cosec}\tfrac{1}{2}t\left[\frac{1}{4\pi}\,\frac{\rho_n^{-m}}{m}\sin(m-\tfrac{1}{2})\,t\right]_{N+1}^{N'+1}$$

$$= O\left(\frac{\delta^2}{t^2}\right) + O(\delta).$$

When $|t|>\delta$, $\delta^2/t^2=O\big(\delta P(\rho, t)\big)$. When $|t|\leqslant\delta$, both terms in (7) are $O(1)$, and therefore so is $O\big(\delta P(\rho, t)\big)$. Hence, for all t, (7) is of the form $O\big(\delta P(\rho, t)\big)+O(\delta)$, which proves (2).

Since

$$\frac{1}{2\pi}\int\delta P(\rho_n, t)\,|\phi'(\rho_n, \theta+t)|\,dt\leqslant\delta\left(\frac{1}{2\pi}\int|\phi'|^2 P\,dt\right)^{\frac{1}{2}},$$

and

$$\frac{1}{2\pi}\int\delta\,|\phi'(\rho_n, \theta+t)|\,dt\leqslant A\int\Phi(\theta+t)\,dt=A\int\Phi(\theta)\,d\theta,$$

we obtain the results (3) to (5) of the lemma. Finally, when we change the variable from t to $t-\theta$ in the integrals in S and S' and expand $\sin(N+\frac{1}{2})(t-\theta)$ and $\sin(N'+\frac{1}{2})(t-\theta)$ by the formula for the sine of a difference, we see that S and S' are combinations of the four V_{ni}'s with multipliers of the form $O(\delta)$. Thus

$$|\Delta_n|\leqslant B\int\Phi\,d\theta+BT_n+B\delta\sum_{i=1}^{4}|V_{ni}|,$$

and (6) follows at once, since

$$\frac{1}{2\pi}\int\Phi\,d\theta\leqslant M_{2k}(\Phi)\leqslant BM_{2k}(f).$$

15. We shall set out the proof for $2k=6$, a case entirely typical. We have, writing M for $M_6(f)$,

$$\int G^6\,d\theta=\int\left(\sum_1^\infty|\Delta_n|^2\right)^3 d\theta$$

$$\leqslant B\int\left(\Sigma\,\delta_n^2(M^2+T_n^2+|V_{n1}|^2+|V_{n2}|^2+|V_{n3}|^2+|V_{n4}|^2)\right)^3 d\theta,$$

by Lemma 4,

$$\leqslant B\int M^6(\Sigma\,\delta_n^2)^3\,d\theta+B\int(\Sigma\,\delta_n^2\,T_n^2)^3\,d\theta+B\int\left(\sum_{n,\,i}\delta_n^2|V_{ni}|^2\right)^3 d\theta$$

$$\leqslant BM^6+B\int(\Sigma\,\delta_n^2\,T_n^2)^3\,d\theta+B(\Sigma\,\delta_n^2|\alpha_{ni}|^2)+B\int\left(\sum_{n,\,i}\delta_n^2|U_{ni}|^2\right)^3 d\theta,$$

by Lemma 3 (and with its notation) applied to the sum $\sum\limits_{n,\,i}\delta_n^2|V_{ni}|^2$.

Now $|U_{ni}| \leqslant B|\phi'(\rho_n, \theta)|$, and

$$|a_{ni}|^2 \leqslant A\Big(\int |U_{ni}|\, d\theta\Big)^2 \leqslant A \int |U_{ni}|^2 d\theta \leqslant A \int |\phi'(\rho_n, \theta)|^2 d\theta,$$

$$(\Sigma\, \delta_n^{\,2} |a_{ni}|^2)^3 \leqslant B\Big(\int \Big(\Sigma\, \delta_n^{\,2} |\phi_n'(\rho_n, \theta)|^2 d\theta\Big)^3 \leqslant B \int \Big(\Sigma\, \delta_n^{\,2} |\phi_n'(\rho_n, \theta)|^2\Big)^3 d\theta,$$

so that

$$(1) \quad \int G^6\, d\theta \leqslant BM^6 + B\int \Big(\Sigma\, \delta_n^{\,2} |\phi_n'(\rho_n, \theta)|^2\Big)^3 d\theta + B\int \Big(\Sigma\, \delta_n^{\,2}\, T_n^{\,2}(\theta)\Big)^3 d\theta.$$

Write now

$$T^2(\rho) = T^2(\rho, \theta) = \frac{1}{2\pi} \int |\phi'(\rho, \theta+t)|^2 P(\rho, t)\, dt,$$

so that $T_n = T(\rho_n)$. Since

$$|\phi'(\rho_n, \theta)|^2 = \left| \frac{1}{2\pi} \int \phi'(\rho_n^{\frac{1}{2}}, \theta+t)\, P(\rho_n^{\frac{1}{2}}, t)\, dt \right|^2$$

$$\leqslant \frac{1}{2\pi} \int |\phi'(\rho_n^{\frac{1}{2}}, \theta+t)|^2 P(\rho_n^{\frac{1}{2}}, t)\, dt = T^2(\rho_n^{\frac{1}{2}}),$$

(1) gives

$$(2) \qquad \int G^6\, d\theta \leqslant BM^6 + B\int \Big(\Sigma\, \delta_n^{\,2} \{T^2(\rho_n^{\frac{1}{2}}) + T^2(\rho_n)\}^3\Big)\, d\theta.$$

Now (for fixed θ) $T(\rho)$ has the property that, if

$$r > \rho \quad \text{and} \quad (1-\rho) < B(1-r),$$

then

$$(3) \qquad T(r, \theta) \geqslant BT(\rho, \theta).$$

For, since $w(\rho, \theta) = |\phi'(\rho, \theta)|^2$ is sub-harmonic, it is not greater than its Poisson integral over a circle $|z| = r > \rho$†;

$$w(\rho, \theta) \leqslant \frac{1}{2\pi} \int w(r, \theta+t)\, P\Big(\frac{\rho}{r}, t\Big)\, dt.$$

Changing θ to $\theta+x$, multiplying by $1/(2\pi)\, P(\rho, x)$, and integrating with respect to x,

---

† F. Riesz, 6.

$$T(\rho, \theta) \leqslant \left(\frac{1}{2\pi}\right)^2 \iint w(r, \theta+t+x)\, P\left(\frac{\rho}{2\pi}, t\right) P(\rho, x)\, dt\, dx$$

$$= \frac{1}{2\pi} \int w(r, \theta+t)\, dt\, \frac{1}{2\pi} \int P\left(\frac{\rho}{r}, t-x\right) P(\rho, x)\, dx$$

$$= \frac{1}{2\pi} \int w(r, \theta+t)\, P\left(\frac{\rho^2}{r}, t\right) dt. \tag{4}$$

But $1-r < 1-(\rho^2/r) < B(1-r)$, and so

$$P\left(\frac{\rho^2}{r}, t\right) \leqslant BP(r, t)$$

(for all t), and substituting from this in (4) we get $T(\rho) \leqslant BT(r)$.

It follows that

$$T^2(\rho_n) + T^2(\rho_n^{\frac{1}{2}}) \leqslant BT^2(\sigma) \quad (\rho_n^{\frac{1}{2}} \leqslant \sigma \leqslant \rho_{n+1}^{\frac{1}{2}})$$

$$\delta_n\left(T^2(\rho_n) + T^2(\rho_n^{\frac{1}{2}})\right) \leqslant B(1-\sigma)\, T^2(\sigma) \quad (\rho_n^{\frac{1}{2}} \leqslant \sigma \leqslant \rho_{n+1}^{\frac{1}{2}}),$$

and so, integrating from $\rho_n^{\frac{1}{2}}$ to $\rho_{n+1}^{\frac{1}{2}}$ and observing that the length of the path exceeds $B\delta_n$,

$$\delta_n^2\left(T^2(\rho_n) + T^2(\rho_n^{\frac{1}{2}})\right) \leqslant B \int_{\rho_n^{\frac{1}{2}}}^{\rho_{n+1}^{\frac{1}{2}}} (1-\sigma)\, T^2(\sigma)\, d\sigma.$$

Hence, finally,

$$\sum_1^\infty \delta_n^2 \left(T^2(\rho_n) + T^2(\rho_n^{\frac{1}{2}})\right) \leqslant B \int_0^1 (1-\sigma)\, T^2(\sigma)\, d\sigma = Bg^{*2},$$

$$\int G^6\, d\theta \leqslant BM^6 + B\int g^{*6}\, d\theta \leqslant B\int |f|^6\, d\theta,$$

by Theorem 7*.

16. We next prove Theorem 4 for the even integer case. We may again suppose that

$$f \sim \sum_{m > \nu_1} c_m e^{m\theta i},$$

and we shall take again the typical case $2k = 6$. Let

$$F_\nu = \sum_1^\nu \eta_m \Delta_m, \quad G_\nu^2 = \sum_1^\nu |\Delta_m|^2.$$

It is enough to prove that

$$(1) \qquad \int |F_\nu|^6 \leqslant B \int G_\nu^6 \, d\theta.$$

For, by our last theorem [(6) of § 13],

$$(2) \qquad \int G_\nu^6 \, d\theta \leqslant \int G^6 \, d\theta \leqslant B \int |f|^6 \, d\theta.$$

Also, if s_m is the sum to m terms of the formal series $\Sigma \eta \Delta$ (written at length), s_m is of the form $F_\nu + \Delta'$, for some ν, where Δ' is a connected piece of $\Delta_{\nu+1}$ and so of the Fourier series of f. By (2) and (L) (i) it follows, assuming the truth of (1), that

$$\int |s_m|^6 \, d\theta \leqslant B \int G_r^6 \, d\theta + B \int |\Delta'|^6 \, d\theta$$

$$\leqslant B \int |f|^6 \, d\theta,$$

and it then follows from (L)(iii) that $\Sigma \eta_m \Delta_m \sim F$, where

$$\int |F|^6 \, d\theta \leqslant B \int |f|^6 \, d\theta.$$

We separate the formal series $\Sigma \eta_m \Delta_m$ into h part series $F^{(a)}$; in each part m ranges through the numbers $\mu h + a$, where a $(= 1, 2, \ldots, h)$ is constant for each part, and $\mu = 1, 2, 3, \ldots$. If now h is a large enough B, each of the $F^{(a)}$ has the property that, if Δ_l, $\Delta_{l'}$, Δ_m, $\Delta_{m'}$, Δ_n, $\Delta_{n'}$ are Δ's belonging to it for which no one of l, m, n is equal to any one of l', m', n', then

$$\Delta_l \bar{\Delta}_{l'} \Delta_m \bar{\Delta}_{m'} \Delta_n \bar{\Delta}_{n'}$$

(where the bar denotes the conjugate complex number), when expanded as a complex Fourier series $\Sigma\, C_\mu e^{\mu \theta i}$, has $C_0 = 0$, so that

$$\int \Delta_l \bar{\Delta}_{l'} \Delta_m \bar{\Delta}_{m'} \Delta_n \bar{\Delta}_{n'} \, d\theta = 0 \dagger.$$

[†] Suppose, *e.g.*, that $l \geqslant m \geqslant n$, and that l', m', n' are less than l. $\Delta_l \Delta_m \Delta_n$ has only positive powers in its expansion, and the least index is at least ν_l. $\bar{\Delta}_{l'} \bar{\Delta}_{m'} \bar{\Delta}_{n'}$ has only negative powers, and the greatest absolute value of an index is at most $\nu_{l'+1} + \nu_{m'+1} + \nu_{n'+1}$. These suffixes are each at most $l - h + 1$, and h has only to satisfy $\nu_l > 3\nu_{l-h+1}$ for all $l \geqslant h$.

It is evident, after a little consideration, that

$$\int |F_\nu^{(a)}|^6 \, d\theta = b_1 \int (\Sigma |\Delta_l|^2 \Delta_m \bar{\Delta}_{m'} \Delta_n \bar{\Delta}_{n'}) \, d\theta \pm b_2 \int (\Sigma |\Delta_l|^2 |\Delta_m|^2 \Delta_n \bar{\Delta}_{n'}) \, d\theta$$

$$\pm b_3 \int (\Sigma |\Delta_l|^4 \Delta_n \bar{\Delta}_{n'}) \, d\theta \pm b_4 \int (\Sigma |\Delta_l|^2 |\Delta_m|^2 |\Delta_n|^2) \, d\theta,$$

where b_1, b_2, b_3, b_4 are integers (of type B), and in each summation all the suffixes concerned range through all values that can be suffixes of Δ's of $F_\nu^{(a)}$,

$$= b_1 \int (G_\nu^{(a)})^2 |F_\nu^{(a)}|^4 \, d\theta \pm b_2 \int (G_\nu^{(a)})^4 |F_\nu^{(a)}|^2 \, d\theta \pm b_3 \int (\Sigma |\Delta_l|^4) |F_\nu^{(a)}|^2 \, d\theta$$

$$\pm b_4 \int (G_\nu^{(a)})^6 \, d\theta.$$

Since $\Sigma |\Delta_l|^4 \leqslant (\Sigma |\Delta_l|^2)^2 = (G_\nu^{(a)})^4$, we get

$$\int |F_\nu^{(a)}|^6 \, d\theta \leqslant B \left(\int (G_\nu^{(a)})^2 |F_\nu^{(a)}|^4 \, d\theta + \int (G_\nu^{(a)})^4 |F_\nu^{(a)}|^2 \, d\theta + \int (G_\nu^{(a)})^6 \, d\theta \right),$$

and so (by an argument by now familiar)

$$\int |F_\nu^{(a)}|^6 \, d\theta \leqslant B \int (G_\nu^{(a)})^6 \, d\theta \leqslant B \int G^6 \, d\theta.$$

Combining the h inequalities for the different a, we get

$$\int |F_\nu|^6 \, d\theta \leqslant B \int G^6 \, d\theta,$$

the desired result.

17. We can at once extend Theorem 4 to general q. For any fixed sequence (η_m), $F\{f\}$ is a linear operation, and "interpolation" is possible for $2k \leqslant q \leqslant 2k+2$ by M. Riesz's theorem.

To complete the proof of Theorem 4, we have still to deal with an index p. This is a matter of "conjugacy".

Given $f \in L^p$, let $h(-\theta)$ be an arbitrary function of $L^{p'}$. Let

$$h(\theta) \sim \Sigma \gamma_m e^{m\theta i}, \quad h_1 = h - \gamma_0,$$

and let $H_1(-\theta)$ be the "F" of $h_1(-\theta)$ (with the same set of η's as F). Then

$$\frac{1}{2\pi} \int s_n(F) \, h(-\theta) \, d\theta = \frac{1}{2\pi} \int s_n(f) \, H_1(-\theta) \, d\theta,$$

since each is equal to $$\sum_{-n}^{n} \eta_\mu c_m \gamma_m,$$

where μ is the suffix of the Δ to which $c_m e^{m\theta i}$ belongs. Hence

$$\left|\frac{1}{2\pi}\int s_n(F)\,h(-\theta)\,d\theta\right| \leqslant M_p\big(s_n(f)\big)\,M_{p'}\big(H_1(-\theta)\big)$$

$$\leqslant BM_p(f)\,BM_{p'}\big(h_1(-\theta)\big),$$

by (L) (i) and the result for q $(=p')$,

$$\leqslant BM_p(f)\,M_{p'}\big(h(-\theta)\big),$$

since $|\gamma_0| \leqslant M_{p'}\big(h(-\theta)\big)$. By the converse of Hölder's inequality,

$$M_p\big(s_n(F)\big) \leqslant BM_p(f),$$

and the desired result follows by (L) (iii).

18. We have still to prove Theorem 3 with general index r; and this we can do in a few lines. Evidently any F is an F of any other (for a fixed system of Δ's and by a suitable choice of η's). In particular, keeping the *same* set of η's, $F\{F\}$ is f. Theorem 4 therefore gives $\left(\text{taking } f=f_n=\sum_{-n}^{n}\Delta_m\right)$

$$\text{(1)} \qquad B\int\left|\sum_{-n}^{n}\eta_m\Delta_m\right|^r d\theta \leqslant \int|f_n|^r\,d\theta \leqslant B\int\left|\sum_{-n}^{n}\eta_m\Delta_m\right|^r d\theta.$$

We now average over the 2^{2n+1} distinct sets of (η_m). Since† the average of

$$\left|\sum_{-n}^{n}\eta_m a_m\right|^r$$

lies between constant multiples A_r of

$$(\Sigma|a_m|^2)^{\frac{1}{2}},$$

the average of

$$\int\left|\sum_{-n}^{n}\eta_m\Delta_m\right|^r d\theta$$

lies between multiples A_r of

$$\int\left(\sum_{-n}^{n}|\Delta_m|^2\right)^{\frac{1}{2}r} d\theta.$$

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† J. E. Littlewood (4), 170; Paley and Zygmund (5).

(1) now gives

$$B\int\left(\sum_{-n}^{n}|\Delta_m|^2\right)^{\frac{1}{2}r}d\theta \leqslant \int |f_n|^r\,d\theta \leqslant B\int\left(\sum_{-n}^{n}|\Delta_m{}^2|\right)^{\frac{1}{2}r}d\theta,$$

and Theorem 3 follows by making $n \to \infty$ [and using (L) (ii) for the middle term].

19. It remains to say a few words about Theorems 1 and 2, that

$$\Sigma\int|\Delta_n|^q\,d\theta \leqslant B\int|f|^q\,d\theta, \qquad \int|f|^p\,d\theta \leqslant \Sigma\int|\Delta_n|^p\,d\theta.$$

They are immediate corollaries of Theorem 3, since

$$\Sigma|\Delta_n|^q \leqslant G^q, \qquad G^p \leqslant \Sigma|\Delta_n|^p$$

in virtue of "$(\Sigma a)^\lambda \gtreqless \Sigma a^\lambda \quad (\lambda \gtreqless 1)$".

They can, however, be deduced from their power series analogues, Theorems 5 and 6. The deduction of Theorem 1 from Theorem 5 is more or less parallel to that of Theorem 3 from Theorem 7*; but it is far easier. We do not need Lemma 3, and Theorem 5 does not need to be strengthened into a "Theorem 5*". We shall not set the argument out in detail, nor the deduction of Theorem 2 from Theorem 6.

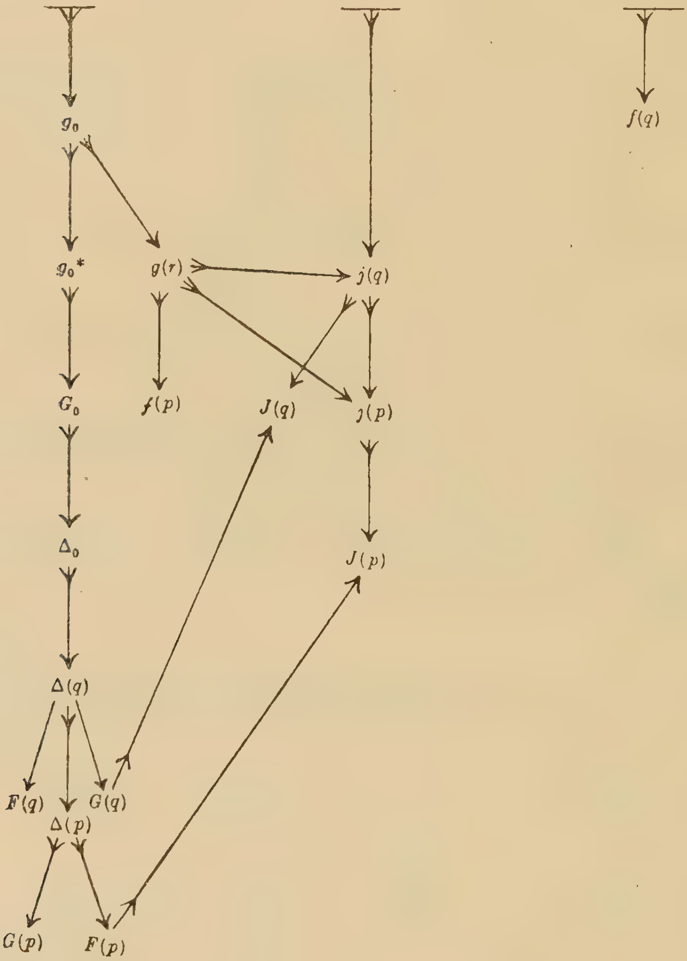
20. Let us give short names to the effectively different propositions that figure in the paper according to the following scheme. We call[†]

"$M_r(g) \leqslant BM_r(f)$"	$g(r)$,	"$M_r(f) \leqslant BM_r(g)$"	$f(r)$;
"$J_q \leqslant BM_q(f)$"	$j(q)$,	"$M_p(f) \leqslant BJ_p$"	$j(p)$;
"$M_r(G) \leqslant BM_r(f)$"	$G(r)$,	"$M_r(f) \leqslant BM_r(G)$"	$F(r)$;
"$\Sigma\int|\Delta_n|^q\,d\theta \leqslant B\int|f|^q\,d\theta$"	$J(q)$,	"$\int|f|^p\,d\theta \leqslant \Sigma\int|\Delta_n|^p\,d\theta$"	$J(p)$;

"Theorem 4" $\Delta(r)$.

[†] Power series propositions have small letters, the "analogous" Fourier ones corresponding large ones. Theorem 4, on the existence of $F \in L^r$, is outside the analogy and is given a distinctive name.

We distinguish where necessary (and usually) between $r = p$, $r = q$, and $r = 2k$, and instead of (*e.g.*) $g(2k)$ we write g_0. Finally, Theorem 7* (which has $r = 2k$ and g^* for g) we call g_0^*. Then the possible chains of argument are shown in the following diagram. Arrows represent deductions (*from* the premises *to* the conclusion in the sense of the arrow); the chains start from horizontal lines representing existing knowledge.



There are no short cuts (so far as we know) on the main chain: a proof of either $F(p)$ or $G(p)$ would occupy 18 pages.

List of papers referred to.

1. G. H. Hardy, " The mean value of the modulus of an analytic function ", *Proc. London Math. Soc.* (2), 14 (1914), 269–277.
2. G. H. Hardy and J. E. Littlewood, " A maximal theorem with function-theoretic applications ", *Acta Math.*, 54 (1930), 81–116.
3. J. E. Littlewood, " On bounded bilinear forms in an infinite number of variables ", *Quart J. of Math.* (2), 1 (1930), 164–174.
4. J. E. Littlewood and R. E. A. C. Paley, " Theorems on Fourier series and power series ", *Journal London Math. Soc.*, 6 (1931), 230–233.
5. R. E. A. C. Paley and A. Zygmund, " On some series of functions ", *Proc. Cambridge Phil. Soc.*, 26 (1930), 337–357, 458–474.
6. F. Riesz, " Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel ", *Acta Math.*, 48 (1926), 329–343.
7. M. Riesz, " Sur les fonctions conjuguées ", *Math. Zeitschrift*, 27 (1927), 218–244.
8. ———, " Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires ", *Acta Math.*, 49 (1926), 465–497.
9. P. Stein, " On a theorem of M. Riesz ", *Journal London Math. Soc.*, 8 (1933), 242–247.

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ON MILNE'S THEORY OF WORLD-STRUCTURE*.

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[Received 18 June, 1936.—Read 18 June, 1936.]

1. *Introduction.*

It was shown in a recent paper† that the model universe constructed by Milne from purely kinematical considerations is very similar in its macroscopic aspect to some of the well-known systems provided by general relativity, viz., those corresponding to space of constant negative curvature. Milne's system is, however, fundamentally different from the general relativity systems corresponding to space of zero or positive curvature, and this at first sight is surprising, for Milne based his work upon certain general principles which are formally satisfied by all the general relativity models. In the paper mentioned above, it was suggested that the elimination of these other possible solutions from Milne's general result is a consequence of the assumption of the Lorentz transformations relating observers in three spatial dimensions, for it can be shown that other more complicated transformations are equally valid when more than one spatial dimension is to be considered. If, however, the Lorentz transformations are to be abandoned, we meet with considerable difficulty in attempting to follow Milne's method, and it is at once evident that some other method of attack is required. The object of the present paper is to show that we can proceed from the same general principles laid down by Milne, without assuming any particular transformations. We shall obtain what I believe to be the general solution based upon these principles and certain other hypotheses, and it will be seen that this solution includes as special cases all of the systems provided by general relativity, as well as Milne's system.

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* An account of the results here obtained was read before the British Association meeting at Norwich, September, 1935, under the title, "The place of geometry in relativistic cosmology". An abstract was published in the *British Association Reports*, 1935. An account was also read at a colloquium at Oxford, May, 1935.

† *Monthly Notices R.A.S.*, 95 (1935), 263–269.

The two basic principles in this work are the *cosmological principle*, as formulated precisely by Milne, and the *principle of symmetry*. Sections 3–7 are devoted to the discussion of a material system satisfying the cosmological principle, after which the restrictions implied by the principle of symmetry are introduced. The light paths are then found by adopting Fermat's principle. At this stage, the general solution involves two arbitrary functions, but it is shown how these functions are restricted in form as a consequence of Milne's dimensional hypothesis (§ 17).

A result of great importance is that there is a quadratic differential form which is invariant under the transformations from one observer to any other. This quadratic leads to a Riemannian "map", possessing, with one important exception, all the properties of the map used in general relativity. The exception is that the paths of free particles do not necessarily correspond to geodesics in the map. It is seen, however, that the free paths can still be obtained from a variational equation, *i.e.* the system satisfies the principle of least action. This does not follow from the cosmological principle alone, but only when the principle of symmetry is added.

It is fully realised that the formulation of the principles taken as our starting point requires a logical background not discussed in this paper. We are at present concerned only with the consequences of these principles. If it is found that these agree with observational results and with the existing laws of matter and motion in the universe, it is then the time for us to seek a set of axioms that will enable us to formulate our principles in the desired logical manner. From the logical point of view, the most unsatisfactory part of the present paper is concerned with the hypothesis that an observer can make instantaneous angle measurements, and that these are Euclidean angles, *i.e.* they can be regarded as determined by a second order tensor. Very fortunately, it appears that the rotation group given by the principle of symmetry leads to a congruency relation similar to that implied by the angle hypothesis, so that the hypothesis does not introduce any further restrictions. The important problem of investigating the concept of angle still remains, however, and it is hoped that this will be examined before long.

Since the results given in the present paper were first announced, a series of three papers by H. P. Robertson has appeared* on the subject. Working from the cosmological principle, Robertson derives the invariant quadratic by appealing to the Helmholtz-Lie theorem, and then obtains the free path equations, in agreement with our results. It is hoped that a detailed comparison of our methods and results with those of Robertson will be given in a future paper.

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* *Astrophys. J.*, 82 (1935), 284–301; 83 (1936), 187–201, 257–271.

The technique here used is that of continuous groups of transformations, and in § 2 will be found a summary of the more important formulae and results required later. Sufficient details are given throughout the paper to enable the reader not familiar with group theory to follow the argument, and all that is required is a knowledge of the tensor calculus and the elements of Riemannian geometry.

2. *Continuous groups of transformations.*

Since this work is largely concerned with the application of group theory, we shall give here the standard formulae required later. Any result quoted without proof may be found in Eisenhart's book* on the subject.

An r-parameter class of transformations in n variables may be written

$$x'^i = f^i(x, a), \tag{1}$$

where x^i $(i = 1, 2, \ldots, n)$ are the variables, and a^α $(\alpha = 1, 2, \ldots, r)$ are the parameters. Denoting by T_a the transformation given by the set of parametric values (a), etc., then the class of transformations (1) form a *group* under the following conditions:

(i) If T_a, T_b are any two transformations of the class, then the transformation $T_b T_a$ belongs to the class;

(ii) $T_c(T_b T_a) = (T_c T_b) T_a$;

(iii) The class includes the identity transformation $x'^i = x^i$;

(iv) If T_a is a transformation of the class, then the reciprocal transformation T_a^{-1} belongs to the class.

It can be shown that, if the transformations (1) form a group, the functions $f^i(x, a)$ satisfy equations of the form

$$\frac{\partial f^i}{\partial a^\alpha} = \xi_\beta{}^i(x') A_\alpha{}^\beta(a), \qquad |A_\alpha{}^\beta| \neq 0 \tag{2}$$

$$(i = 1, 2, \ldots, n; \quad \alpha, \beta = 1, 2, \ldots, r).$$

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* L. P. Eisenhart, *Continuous groups of transformations* (1933). This work will be quoted as C.G.T.

It is convenient to choose the parameters so that the identity transformation is given by $a^\alpha = 0$, and so that $A_\alpha{}^\beta(0) = \delta_\alpha{}^\beta$, whence

$$f^i(x, 0) = x^i, \qquad \left(\frac{\partial f^i}{\partial a^\alpha}\right)_{a=0} = \xi_\alpha{}^i(x). \tag{3}$$

The *infinitesimal group* is now derived from (1) by letting the parameters a^α take small values and neglecting a^2. From (1) and (3), the transformations of the infinitesimal group are given by

$$x'^i = x^i + \xi_\alpha{}^i(x)\, a^\alpha. \tag{4}$$

It can be shown that the *group vectors* $\xi_\alpha{}^i(x)$ occurring in (2) satisfy equations of the form

$$\xi_\alpha{}^j \xi^i_{\beta.j} - \xi_\beta{}^j \xi^i_{\alpha.j} = C^\gamma_{\alpha\beta} \xi_\gamma{}^i \quad (\xi^i_{\alpha.j} = \partial \xi_\alpha{}^i / \partial x^j), \tag{5}$$

where the C's are constants, the *constants of structure*, satisfying

$$C^\gamma_{\alpha\beta} + C^\gamma_{\beta\alpha} = 0, \tag{6}$$

$$C^\delta_{\alpha\epsilon} C^\epsilon_{\beta\gamma} + C^\delta_{\beta\epsilon} C^\epsilon_{\gamma\alpha} + C^\delta_{\gamma\epsilon} C^\epsilon_{\alpha\beta} = 0. \tag{7}$$

The group (1) is *simply transitive* if $r = n$, and if, for any two points (x), (x'), parameters a^α can be found such that (1) holds. In this case it is seen that $|\xi_\alpha{}^i| \not\equiv 0$, and we can define covariant vectors $\xi_i{}^\alpha$ by means of the equations

$$\xi_\alpha{}^i \xi_j{}^\alpha = \delta_j{}^i, \quad \xi_\alpha{}^i \xi_i{}^\beta = \delta_\alpha{}^\beta.$$

Writing
$$\Lambda^i_{jk} = -\xi_k{}^\alpha \xi^i_{\alpha.j} = \xi_\alpha{}^i \xi^\alpha_{k.j}, \tag{8}$$

we see that the quantities Λ^i_{jk} transform as connection coefficients in transport theory for changes in the variables. Writing

$$\Omega^i_{jk} = \tfrac{1}{2}(\Lambda^i_{jk} - \Lambda^i_{kj}), \tag{9}$$

then, from (5) and (8),

$$C^\alpha_{\beta\gamma} = -2\xi_i{}^\alpha \xi_\beta{}^j \xi_\gamma{}^k \Omega^i_{jk}, \quad \Omega^i_{jk} = -\tfrac{1}{2}\xi_\alpha{}^i \xi_j{}^\beta \xi_k{}^\gamma C^\alpha_{\beta\gamma}, \tag{10}$$

whence, from (7),

$$\Omega^l_{im} \Omega^m_{jk} + \Omega^l_{jm} \Omega^m_{ki} + \Omega^l_{km} \Omega^m_{ij} = 0. \tag{11}$$

From (8), we find

$$\Lambda^i_{jl.k} - \Lambda^i_{kl.j} + \Lambda^i_{km} \Lambda^m_{jl} - \Lambda^i_{jm} \Lambda^m_{kl} = 0, \tag{12}$$

where the full point denotes partial differentiation, and from (12), (9), (10), and (11),

$$\Lambda^{i}_{lj\,.\,k}-\Lambda^{i}_{lk\,.\,j}+\Lambda^{i}_{mk}\Lambda^{m}_{lj}-\Lambda^{i}_{mj}\Lambda^{m}_{lk}=0. \tag{13}$$

Equations (13) are exactly the conditions of integrability of the set of differential equations

$$\zeta^{i}_{.j}+\Lambda^{i}_{kj}\zeta^{k}=0. \tag{14}$$

These equations therefore admit n independent vector solutions ${\zeta_a}^i$ $(a=1, 2, \ldots, n)$, and the general solution is of the form $\alpha^a {\zeta_a}^i$, where the α's are arbitrary constants. It can be shown that the vectors ${\zeta_a}^i$ are the group-vectors of another simply transitive group, the *reciprocal group*. For this group we have

$$\bar{\Lambda}^{i}_{jk}=-{\zeta_k}^{a}\zeta^{i}_{a\,.\,j}=\Lambda^{i}_{kj}, \tag{15}$$

where ${\zeta_i}^a$ are the covariant vectors defined by the contravariant set ${\zeta_a}^i$.

Hence $$\bar{\Omega}^{i}_{jk}=\tfrac{1}{2}(\bar{\Lambda}^{i}_{jk}-\bar{\Lambda}^{l}_{kj})=-\Omega^{i}_{jk}, \tag{16}$$

whence the constants of structure are

$$\bar{C}^{a}_{bc}=2{\zeta_i}^{a}{\zeta_b}^{j}{\zeta_c}^{k}\,\Omega^{i}_{jk}. \tag{17}$$

That the expression on the right in (17) is constant can easily be verified from (10), (8), and (14).

3. *The cosmological principle and the group of transformations.*

Following the course adopted by Milne*, we consider a "world" consisting of matter in motion idealized as a system of fundamental particles so that each world-event is occupied by one particle, and one only. Attached to each fundamental particle is an observer equipped with a theodolite and an arbitrary clock, and all observations are made by these observers. Out of all possible material systems, we now consider only those which satisfy the *cosmological principle*, *i.e.* our system is such that the totality of observations that any observer can make on the system is identical with the totality of observations that any other observer can make on the system.

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* E. A. Milne, *Relativity, gravitation, and world-structure* (1935). This work will be quoted as W.S.

Milne has shown that any observer can adopt a definable convention for assigning four coordinates to each event by means of theodolite and clock measurements made at himself; Milne considered in particular a convention for assigning one "temporal" and three "spatial" coordinates to an event, but we are now only concerned with the fact that a definable convention does exist. Consider a particular observer A equipped with a clock in accordance with which he assigns to each event coordinates x^r $(r = 0, 1, 2, 3)$. Then, from the cosmological principle, any other observer A' can choose a clock in such a way that, by the same convention as A, he adopts a coordinate system identical with that of A, so that if there is an occurrence at the event (x) in A's system, there is an exactly similar occurrence at the event (x) in the system of A'. In this way each observer can provide himself with a clock and can choose a coordinate system so that for any two observers, their clocks are "similar" and their coordinate systems identical. Their clocks are "similar" in the sense described by Milne, for an observer can theoretically observe the clocks associated with other observers, and from the cosmological principle it follows that each pair of observers are equivalent in the sense $A \equiv B$. It may be assumed, therefore, that the observers' clocks are rated and synchronized with A's clock and that each clock records time τ, so that the world-pictures seen by all the observers, each at his own time τ, are identical. Milne has shown explicitly how this synchronization can be carried out by means of signal experiments in the case when each pair of observers intersect. Later (§ 17) we shall see that this is the case under certain hypotheses.

Since there is a four-fold infinity of events, we may set up a (1, 1) correspondence between events and points of a four-dimensional manifold M_4. Each fundamental particle occupies a single infinity of events during its history and therefore corresponds to a curve, or *world-line*, in M_4. There is a world-line corresponding to each particle, and, since each event is occupied by one particle, there is just one fundamental world-line passing through each point of M_4. The points of each world-line may be specified by the times recorded by the clock associated with the corresponding particle, whence τ may be regarded as a parameter for each world-line. This parameter defines a family of three-dimensional surfaces S_τ, each one of which meets each world-line once and once only; the surface S_τ corresponds to the continuum of events, one in the history of each particle, at which the clocks record the particular time τ, and it may be assumed that the different observers' clocks are so synchronized that the surfaces S_τ are continuous. Thus τ, which may be regarded as a "cosmic time", serves as a coordinate of M_4.

An observer A can assign coordinate numbers x^i $(i=1, 2, 3)$ to each fundamental particle from observations carried out at a particular time τ_0, and τ, x^i may then be regarded as A's coordinate system, the coordinates of an event (or point) E being determined uniquely by the surface S_τ through E and by the world-line through E. With this coordinate system, the surfaces S_τ are the surfaces $\tau=$ constant, and the world-lines are the curves $x^i=$ constant. Each other observer A' can choose a similar coordinate system by means of observations at time τ_0 similar to those carried out by A, so that there is a coordinate system for M_4 associated with each particle-observer. These systems have one coordinate in common, for each observer will assign the same coordinate τ to an event.

Now consider two observers A, A' and any fundamental particle P. Then A assigns to P coordinates x^i, and A' assigns to P coordinates x'^i, in general different from x^i. Hence, to each set x^i there corresponds a set x'^i, so that there is a transformation from one set to the other of the form $x'^i=f^i(x)$, this, with $\tau'=\tau$, being the transformation from A's coordinate system to that of A'. Keeping A fixed and varying A', we get a class of transformations of the form

$$x'^i=f^i(x, a) \quad (i=1, 2, 3), \tag{18}$$

where a^α $(\alpha=1, 2, 3)$ are parameters specifying A'. Similarly, there is a class of transformations connecting the coordinate systems of another observer A_1 and any observer A_1', and from the cosmological principle and from the fact that the coordinate systems of all observers are defined in the same way, it follows that this second class of transformations must be identical with the class (18). This must be true for each observer A_1, whence the class (18) must form a simply transitive group of transformations, in accordance with § 2. It is easily verified that the conditions of § 2 must be satisfied, for a transformation of the class (18) is simply equivalent to passing from one observer to another.

We may regard x^i as the coordinates of a three-dimensional manifold M_3 (*e.g.*, any one of the surfaces S_τ), in which case the transformations (18) arise as a change of base, or coordinate system, the point x'^i being the same geometric point x^i. It is more convenient, however, to adopt the dual interpretation of a transformation, so that x^i, x'^i are different points referred to the same coordinate system, a transformation now being equivalent to a displacement of each point in the space M_3. This is permissible provided results are interpreted in accordance with the change of base conception.

We shall now be dealing with a simply transitive group of tranformations of the form described in § 2 with $n=3$, and the notation of § 2 will be used without further description. Suffixes i, j, k, l, m will refer to the coordinates x and to the components of vectors, tensors, connections, etc., in this coordinate system; $\alpha, \beta, \gamma, \delta, \epsilon$ will refer to the parameters a^α and the corresponding group vectors $\xi_\alpha^{\ i}$, $\xi_i^{\ \alpha}$, etc.; and a, b, c, d will refer to the reciprocal group vectors $\zeta_a^{\ i}$, $\zeta_i^{\ a}$, etc. All suffixes will take the values 1, 2, 3.

4. *Distribution of fundamental particles.*

Since the fundamental particles are specified by coordinates x^i relative to an observer A, the distribution relative to A will be given by an expression of the form

$$dN = \psi(x)\, dx^1\, dx^2\, dx^3, \tag{19}$$

dN being the number of particles with coordinates in the range $(x, x+dx)$. From the cosmological principle, any other observer A' will observe a similar distribution given by the expression (19) in the coordinates of A'. Hence, since A and A' are observing the same system of particles from different viewpoints, the expression (19) must be invariant under the transformation from A to A', *i.e.*

$$\psi(x')\, dx'^1\, dx'^2\, dx'^3 \equiv \psi(x)\, dx^1\, dx^2\, dx^3, \tag{20}$$

where $x'^i = f^i(x, a)$, the identity holding for all values of the parameters a^α. It is well known that in dealing with invariants under a group, it is sufficient to consider the infinitesimal group. Since in (4) the a's are small, we have

$$\psi(x') = \psi(x) + \frac{\partial\psi}{\partial x^i}\, \xi_\alpha^{\ i}(x)\, a^\alpha$$

and
$$\frac{\partial(x'^1, x'^2, x'^3)}{\partial(x', x^2, x^3)} = |\delta_j^{\ i} + \xi^i_{\alpha.j}\, a^\alpha| = 1 + \xi^j_{\alpha.j}\, a^\alpha.$$

Hence, (20) becomes

$$\left(\frac{\partial\psi}{\partial x^i}\, \xi_\alpha^{\ i} + \psi\, \xi^j_{\alpha.j}\right) a^\alpha = 0.$$

Since this equation must hold for all a's, it follows that ψ must satisfy the equations

$$\frac{\partial\psi}{\partial x^i}\, \xi_\alpha^{\ i} + \psi \xi^j_{\alpha.j} = 0 \quad (\alpha = 1, 2, 3),$$

which, from (8), are equivalent to

$$\frac{\partial\psi}{\partial x^i} - \psi\Lambda^j_{ji} = 0 \quad (i = 1, 2, 3). \tag{21}$$

These equations serve to determine ψ when the group is known and are completely integrable, for, on contracting (13) by writing $i = l$, we have $\Lambda^l_{lj.k} = \Lambda^l_{lk.j}$. From the form of (21) it follows that, when the group is known, ψ is completely determinate except for an arbitrary multiplicative constant, *i.e.* the distribution of fundamental particles can be found to within an arbitrary constant when the group is known. The solution of equations (21) can be expressed in terms of the solutions of equations (14). For, if $\xi_a^{\;i}$ are three independent vector solutions of (14), and if $\Delta = |\zeta_i^a|$, where ζ_i^a are the corresponding covariant vectors, then, from (14),

$$\frac{1}{\Delta}\,\frac{\partial\Delta}{\partial x^i} = \zeta_a^{\;j}\,\zeta^a_{j.i} = -\zeta_j^{\;a}\,\zeta^j_{a.i} = \Lambda^j_{ji},$$

whence $\psi = B\Delta$, where B is an arbitrary constant. The distribution (19) can therefore be written

$$dN = B\,|\zeta_j^a|\,dx^1\,dx^2\,dx^3. \tag{22}$$

5. *Free paths.*

Now that the group of transformations from one observer to another has been introduced, the cosmological principle may be further applied to consider in its kinematical aspect the "gravitational field" of the system of fundamental particles. It may be assumed that, at any desired event, there is a free test-particle moving in any direction and with any velocity within certain bounds, and in accordance with experience we assume that the motion of the particle throughout its history is determined by these initial conditions. It is to be expected that the behaviour of free test-particles in the presence of a material system depends upon the distribution of matter, and may therefore be considered to constitute the gravitational field of the material system, in accordance with the results of classical mechanics.

In the manifold M_4 corresponding to the continuum of events, a free test-particle will correspond to a curve as in the case of a fundamental particle; such a curve we shall call a *free path*. Since the behaviour of a test-particle is determined by the conditions of motion at any one event in its history, a free path in M_4 is determined by the direction of the tangent vector at any point of the path. The set of paths may, therefore, be considered as the solutions of a set of second order differential equations, as in the geometrical theory of paths. Hence, regarding the coordinate τ as an independent variable for the description of a path, the paths may be regarded as the solutions of a set of differential equations of the form

$$\frac{dx^i}{d\tau} = p^i, \qquad \frac{dp^i}{d\tau} = G^i(\tau, x, p) \quad (i = 1, 2, 3). \tag{23}$$

Here, p^i are the components of coordinate-velocity, and $dp^i/d\tau$ the components of coordinate-acceleration. Equations (23) simply state that the acceleration of a free particle at any event is determined by the position and velocity of the particle at this event.

If the range of velocities of free particles is restricted at each event, as in the special theory of relativity, another condition, of the form

$$\phi(\tau, x, p) < 0, \tag{24}$$

must be satisfied at each point of every path, the free paths then being the solutions of (23) satisfying the inequality (24) at every point. Restriction of this form will be considered later when light paths are introduced.

It is convenient to adopt τ as independent variable, for by this means we obtain a representation of the paths in M_4 by paths in the three dimensional manifold M_3 defined by the coordinate system (x). These paths in M_3 do not, of course, form a geometric set as do the paths in M_4, for there are an infinite number of paths connecting two given points of M_3. This representation is convenient because the coordinate τ is invariant under transformations from one observer to another, so that, with τ as independent variable, results can be expressed concisely in tensor form with regard to the coordinate system of M_3.

Since τ, x^i constitute the coordinate system of an observer A, equations (23) may be regarded as A's description of the set of free paths and can be considered as constructed out of observations made by A. From the cosmological principle, it follows that any other observer A' will obtain an identical set of equations for his description of the set of paths in terms of his own coordinate system. Hence, since A and A' are describing the same set of paths, each in his own coordinate system, equations (23) must be invariant under the transformation from A to A', *i.e.* equations (23) must imply

$$\frac{dx'^i}{d\tau} = p'^i, \qquad \frac{dp'^i}{d\tau} = G^i(\tau, x', p'), \tag{25}$$

where $x'^i = f^i(x, a)$, for all values of the parameters a^α. It is again sufficient to consider only the infinitesimal group, whence, from (4),

$$p'^i = p^i + \xi^i_{\alpha.j}\, p^j a^\alpha,$$

and
$$\frac{dp'^i}{d\tau} = \frac{dp^i}{d\tau} + \left(\xi^i_{\alpha.j}\frac{dp^j}{d\tau} + \xi^i_{\alpha.jl}p^j p^l\right) a^\alpha.$$

Also, since the a's are now small,

$$G^i(\tau, x', p') = G^i + \left(\frac{\partial G^i}{\partial x^k}\xi_\alpha{}^k + \frac{\partial G^i}{\partial p^l}\xi^l_{\alpha.j}p^j\right) a^\alpha.$$

Substituting in (25), using (23), and remembering that the resulting equations must hold for all values a^α, we get

$$\frac{\partial G^i}{\partial x^l}\xi_\alpha{}^l + \frac{\partial G^i}{\partial p^l}\xi^l_{\alpha.j}p^j - G^j\xi^i_{\alpha.j} - \xi^i_{\alpha.jl}p^j p^l = 0 \quad (i, \alpha = 1, 2, 3). \tag{26}$$

Multiplying by $\xi_k{}^\alpha$ and summing for α, these equations are, from (8), equivalent to

$$\frac{\partial G^i}{\partial x^k} - \frac{\partial G^i}{\partial p^l}\Lambda^l_{jk}p^j + G^j\Lambda^i_{jk} + (\Lambda^i_{jk.l} - \Lambda^i_{jm}\Lambda^m_{lk})p^j p^l = 0. \tag{27}$$

The functions G^i are not components of a vector in M_3, but we can form a vector by using the quantities Λ^i_{jk} which behave as connection coefficients in M_3. We should, therefore, expect equations (27) to be simplified by the substitution

$$g^i(\tau, x, p) = G^i + \Lambda^i_{jk}p^j p^k. \tag{28}$$

By means of (13), it is easily verified that equations (27) become

$$\frac{\partial g^i}{\partial x^k} - \frac{\partial g^i}{\partial p^l}\Lambda^l_{jk}p^j + g^j\Lambda^i_{jk} = 0. \tag{29}$$

The vectors $\zeta_a{}^i$, $\zeta_i{}^a$ have already been introduced. Now make the substitution

$$g^a(\tau, x, p) = \zeta_i{}^a g^i \quad (g^i = \zeta_a{}^i g^a). \tag{30}$$

Then, from (14), we find that equations (29) become simply

$$\frac{\partial g^a}{\partial x^k} - \frac{\partial g^a}{\partial p^l}\Lambda^l_{jk}p^j = 0 \quad (a, k = 1, 2, 3). \tag{31}$$

Each solution of these equations for g^a is an integral of the equations

$$dp^i + \Lambda^i_{jk}p^j dx^k = 0, \tag{31'}$$

the general solution of which is, by comparison with (14), $p^i = \zeta_a{}^i \alpha^a$, where $\zeta_a{}^i$ are three independent vector solutions of (14) and the α's are arbitrary constants. The integrals of (31′) are, therefore, $\zeta_i{}^a p^i = \text{constant}$, whence the functions g^a involve the x's and p's in the combinations $\zeta_i{}^a p^i$. Writing

$$q^a = \zeta_i{}^a p^i, \tag{32}$$

it follows that the general solution of equations (31) is of the form

$$g^a = g^a(\tau, q^1, q^2, q^3). \tag{33}$$

Hence in order that the cosmological principle shall be satisfied, the free paths must be given by equations of the form

$$\frac{dx^i}{d\tau} = p^i, \quad \frac{dp^i}{d\tau} + \Lambda^i_{jk} p^j p^k = \zeta_a{}^i g^a(\tau, q) \quad (q^a = \zeta_i{}^a p^i). \tag{34}$$

The functions g^a of arguments τ, q^1, q^2, q^3 are not further determined by the restrictions already imposed upon the material system.

We observe that the existence of free paths in accordance with the cosmological principle does not impose any restrictions upon the group of transformations from one observer to another. Further, since the distribution of fundamental particles is determined by the group to within the arbitrary constant B, it is probable that there are other arguments of a fundamental nature that serve to determine the functions g^a in (34) in terms of the group and the constant B. For example, there is Milne's hypothesis that no constant having physical dimensions should occur except constants chosen arbitrarily by the observers. There is also the tentative argument put forward by Milne in a recent paper, where systems of fundamental particles are held to be essentially distinct from systems of elementary particles, and where it is suggested that the solution of the Boltzmann equations relating the distribution and motion of elementary particles should be indeterminate in the case of a system of fundamental particles. These hypotheses will be considered later.

6. *The principle of least action.*

The free paths satisfy a principle of least action if a function $W(\tau, x, p)$ can be found so that the equations (23) are reproduced by the variational equation

$$\delta \int W \, d\tau = 0.$$

Since the paths are required to be invariant under the group of transformations and since τ is invariant, it at once follows that W must be invariant in form under the group, *i.e.*

$$W(\tau,\, x',\, p') \equiv W(\tau,\, x,\, p),$$

where $x'^i = f^i(x,\, a)$, for all values a^α. Considering the infinitesimal group, we have

$$x'^i = x^i + \xi_\alpha{}^i a^\alpha, \quad p'^i = p^i + \xi^{i}_{\alpha . j} p^j a^\alpha,$$

whence W must satisfy the equations

$$\frac{\partial W}{\partial x^k}\, \xi_\alpha{}^k + \frac{\partial W}{\partial p^i}\, \xi^{i}_{\alpha . j} p^j = 0 \quad (\alpha = 1,\, 2,\, 3).$$

These are equivalent to

$$\frac{\partial W}{\partial x^k} - \frac{\partial W}{\partial p^i}\, \Lambda^i_{jk} p^j = 0 \quad (k = 1,\, 2,\, 3),$$

which is of the form (31). Hence, W is an integral of equations (31′), *i.e.* involves the x's and p's in the combinations $\zeta_i{}^a p^i$. The general solution is therefore of the form

$$W = W(\tau,\, q), \quad q^a = \zeta_i{}^a p^i,$$

so that, if the free paths satisfy the principle of least action, they are given by an equation of the form

$$\delta \int W(\tau,\, q)\, d\tau = 0. \tag{35}$$

Writing $W_a = \partial W / \partial q^a$, $W_{ab} = \partial^2 W / \partial q^a \partial q^b$, and defining W^{ab} by the equations $W^{ab} W_{ac} = \delta_c{}^b$, it can be verified that the variational equation (35) leads to equations (34), where

$$g^a = -W^{ab}\left(W_c\, \bar{C}^c_{bd}\, q^d + \frac{\partial W_b}{\partial \tau}\right), \tag{36}$$

the constants $\bar{C}^c_{bd}$ being given by (17). Thus g^a is a function of τ, q^1, q^2, q^3, as we should expect. We observe that the principle of least action restricts the possible range of solutions, for this principle is satisfied only when the functions $g^a(\tau,\, q)$ are such that a single function $W(\tau,\, q)$ can be found for which equations (36) are satisfied. Thus, the cosmological principle alone does not imply the principle of least action.

7. *Absolute parallelism in M_3.*

It is seen that the vectors $\zeta_a{}^i$ play an important part in this work. From (14) it follows that these vectors constitute absolutely parallel vector fields with respect to the connection Λ^i_{jk} in the manifold M_3, the vector fields being determined by an arbitrary triad $(\zeta_a{}^i)_O$ at any point O. The triad at each point of M_3 may be considered as a local base, and geometrical elements at a point P may be referred to the base at P. Thus if λ^i are the coordinate components of a contravariant vector at P, the base components are $\lambda^a = \lambda^i(\zeta_i{}^a)_P$, and, similarly, $\lambda_a = \lambda_i(\zeta_a{}^i)_P$ are the base components of a covariant vector λ_i. Thus the quantities $q^a = \zeta_i{}^a p^i$ are the base components of the velocity vector p^i. The components of acceleration $dp^i/d\tau$ do not form a coordinate vector, but we may define the *relative acceleration* to be given by the base vector $dq^a/d\tau$. It is easily verified that the coordinate components of this vector are

$$\frac{dp^i}{d\tau} + \Lambda^i_{jk} p^j p^k,$$

whence, from (34), the functions g^a are simply the base components of the relative acceleration vector.

8. *Principle of symmetry, and the rotation group.*

We shall now consider those material systems which, in addition to the cosmological principle, satisfy the principle of symmetry, *i.e.* are such that each fundamental observer sees himself to be at a centre of spherical symmetry. Thus we may assume that there are ∞^3 observers attached to each fundamental particle, and these observers are equivalent as were the observers attached to different fundamental particles.

Since it is assumed that the cosmological principle is satisfied, it is only necessary to consider the relations between the different observers attached to one particular fundamental particle P_0. If A is the original observer attached to this particle and τ, x^i the coordinate system of A as described above, then the particle P_0 corresponds to a point O, coordinates $x_0{}^i$, in the manifold M_3 with coordinate system (x). Now, if A' is any other observer attached to the particle P_0, then, since A and A' are equivalent, A' has a coordinate system τ, x^i similar to that of A, the event (τ, x^i) relative to A' being exactly similar to the event (τ, x^i) relative to A. If E is the event (τ, x^i) relative to A, then A' will assign coordinates τ', x'^i to E, where $\tau' = \tau$ but x'^i is not necessarily equal to x^i. Thus there is a transformation of the form $\tau' = \tau$, $x'^i = \phi^i(x)$ relating the coordinate systems of A and A', and by varying A' we get a class of transformations

$$x'^{i}=\phi^{i}(x,\,a), \tag{37}$$

where a^{λ} $(\lambda=4,\,5,\,6)$ are parameters determining A'. As in § 3, this class of transformations will form a 3-parameter group, with variables x^{i} $(i=1,\,2,\,3)$ and parameters a^{λ} $(\lambda=4,\,5,\,6)$. This group may be regarded as a group of rotations, for each transformation is equivalent to a rotation of the manifold M_3 about the point O as centre.

The group of rotations (37) is not transitive but is *intransitive*, for, under any rotation of the group, the point O remains invariant, and any other point P is displaced to a point of a certain surface through P. There is a family of such surfaces, one through each point of M_3, and these are called the *invariant varieties** for the group; they correspond to the family of invariant spheres for the group of Euclidean rotations in 3-space. If the invariant varieties are given by the equations $F(x)=\text{constant}$, then, under any transformation of the group (37),

$$F(x')\equiv F(x). \tag{38}$$

Since we are assuming that there is complete spherical symmetry, it is evident that these invariant varieties must be closed surfaces surrounding the point O; the point O is itself a member of the family of surfaces.

In the notation of § 2, the group of rotations gives rise to three group-vectors $\xi_{\lambda}{}^{i}(x)$ $(\lambda=4,\,5,\,6)$, and, choosing the parameters a^{λ} as in § 2, the infinitesimal rotation group is given by

$$x'^{i}=x^{i}+\xi_{\lambda}{}^{i}(x)\,a^{\lambda}, \tag{39}$$

where the a's are small. Since the point O is invariant under each rotation, we must have $x'^{i}=x_0^{i}$ when $x^{i}=x_0^{i}$, whence

$$\xi_{\lambda}{}^{i}(x_0)=0. \tag{40}$$

Further, since the group is intransitive,

$$|\xi_{\lambda}{}^{i}|\equiv 0, \tag{41}$$

for otherwise a function $F(x)$ satisfying (38) could not exist. Thus we cannot form covariant vectors $\xi_{i}{}^{\lambda}$ as for the transitive group. The constants of structure $C^{\nu}_{\lambda\mu}$ $(\lambda,\,\mu,\,\nu=4,\,5,\,6)$ for the rotation group are given by equations of the form (5), *i.e.*

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* C.G.T., § 19.

$$\xi_\lambda^{\,j}\,\xi^i_{\mu\,.\,j}-\xi_\mu^{\,j}\,\xi^i_{\lambda\,.\,j}=C^\nu_{\lambda\mu}\,\xi_\nu^{\,i} \tag{42}$$

and satisfy relations of the form (6) and (7).

There are ∞^3 observers attached to any other fundamental particle, and, from the cosmological principle, the coordinate systems of these observers will be related by a rotation group identical with that considered above. Thus there are ∞^6 observers throughout the system, each with his own coordinate system, and it is at once seen that the class of transformations in the coordinates (x), relating one observer and any other, form a 6-parameter group G_6. The simply transitive group G_3 and the rotation group R_3 are sub-groups of G_6, for, if T_a, T_b both belong either to G_3 or to R_3, then $T_c=T_b\,T_a$ belongs to the same group. Choosing the parameters of G_6 to be a^α, a^λ, where a^α $(\alpha=1, 2, 3)$ are the parameters of G_3 and a^λ $(\lambda=4, 5, 6)$ are the parameters of R_3, we see that the group-vectors of G_6 are the vectors $\xi_\alpha^{\,i}$, $\xi_\lambda^{\,i}$ considered above.

The constants of structure for G_6 are given by

$$\xi_\theta^{\,j}\,\xi^i_{\phi\,.\,j}-\xi_\phi^{\,j}\,\xi^i_{\theta\,.\,j}=C'^\psi_{\theta\phi}\,\xi_\psi^{\,i}, \tag{43}$$

where θ, ϕ, ψ take the values 1, 2, ..., 6. It is known that the vectors of a group must be linearly independent as far as constant coefficients are concerned, whence, since equations of the form (43) are already satisfied separately by the vectors $\xi_\alpha^{\,i}$, $\zeta_\lambda^{\,i}$ and the corresponding constants $C^\gamma_{\alpha\beta}$, $C^\nu_{\lambda\mu}$, it follows that

$$C'^\gamma_{\alpha\beta}=C^\gamma_{\alpha\beta},\quad C'^\nu_{\lambda\mu}=C^\nu_{\lambda\mu},\quad C'^\nu_{\alpha\beta}=0=C'^\gamma_{\lambda\mu}$$

$$(\alpha,\beta,\gamma=1, 2, 3;\ \lambda,\mu,\nu=4, 5, 6).$$

Thus, the constants of structure for G_6 are the existing constants $C^\gamma_{\alpha\beta}$, $C^\nu_{\lambda\mu}$ together with new constants $C^\gamma_{\alpha\mu}=-C^\gamma_{\mu\alpha}$, $C^\nu_{\alpha\mu}=-C^\nu_{\mu\alpha}$; the remaining constants $C^\nu_{\alpha\beta}$, $C^\gamma_{\lambda\mu}$ all vanish. The constants $C^\psi_{\theta\phi}$ $(\theta,\phi,\psi=1, 2, ..., 6)$ for G_6 must satisfy relations of the form (6) and (7).

In the following work, we shall follow the notation that suffixes α, β, γ take values 1, 2, 3 and refer to G_3, suffixes λ, μ, ν take values 4, 5, 6 and refer to R_3, and suffixes θ, ϕ, ψ take values 1, 2, ..., 6 and refer to G_6.

9. *The measurement of angle.*

It has been assumed that each observer is equipped with instruments with which he can measure Euclidean angles at himself, an observed angle being regarded as the angle subtended at the observer by two neighbouring fundamental particles. If the particle P_0 corresponds to the point O, coordinates x_0^i, in the manifold M_3 defined by the coordinate system (x) of an observer A attached to P_0, this assumption implies the existence of a tensor a_{ij} at the point O of M_3, the components of which may involve τ. Then, if P_1, P_2 are neighbouring fundamental particles with coordinates $x_0^i+\delta_1 x^i$, $x_0^i+\delta_2 x^i$ respectively, where $\delta_1 x$, $\delta_2 x$ are small, the angle subtended at P_0 by P_1, P_2 at time τ is given by

$$\cos\theta = \frac{a_{ij}(\tau)\,\delta_1 x^i\,\delta_2 x^j}{(a_{ij}\,\delta_1 x^i\,\delta_1 x^j)^{\frac{1}{2}}\,(a_{ij}\,\delta_2 x^i\,\delta_2 x^j)^{\frac{1}{2}}}. \tag{44}$$

It is, of course, understood that the tensor a_{ij} at each epoch must be such that the quadratic $a_{ij} X^i X^j$ is positive definite.

We observe that the above expression remains unaltered when a_{ij} is replaced by λa_{ij}. The tensor a_{ij} at each epoch is, therefore, determinate only to within an arbitrary multiplier.

We shall now show that, if the principle of symmetry is satisfied, the tensor a_{ij} for any observer can involve τ only in a multiplier, so that a suitable multiplier can be found to make a_{ij} independent of τ. Let a_{ij} be the components at time τ of the tensor for the observer A, and a'_{ij} the components at time $\tau+\delta\tau$, where $\delta\tau$ is small. Then the equations

$$(a'_{ij}-\rho a_{ij})\,\mu^j = 0, \tag{45}$$

where ρ satisfies the cubic equation

$$|a'_{ij}-\rho a_{ij}| = 0, \tag{46}$$

determine the principal directions of the tensor a'_{ij} with respect to the tensor a_{ij}. If the roots of equation (46) are unequal, the three principal directions are unique and are mutually orthogonal with respect to both a_{ij} and a'_{ij}. If two roots of (46) are equal but unequal to the third root, equations (45) define a unique direction, and a plane orthogonal to this direction. In each of these cases the unique directions could be found by A from observation, for he could seek those pairs of neighbouring fundamental particles subtending a right angle at himself throughout the interval of time $(\tau, \tau+\delta\tau)$. Thus, unless the roots of equation (46) are all equal, there is at least one preferential direction that can be detected from observation. The existence of such a preferential direction is contrary to the principle of symmetry, whence it follows that the roots of equation (46) must all be equal, *i.e.* $a'_{ij} \propto a_{ij}$. Repeating the argument at each time τ, we see that the components a_{ij} can involve τ only in a multiplier. Hence we may assume that for each observer the components a_{ij} are constants. We conclude from this argument that the angles subtended at an observer by neighbouring fundamental particles are constant in time.

There is a similar tensor for each observer in his own coordinate system, and since the coordinate systems for all observers were assumed to be constructed from observations in the same way, the tensors a_{ij}, $\bar{a}_{ij}$ for any two observers must be such that $\bar{\theta} = \theta$, where θ is the angle given by any two vectors $\delta_1 x^i$, $\delta_2 x^i$ by (44) with the tensor a_{ij}, and $\bar{\theta}$ the angle given by the same two vectors with the tensor $\bar{a}_{ij}$. It can be verified algebraically that this is true only when $\bar{a}_{ij} \propto a_{ij}$, whence the tensors for all observers are identical.

It is evident that the tensor a_{ij} at O must be related in some way to the group of rotations about O, for if A, A' are two observers attached to the same fundamental particle P_0, and if P_1, P_2 are neighbouring fundamental particles, then A and A' will observe P_1, P_2 to subtend the same angle at P_0. Hence, if the rotation from A to A' is given by $x'^i = \phi^i(x, a)$, and if the coordinates of P_1, P_2 relative to A are $x_0{}^i + \delta_1 x^i$, $x_0{}^i + \delta_2 x^i$ respectively, then

$$\frac{a_{ij}\,\delta_1 x'^i\,\delta_1 x'^j}{(a_{ij}\,\delta_1 x'^i\,\delta_1 x'^j)^{\frac{1}{2}}\,(a_{ij}\,\delta_2 x'^i\,\delta_2 x'^j)^{\frac{1}{2}}} = \frac{a_{ij}\,\delta_1 x^i\,\delta_2 x^j}{(a_{ij}\,\delta_1 x^i\,\delta_1 x^j)^{\frac{1}{2}}\,(a_{ij}\,\delta_2 x^i\,\delta_2 x^j)^{\frac{1}{2}}},$$

where $\delta x'^i = (\partial\phi^i/\partial x^j)_0\,\delta x^j$. This relation must hold for all values of the parameters a^λ in the transformation and for all vectors $\delta_1 x^i$, $\delta_2 x^i$, whence, by considering the infinitesimal rotations, it can be verified that relations of the form

$$a_{il}(\xi^l_{\lambda . j})_0 + a_{jl}(\xi^l_{\lambda . i})_0 = B_\lambda\, a_{ij} \tag{48}$$

must hold. If a^{ij} are the components of the contravariant tensor given by the covariant tensor a_{ij}, then, on multiplying (48) through by a^{ij} and summing for i and j, we find that $B_\lambda = 2(\xi^l_{\lambda . l})_0$. We shall see that $(\xi^l_{\lambda . l})_0 = 0$, whence $B_\lambda = 0$, and we have

$$a_{il}(\xi^l_{\lambda . j})_0 + a_{jl}(\xi^l_{\lambda . i})_0 = 0. \tag{49}$$

To prove that $(\xi^l_{\lambda . l})_0 = 0$, consider the invariant varieties given by (38), and take $x^i = x_0{}^i + \delta x^i$, where δx is small. Then $x'^i = x_0{}^i + \delta x'^i$, and, on substituting in (38) and expanding to first order terms, we get

$$\left(\frac{\partial F}{\partial x^i}\right)_0 \delta x'^i = \left(\frac{\partial F}{\partial x^i}\right)_0 \delta x^i.$$

This relation shows that the plane $(\partial F/\partial x^i)_0\,\mu^i = 0$ at the point $x_0{}^i$ is invariant under all rotations, *i.e.* a direction μ^i in this plane always rotates into a direction in the same plane. This is contrary to the assumption of complete symmetry, for it must be possible to find a rotation taking any direction into any other direction. Hence F must satisfy $(\partial F/\partial x^i)_0 = 0$. Expanding therefore to second order terms and writing $b_{ij} = (\partial^2 F/\partial x^i \partial x^j)_0$, we have

$$b_{ij}\,\delta x'^i \delta x'^j = b_{ij}\,\delta x^i \delta x^j. \tag{50}$$

The argument against the first expansion does not hold here provided the quadratic $b_{ij} X^i X^j$ is definite, for then no real vector δx^i can be found to make the right-hand side of (50) vanish. Equation (50) must hold for all rotations about O and for all vectors δx^i, whence, by considering the rotations of the infinitesimal group, we find that

$$b_{il}(\xi^l_{\lambda.j})_0 + b_{jl}(\xi^l_{\lambda.i})_0 = 0.$$

Since $b_{ij} X^i X^j$ must be definite, $|b_{ij}| \neq 0$, whence the tensor b^{ij} satisfying $b^{ij} b_{jk} = \delta_k{}^i$ exists. Multiplying the above equation by b^{ij} and summing for i and j, we get $(\xi^l_{\lambda.l})_0 = 0$, as required*.

Hence the tensor a_{ij} at O is related to the group of rotations about O by means of the equations (49). The assumption concerning angle measurements does not restrict the rotation group, for we have shown that the existence of a tensor b_{ij} similar to a_{ij} is a consequence of complete symmetry.

10. *The invariant quadratic form.*

We are now in a position to prove that there is a quadratic differential form that is invariant in form under all transformations belonging to the group G_6. If the quadratic is $g_{ij}\,dx^i dx^j$, then, considering the infinitesimal group

$$x'^i = x^i + \xi_\theta{}^i(x)\,a^\theta, \tag{51}$$

we must prove that $g_{ij}(x')\,dx'^i dx'^j \equiv g_{ij}(x)\,dx^i dx^j$ for all small values of the parameters a^θ, *i.e.* that

$$(g_{ij.l}\,\xi_\theta{}^l + 2g_{il}\,\xi^l_{\theta.l})\,dx^i dx^j = 0.$$

This must hold for all vectors dx^i, whence we must prove that there is a tensor $g_{ij}(x)$ satisfying the equations

$$g_{ij.l}\,\xi_\theta{}^l + g_{il}\,\xi^l_{\theta.j} + g_{jl}\,\xi^l_{\theta.l} = 0 \tag{52}$$

$$(i, j = 1, 2, 3;\ \theta = 1, 2, \ldots, 6).$$

These are the well-known Killing equations.

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* It is probable that there is a more direct proof of these equations.

Let a_{ij} be the "angle" tensor at the point $O(x_0)$, let the vector fields $\zeta_a{}^i$, $\zeta_i{}^a$ be determined by a triad $(\zeta_a{}^i)_0$ at O as in § 7, and write

$$a_{ab} = (a_{ij}\,\zeta_a{}^i\,\zeta_b{}^j)_0, \quad g_{ij} = a_{ab}\,\zeta_i{}^a\,\zeta_j{}^b. \tag{53}$$

Then we shall prove that the tensor g_{ij} satisfies equations (52). From the equations defining $\zeta_a{}^i$, $\zeta_i{}^a$, we have

$$\zeta^i_{a\,.\,j} = -\zeta_a{}^l\Lambda^i_{lj}, \quad \zeta^a_{i\,.\,j} = \zeta_l{}^a\Lambda^l_{ij}, \tag{54}$$

whence, since the components a_{ab} are constants,

$$g_{ij\,.\,k} - g_{il}\Lambda^l_{jk} - g_{jl}\Lambda^l_{ik} = 0. \tag{55}$$

These equations are equivalent to equations (52) for $\theta = \alpha = 1, 2, 3$, from the definition of Λ^i_{jk}, and may be written

$$g_{ij/k} = 0, \tag{56}$$

where a solidus before a suffix denotes covariant differentiation with respect to the connection Λ^i_{jk}.

It remains to prove that equations (52) hold for $\theta = \lambda = 4, 5, 6$. Substituting for $g_{ij\,.\,k}$ from (55), these equations may be written

$$g_{il}\,\xi^l_{\lambda;j} + g_{jl}\,\xi^l_{\lambda;i} = 0, \tag{57}$$

where a semi-colon before a suffix denotes covariant differentiation with respect to the connection $\bar{\Lambda}^i_{jk} = \Lambda^i_{kj}$. Writing

$$T_{\lambda ij} = g_{il}\,\xi^i_{\lambda;j} + g_{jl}\,\xi^l_{\lambda;i}, \tag{58}$$

then, since $g_{ij/k} = 0$, by (56),

$$T_{\lambda ij/k} = g_{il}\,\xi^l_{\lambda;j/k} + g_{jl}\,\xi^l_{\lambda;i/k}. \tag{59}$$

Now it can easily be verified that, for any vector λ^i,

$$\lambda^i_{;j/k} = \lambda^i_{;kj} + 2\lambda^l_{;j}\,\Omega^i_{lk}, \tag{60}$$

where Ω^i_{lk} is given by (9). Now from (43), with the constants of structure found in § 7,

$$\xi_\alpha{}^j\,\xi^i_{\lambda\,.\,j} = \xi_\lambda{}^j\,\xi^i_{\alpha\,.\,j} + C^\gamma_{\alpha\lambda}\,\xi_\gamma{}^i + C^\nu_{\alpha\lambda}\,\xi_\nu{}^i,$$

whence, multiplying by $\xi_k{}^\alpha$ and summing over $\alpha = 1, 2, 3$, we find, using (8) and (9),

$$\xi^i_{\lambda;k} = -2\Omega^i_{lk}\,\xi_\lambda{}^l + C^\gamma_{\alpha\lambda}\,\xi_\gamma{}^i\,\xi_k{}^\alpha + C^\nu_{\alpha\lambda}\,\xi_\nu{}^i\,\xi_k{}^\alpha. \tag{61}$$

From (8) and (10), we have identities

$$\xi^i_{\alpha;j}=0, \quad \xi^\alpha_{i;j}=0, \quad \Omega^i_{jk;l}=0,$$

whence, differentiating (61) covariantly with respect to the connection $\bar{\Lambda}^i_{jk}=\Lambda^i_{kj}$,

$$\xi^i_{\lambda;kj}=-2\Omega^i_{lk}\,\xi^l_{\lambda;j}+C^\nu_{\alpha\lambda}\,\xi^i_{\nu;j}\,\xi_k{}^\alpha.$$

Hence, from the identity (60),

$$\xi^i_{\lambda;j/k}=C^\nu_{\alpha\lambda}\,\xi^i_{\nu;j}\,\xi_k{}^\alpha.$$

Substituting in (59) and using (58), we finally have

$$T_{\lambda ij/k}=T_{\nu ij}\,C^\nu_{\alpha\lambda}\,\xi_k{}^\alpha. \tag{62}$$

Now at the point O we have, from (40), $(\xi^i_{\lambda;j})_0=(\xi^i_{\lambda.j})_0$, and, from (53), $(g_{ij})_0=a_{ij}$. It follows from (58) and (49) that

$$(T_{\lambda ij})_0=0.$$

The identities (62) show that the derivatives $\partial T_{\lambda ij}/\partial x^k$ are linear expressions in the T's, whence, since the T's all vanish at one point, they vanish everywhere, *i.e.*

$$T_{\lambda ij}=0. \tag{63}$$

Thus, the remaining conditions (57) are satisfied, and the quadratic $g_{ij}\,dx^i\,dx^j$, with g_{ij} defined by (53), is invariant in form under all transformations belonging to the group G_6.

The above quadratic may be regarded as defining a Riemannian space V_3, with metric

$$d\sigma^2=g_{ij}\,dx^i\,dx^j, \tag{64}$$

and we have proved that this space admits a group of motions G_6. It is a well-known result that, if a Riemannian space V_n admits a group of motions G_r, where $r=n(n+1)/2$, then the space is one of constant curvature. Hence the space V_3 with metric (64) is a space of constant curvature. Since the tensor a_{ij} gives a positive definite quadratic, it follows from (53) that the metric of V_3 is positive definite.

The coordinate τ is also invariant under the transformations from one observer to another, whence any quadratic of the form

$$ds^2=L(\tau)\,d\tau^2-M(\tau)\,d\sigma^2 \quad (d\sigma^2=g_{ij}\,dx^i\,dx^j) \tag{65}$$

is invariant under the transformations from one observer to another. If the observers have other similar coordinate systems X^r $(r=0,\ 1,\ 2,\ 3)$, then the coordinates X^r are related to the coordinates τ, x^i by some transformation $X^r = X^r(\tau, x)$. It follows that if $G_{rs}\,dX^r dX^s$ is the transform of (65), then this quadratic is invariant in form under the transformations from the coordinates (X) of one observer to the similar coordinates of any other observer. Thus there is an invariant quadratic differential form for each coordinate system constructed by the observers. There is also an invariant function, for, if $\tau = F(X)$, then $F(X)$ must be invariant under the transformation from one observer to another. Examples of these invariants are given by the well-known group of Lorentz transformations, for which the invariants are

$$c^2\,dt^2 - dx^2 - dy^2 - dz^2, \quad c^2 t^2 - x^2 - y^2 - z^2.$$

A quadratic of the form (65) may be regarded as the metric of a Riemannian space V_4 with linear element ds, and this space provides a map for the representation of the material system. We shall demonstrate later the many advantages to be derived from using such a map.

11. *The distribution.*

It was found in § 4 that the distribution of fundamental particles is given by (22). For the principle of symmetry to be satisfied, this expression must be invariant under transformations belonging to the rotation group, *i.e.*, by the argument of § 4, $\Delta = |\zeta_i{}^a|$ must satisfy

$$\frac{\partial \Delta}{\partial x^i}\,\xi_\lambda{}^i + \Delta \xi^j_{\lambda.j} = 0.$$

It was shown that $\partial\Delta/\partial x^i = \Delta \Lambda^j_{ji}$, whence it is required that

$$\xi^j_{\lambda.j} + \xi_\lambda{}^i \Lambda^j_{ji} = 0.$$

The expression on the left-hand side is simply $\xi^j_{\lambda;j}$, and this vanishes identically as a consequence of (57). The required conditions are therefore satisfied.

From (53), $|g_{ij}| = |a_{ab}| \cdot |\zeta_i{}^a|^2$. Writing $g = |g_{ij}|$ and absorbing the constant $|a_{ab}|^{-\frac{1}{2}}$ in the constant B of (22), we see that the distribution is given by

$$dN = B g^{\frac{1}{2}}\,dx^1 dx^2 dx^3. \tag{66}$$

12. *The new free-path equations.*

For the principle of symmetry to be satisfied, the free paths considered in § 5 must be invariant under all transformations of the rotation group, whence the functions G^i must satisfy equations (26) with suffix λ written for α. It has already been shown that these functions must be of the form

$$G^i = -\Lambda^i_{jk} p^j p^k + \zeta_a{}^i g^a(\tau, q),$$

but now that a tensor g_{ij} has been introduced it is convenient to replace the coefficients Λ^i_{jk} by the Christoffel symbols Γ^i_{jk} derived from the tensor g_{ij}. From (55), (9), (17), and (53), we find

$$\begin{aligned} \zeta_i{}^a (\Gamma^i_{jk} - \Lambda^i_{jk}) p^j p^k &= 2\zeta_i{}^a g^{il} g_{jm} \Omega^m_{lk} p^j p^k \\ &= a^{ab} a_{cd} \bar{C}^d_{be} q^c q^e, \end{aligned}$$

so that these expressions are functions of the q's only. We may therefore write

$$G^i = -\Gamma^i_{jk} p^j p^k + \zeta_a{}^i h^a(\tau, q), \tag{67}$$

where h^a $(a = 1, 2, 3)$ are some functions of τ, q^1, q^2, q^3. Substituting in (26) and writing λ for α, we find that the equations to be satisfied by the functions h^a can be expressed in the form

$$\mu^b_{\lambda c} q^c \frac{\partial h^a}{\partial q^b} - \mu^a_{\lambda c} h^c = \zeta_i{}^a (\xi^i_{\lambda, jk} - \xi_\lambda{}^l R^i_{jkl}) p^j p^k \quad (\mu^b_{\lambda c} = \zeta_i{}^b \zeta_c{}^j \xi^i_{\lambda;j}), \tag{68}$$

where R^i_{jkl} are the components of the curvature tensor constructed from the tensor g_{ij}, and the comma denotes covariant differentiation with respect to the Christoffel symbols Γ^i_{jk}. Equations (52) for $\theta = \lambda = 4, 5, 6$ can be written

$$\xi_{\lambda i, j} + \xi_{\lambda j, i} = 0 \quad (\xi_{\lambda i} = g_{ik} \xi_\lambda{}^k),$$

and a consequence of these equations is the set of equations*

$$\xi_{\lambda i, jk} = -\xi_{\lambda l} R^l_{kij},$$

which are equivalent to

$$\xi^i_{\lambda, jk} = \xi_\lambda{}^l R^i_{jkl}.$$

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* C.G.T., 213.

Substituting in (68), we find that the equations to be satisfied by the functions h^a take the form

$$\mu^b_{\lambda c}\left(\frac{\partial h^a}{\partial q^b}\, q^c - \delta_b{}^a\, h^c\right) = 0. \tag{69}$$

It is well known that, if g_{ij} is the fundamental tensor of a Riemannian space of constant curvature, then the general solution of the set of equations

$$\frac{\partial \xi^i}{\partial x^j} = \xi^i_{.j},$$

$$g_{ij\,.\,l}\,\xi^l + g_{il}\,\xi^l_{.j} + g_{jl}\,\xi^l_{.i} = 0$$

in the quantities $(\xi^i,\ \xi^i_{.j})$ involves the maximum number of arbitrary constants, *i.e.* six in the present case. Thus the general solution can be expressed in terms of the six known solutions given by $\xi_\alpha{}^i$, $\xi_\lambda{}^i$, and there can be no linear relations of the form

$$P_i\,\xi^i_\theta + P_i{}^j\,\xi^i_{\theta\,.\,j} = 0 \quad (\theta = 1, 2, \ldots, 6)$$

other than those given by (52). These relations can be written

$$\xi^i_{\alpha;j} = 0, \quad g_{il}\,\xi^l_{\lambda;j} + g_{jl}\,\xi^l_{\lambda;i} = 0,$$

whence, if we are given that quantities $P_i{}^j$ are such that

$$P_i{}^j\,\xi^i_{\lambda;j} = 0 \quad (\lambda = 4, 5, 6),$$

it must follow that $g_{il}\,P_j{}^l = g_{jl}\,P_i{}^l$. We see therefore, from (53) and (68), that, if $P_b{}^c$ are such that $P_b{}^c\,\mu^b_{\lambda c} = 0$ $(\lambda = 4, 5, 6)$, then $a_{bd}\,P_c{}^d = a_{cd}\,P_b{}^d$. Hence, for (69) to be satisfied, the functions h^a must satisfy the equations

$$a_{bd}\left(\frac{\partial h^a}{\partial q^c}\, q^d - \delta_c{}^a\, h^d\right) = a_{cd}\left(\frac{\partial h^a}{\partial q^b}\, q^d - \delta_b{}^a\, h^d\right). \tag{70}$$

First contracting by writing $a = b$, and then multiplying equations (70) by $a_{al}\,q^l\,q^b$ and summing for a and b, we obtain relations showing that h^a must be of the form

$$h^a = q^a f(\tau, q).$$

Substituting in (69), we find that the function f must satisfy the equations

$$a_{bd}\,q^d\,\frac{\partial f}{\partial q^c} = a_{cd}\,q^d\,\frac{\partial f}{\partial q^b},$$

whence f must be a function of τ and ω, where

$$\omega = a_{bc} q^b q^c = g_{ij} p^i p^j \tag{71}$$

from (53). Hence the free paths are given by equations of the form

$$\frac{dx^i}{d\tau} = p^i, \qquad \frac{dp^i}{d\tau} + \Gamma^i_{jk} p^j p^k = p^i f(\tau, \omega) \quad (\omega = g_{ij} p^i p^j), \tag{72}$$

where g_{ij} is the fundamental tensor of a Riemannian space V_3 of constant curvature and positive definite form, and Γ^i_{jk} are the Christoffel symbols formed from g_{ij}.

13. *Least action.*

If the principle of least action is satisfied, the function $W(\tau, q)$ derived in § 6 must be invariant under all transformations belonging to the rotation group, *i.e.*, by an argument similar to that of § 6, W must satisfy the equations

$$\frac{\partial W}{\partial x^l} \xi_\lambda{}^l + \frac{\partial W}{\partial p^l} \xi^l_{\lambda . j} p^j = 0.$$

Since W is a function of τ and the q's, where $q^a = \zeta_i{}^a p^i$, these equations become

$$\mu^a_{\lambda b} q^b \frac{\partial W}{\partial q^a} = 0,$$

whence, from the argument used in § 10, W must be such that

$$a_{bc} q^c \frac{\partial W}{\partial q^a} = a_{ac} q^c \frac{\partial W}{\partial q^b}.$$

Hence W is a function of τ and ω, where ω is given by (71). Thus, if the system satisfies the principle of least action, the paths must be given by a variational equation of the form

$$\delta \int W(\tau, \omega)\, d\tau = 0 \quad (\omega = g_{ij} p^i p^j). \tag{73}$$

It can be verified that the equations for the paths given by (73) are of the form (72) with $f(\tau, \omega)$ given by

$$f = -\frac{\partial V}{\partial \tau} \bigg/ 2\omega \frac{\partial V}{\partial \omega}, \qquad V = \omega^{\frac{1}{2}} \frac{\partial W}{\partial \omega}. \tag{74}$$

If the function $f(\tau, \omega)$ in (72) is given, the system satisfies the principle of least action if a function $W(\tau, \omega)$ can be found such that the variational equation (73) reproduces the equations (72). Such a function can always be found, for, if $V(\tau, \omega)$ is an integral of the equation

$$\frac{d\omega}{d\tau} = 2\omega f(\tau, \omega), \tag{75}$$

and if $W(\tau, \omega)$ is found by integrating $\omega^{-\frac{1}{2}} V(\tau, \omega)$ with respect to ω, then, from (74), it follows that W is the function required to reproduce the given equations. Hence *the cosmological principle and the principle of symmetry together imply the principle of least action.* Thus, when these two principles are satisfied, complete generality is obtained by taking the free paths to be given by an equation of the form (73).

14. *Integration of the equations of motion.*

To see how the equations given by (72) or (73) may be integrated, we observe from the form of equations (72) that the projections of the free paths are geodesics in the Riemannian space V_3 with metric (64), *i.e.* if a free path is given by $x^i = x^i(\tau)$, then, treating τ as a parameter, the curve in V_3 given by these equations is a geodesic. The first step in the integration is, therefore, to find the geodesics of V_3 explicitly by one of the standard methods. Each geodesic may be expressed in the form $x^i = x^i(\sigma)$, where σ is the arc measured from some point of the curve, and the integration is complete if we find the relation between σ and τ.

The expression (75) for the derivative of ω along a free path is satisfied identically in consequence of equations (72). If, therefore, $V(\tau, \omega)$ is an integral of (75), or is given in terms of $W(\tau, \omega)$ by (74), it follows that V is constant along each free path, *i.e.*

$$V(\tau, \omega) = \alpha, \tag{76}$$

where α is a constant varying from path to path. Solving this equation for ω, we get an expression of the form

$$\omega = \phi(\tau, \alpha). \tag{77}$$

Now, from (64), $\omega = (d\sigma/d\tau)^2$, whence

$$\sigma = \pm \int_{\beta}^{\tau} \{\phi(\tau, \alpha)\}^{\frac{1}{2}} d\tau. \tag{78}$$

Substituting for σ in the geodesic equations $x^i = x^i(\sigma)$, we have the equations for the free paths.

Equation (78) shows that there are ∞^2 free paths corresponding to each geodesic of V_3; the alternative sign is due to the fact that the free path may or may not be described in the same sense as the corresponding geodesic in V_3. It would seem that there may be more than one expression (77) for ω resulting from (76). This, however, is not the case if we assume that there is only one free path joining two world-events, for there would be a path corresponding to each expression $\phi(\tau, a)$. We assume, therefore, that the function $f(\tau, \omega)$, or $W(\tau, \omega)$, must be such that (76) gives only one real expression for ω.

16. *Light paths.*

In the coordinate system (x) of an observer A attached to a particle $P_0(x_0)$, there will be at the point (x_0) a cone of ∞^2 directions p^i at each time τ giving the directions of incoming and outgoing light rays. There will be a similar cone at time τ for any other observer A' attached to the same particle, and, since A and A' observe the same systems of light rays, the light cone for A' can be determined from the light cone for A by the transformation from A to A'. Hence, if the light cone at time τ is given by

$$L(\tau, p) = 0, \tag{79}$$

then $L(\tau, p') \equiv L(\tau, p)$, where $p'^i = p^i + (\xi^i_{\lambda.j})_0 \, p^j a^\lambda$, whence

$$\frac{\partial L}{\partial p^i} p^j (\xi^i_{\lambda;j})_0 = 0, \tag{80}$$

since $(\xi^i_{\lambda.j})_0 = (\xi^i_{\lambda;j})_0$. From the argument used in § 10, equations (80) lead to

$$(g_{il})_0 \, p^l \frac{\partial L}{\partial p^j} = (g_{jl})_0 \, p^l \frac{\partial L}{\partial p^i},$$

whence L must be of the form $L(\tau, \omega_0)$, $\omega_0 = (g_{ij})_0 p^i p^j$. The light cone at time τ is therefore given by an expression of the form $(g_{ij})_0 p^i p^j = \{H(\tau)\}^2$, where H is some function of τ, the same for each observer. It is evident that $H(\tau)$ must be a unique function, for otherwise it would be possible for an observer to send two light signals at the same time in the same direction to reach a neighbouring observer at different times. It has been proved that the quadratic $g_{ij} p^i p^j$ is invariant under transformations from one observer to another. Hence the light paths at any point (x) in A's coordinate system satisfy the relation

$$g_{ij}(x)\,p^i p^j = \{H(\tau)\}^2. \tag{81}$$

To find the actual paths we shall assume that Fermat's principle is satisfied, *i.e.* that, if a light signal is sent by an observer A_0 at time τ_0 and received by an observer A at time τ, then, out of all possible light paths to A from A_0 at time τ_0, the actual path is the one for which τ is a minimum. By a possible light path, we mean a path for which (81) is satisfied. The actual light paths are therefore given by (81) and

$$\delta \int d\tau = 0,$$

i.e., since $p^i = dx^i/d\tau$ and $H(\tau) \not\equiv 0$, the light paths are given by (81) and

$$\delta \int (g_{ij}\,dx^i\,dx^j)^{\frac{1}{2}} = 0.$$

The light paths therefore correspond to geodesics in the Riemannian space V_3, and, since (81) must be satisfied, it can easily be verified that the light paths are the null geodesics in the Riemannian space V_4 with metric

$$\{H(\tau)\}^2\,d\tau^2 - g_{ij}\,dx^i\,dx^j. \tag{82}$$

We observe that the light paths are similar to the free paths in that they correspond to geodesics in the space with metric $d\sigma^2 = g_{ij}\,dx^i\,dx^j$. It does not, however, follow that the light paths satisfy the equations of free paths, and there is no fundamental reason for assuming that this is the case. If the light paths do not satisfy the path equations, the principle of least action in its wider form is not satisfied, for it is generally understood that this principle implies that the free paths are given by a variational equation $\delta \int W d\tau = 0$ and that the light paths are those free paths satisfying $W = 0$, the form of W being assumed to be such that $W = 0$ is a particular integral of the free path equations. The importance of the case when light paths are not free paths is clearly shown in Milne's theory, for a consequence is that free particles can theoretically attain the speed of light, the model thus providing high speed particles similar to those giving rise to the phenomenon of cosmic radiation.

Since light paths and free paths both correspond to geodesics in V_3, the necessary and sufficient condition for light paths to satisfy the free path equation is that $\omega - H^2 = 0$ should be a particular integral of equations

(72). Equation (75) is a consequence of (73), whence a necessary condition is

$$f(\tau, H^2) \equiv \frac{1}{H} \frac{dH}{d\tau}. \tag{83}$$

If it is assumed that light paths are free paths, equation (83) serves to determine the form of $H(\tau)$ when $f(\tau, \omega)$ is known.

It is reasonable to assume that at each world-event there can be free particles with velocities up to but not exceeding that of light, in which case the free paths issue all inside or all outside the cone $g_{ij} p^i p^j = H^2$. The fundamental particles themselves behave as free particles, for the curves $x^i =$ constant satisfy equations (72) identically. For these curves, $\omega = 0$, *i.e.* $\omega > H^2$, whence all free paths must satisfy the inequality

$$g_{ij} p^i p^j \leqslant H^2 \tag{84}$$

at each point. The solutions of (72) for which this inequality is not satisfied must now be regarded as having no physical significance. When (83) holds, it is easily seen that (84) is satisfied at all points of a path if it is satisfied at one point. When, however, the light paths are not free paths, the forms of $H(\tau)$, $f(\tau, \omega)$ must be such that (84) is satisfied at all points of a path if it is satisfied at one point. It can be verified that this restriction is, in general, equivalent to

$$f(\tau, H^2) \leqslant \frac{1}{H} \frac{dH}{dt}. \tag{85}$$

16. *The Riemannian map.*

We have now constructed the most general possible model of a material system satisfying certain principles and possessing certain fundamental properties in accordance with experience. It is possible that our universe, or, rather, the ideal material system most closely approximating to our universe, can be described by one of these models, and a test of this possibility is afforded by a comparison of the observed relations between various astronomical observables with the equivalent relations calculated from the models. Since the models we have considered are described in terms of coordinates τ, x^i, it is therefore desirable to find the relations that would exist between these coordinates and the various observations we can make with our measuring instruments under the assumption that one of the models approximates to our universe.

In § 3 the clock recording time τ was any clock constructed by one observer. We, however, already possess a clock recording time t, which we may call "physical time", and all our time measurements are made with this clock. It is therefore desirable to refer to time t, and, since the clock recording time τ was arbitrary, we may now assume that $\tau = t$.

To formulate a definition of "rigid body" in agreement with the "rigid scale" of physics, we make use of the experimental result that the velocity of light over small distances is constant, the velocity being measured by means of physical time and rigid scales. Thus, if a light signal sent at time t_1 is reflected at a neighbouring event E and returns at time t_2, then the (small) distance of E measured by our rigid scale is $\frac{1}{2}c(t_2-t_1)$, where c is the constant adopted for the velocity of light. It has been shown that light paths are null geodesics in the Riemannian space V_4 with metric (82), and they are therefore null geodesics in any space conformal to this space. Hence, writing $t = \tau$ and $R(t) = 1/H(t)$, the light paths are null geodesics in the space with metric

$$ds^2 = dt^2 - \{R(t)\}^2 d\sigma^2 \quad (d\sigma^2 = g_{ij}\, dx^i\, dx^j). \qquad (86)$$

Thus $dt = R(t)\, d\sigma$ for outgoing light signals, and $dt = -R(t)\, d\sigma$ for incoming light signals, whence, since distance is given by $\frac{1}{2}c(t_2-t_1)$, we have at once that $cR(t)\, d\sigma$ is the element of distance measured by a rigid scale. Since the metric (86) is invariant in form under transformations from one observer to another, it follows that, at any event (t, x), the element of distance measured with a rigid scale by the observer at this event is $cR(t)\, d\sigma$.

If a light signal sent at time t_1 is reflected at the neighbouring event E and returns at time t_2, the observer may assign to the event E an epoch $\frac{1}{2}(t_2+t_1)$ and say that this is the time at which the distance is $\frac{1}{2}c(t_2-t_1)$. For a neighbouring event, t_2-t_1 is small, whence the time at which the distance of an event is $cR(t)\, d\sigma$ is just the time t. Thus the neighbouring events whose distances are measured at time t lie in the surface $t =$ constant. With the usual terminology, the part of the surface $t =$ constant in the neighbourhood of an observer is called the observer's *instantaneous space* at time t, and an observer may be considered to have direct experience of events occurring in his instantaneous space.

We have now the mechanism for finding what, in the model, correspond to actual observations that can be carried out by one observer, for we have the observer's local measuring system and we know the light paths.

Milne* and McCrea† have considered the problem of combining and relating observations in such a way that the resulting laws calculated for a particular model can be tested observationally. This process can be carried out for the general model derived above, and the arbitrary elements in this model can then be chosen so that the model describes the universe as accurately as possible. The actual observations that can be carried out are all concerned with the extra galactic nebulae, with which we identify the fundamental particles. We are, therefore, only in a position to test those features of the model which, with one exception, are described in the map given by the metric (86). The exception is the local gravitational field, a knowledge of which can be derived from the equations of motion of free test particles. With this exception, we cannot discriminate between the models we have derived and the models provided by general relativity, for the general theory of relativity leads to the metric (86) and interprets this metric in exactly the way we have indicated. The fundamental differences between the models given here and the models of general relativity are concerned with the gravitation field and the free paths. For according to general relativity the free paths are the geodesics given by the metric (86), whereas we see at once that this is not necessarily the case for the present model. The paths given by (72) are the geodesics only if $f(t, \omega)$ is restricted to take the form

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* W.S., ch. 15.

† *Zeitschrift für Astrophys.*, 9 (1935), 290–314.

$$f(t, \omega) \equiv (R^2\omega - 2)\frac{1}{R}\frac{dR}{dt}. \tag{87}$$

17. *Milne's dimensional hypothesis.*

It is convenient to adopt the Riemannian space V_4 with metric (86) as a map, for the geometry of this space describes the model completely except for the free paths. The light paths are the null geodesics, the fundamental world-lines are the geodesics $x^i =$ constant, the linear element and metric give the time, distance, and angle measurements that can be made by the fundamental observers, and the distribution of fundamental particles is given by (79). One extra function is required to specify the free paths. Writing $\tau = t$ and

$$\Omega = \left(\frac{ds}{dt}\right)^2 = 1 - R^2 g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}, \tag{88}$$

then, since Ω involves only t and ω, equations (72) can be written

$$\frac{d^2x^i}{dt^2} - \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = G(t, \Omega)\frac{dx^i}{dt}. \tag{89}$$

The form of $G(t, \Omega)$ is not given by the geometry of V_4.

Since $g_{ij}\,dx^i\,dx^j$ is the metric of a space of constant curvature, it is convenient to choose for x^i the particular coordinates r, θ, ϕ which are such that (86) takes the form

$$ds^2 = dt^2 - R^2\left(\frac{dr^2}{1-Kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right). \tag{90}$$

Here θ, ϕ are angular coordinates measured by the observer with world-line given by $r=0$, and K is a constant which is at our disposal except for sign. The distribution (79) can now be written

$$dN = B(1-Kr^2)^{-\frac{1}{2}}\,r^2\sin\theta\,dr\,d\theta\,d\phi. \tag{91}$$

We observe that the arbitrary elements in the general model are contained in the constants B, K, and the functions $R(t)$, $G(t, \Omega)$. Milne has suggested that, if a suitable model for the universe can be constructed out of such general principles as have here been considered, it is reasonable to assume that it should not be necessary to introduce any constants having physical dimensions. To avoid such constants, it is necessary to restrict the forms of the functions $R(t)$, $G(t, \Omega)$, for, in general, a function of an argument having physical dimensions must involve at least one dimensional constant.

Adopting Milne's hypothesis, the constant B in (91) must be dimensionless. Hence, since dN is a pure number, it follows that the coordinate r is dimensionless. Writing $r=0$, we see, from (90), that the element of distance measured at time t by the observer at the spatial origin $r=0$ is $cR(t)dr$, whence $cR(t)$ must have the dimensions of length. It follows at once that $R(t)$ must be of the form* αt, where α is a pure constant, and we may choose r and K in (90) so that $\alpha = 1$. Further, we see from (89) that $G(t, \Omega)$ must have dimensions T^{-1}, *i.e.* must be of the form $t^{-1}G(\Omega)$, since, from (88), Ω is dimensionless. Thus Milne's hypothesis leads to the restrictions

$$R(t) \equiv t, \quad G(t, \Omega) \equiv t^{-1}G(\Omega). \tag{92}$$

The arbitrary elements in the model are now contained in the pure constants B, K, and the function $G(\Omega)$.

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* The most general possible form is $\alpha(t-t_0)$, where t_0 is a constant. We may, however, assume that t is measured so that $t_0 = 0$.

An immediate result is that all measured distances are zero at the epoch $t=0$, since $dl = cR(t)\,d\sigma = ct\,d\sigma$. Thus the fundamental observers all occupy the same event at this epoch, as was assumed by Milne.

Substituting in (83), we see that a necessary condition for light paths to be free paths is

$$G(0) = -1. \tag{93}$$

When the restrictions (92) are imposed, the free path equations (89) admit a useful first integral*, for it can be verified that

$$t^2 \exp\int^{\Omega} \frac{d\Omega}{(1-\Omega)\{1+G(\Omega)\}} = \text{constant} \tag{94}$$

is satisfied along each path. Since a free particle is moving with the velocity of light at an event E if its path satisfies $\Omega = 0$ at E, it follows from (94) that each free particle reaches the velocity of light in finite time if

$$\int_0 \frac{d\Omega}{1+G(\Omega)} \tag{95}$$

is convergent. It is evident that this condition cannot be satisfied if the light paths are free paths, for a free path is completely determined by the coordinate velocity at any one event, so that two different paths cannot touch at any common point.

18. *Transformation to Milne's coordinates.*

Referred to any observer A attached to the particle with world-line given by $r=0$ in the map V_4 with metric (90), the world-lines of the other particles are the geodesics $r, \theta, \phi =$ constant. For neighbouring particles, r is small and from (90), with $R(t) \equiv t$, the distance of such a particle measured by A at time t is $l = ctr$. Since r is constant for each particle, we have $l/t =$ constant, whence each observer sees the neighbouring fundamental particles to be receding with constant velocities.

The same result holds for distant particles if we define distance and epoch in accordance with Milne's convention, *i.e.* if we assign a distance r' and an epoch t' to an event E where

$$t' = \tfrac{1}{2}(t_2+t_1), \quad r' = \tfrac{1}{2}c(t_2-t_1),$$

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* This corresponds to Milne's X-integral, W.S., 143, equation 9. It will be seen later that Milne's X, ξ correspond, in our notation, to t^2, Ω^{-1} respectively.

t_1, t_2 being the times of emission and reception by A of a light signal reflected at E. A typical fundamental particle has the world-line $\theta = 0$, $r =$ constant, and the light paths between A and this particle are given by $dt^2 = t^2 dr^2/(1-Kr^2)$. Hence, if the event of reflection is given by the coordinate value t, we have

$$t_1 = te^{-\sigma(r)}, \quad t_2 = re^{\sigma(r)}, \quad \sigma(r) = \int_0^r \frac{dr}{(1-Kr^2)^{\frac{1}{2}}}, \tag{96}$$

whence $$t' = t \cosh \sigma(r), \quad r' = ct \sinh \sigma(r). \tag{97}$$

Thus, since r is constant for a particular particle, r'/t' is constant and the particle is receding with constant velocity, the velocity associated with r being $c \tan \sigma(r)$. This result supports Milne's assumption of constant velocities, but the argument here given is different from any of the arguments supplied by Milne.

Throughout his work Milne uses coordinates t', x', y', z', where t' and $r' \{= (x'^2+y'^2+z'^2)^{\frac{1}{2}}\}$ are the coordinates defined as above and x', y', z' are Cartesian coordinates. Transforming to polar coordinates r', θ', ϕ', we see that θ', ϕ' are the coordinates θ, ϕ in (90). Hence, from (97), the transformation from the coordinates t, r, θ, ϕ to Milne's coordinates t', r', θ', ϕ' is given by

$$t' = t \cosh \sigma(r), \quad r' = ct \sinh \sigma(r), \quad \theta' = \theta, \quad \phi' = \phi. \tag{98}$$

Thus we have $$t^2 = t'^2 - r'^2/c^2, \tag{99}$$

and, from (92),

$$ds^2 = dt'^2 - \frac{1}{c^2}\{dr'^2 + \rho^2(d\theta'^2 + \sin^2\theta' d\phi'^2)\}, \tag{100}$$

where ρ is ctr expressed in terms of t' and r'. Hence, since t^2 and ds^2 are invariants under transformations from one observer to another, we have that the corresponding invariants expressed in terms of Milne's coordinates are given by (99) and (100), the fundamental world-lines now being $r'/t' =$ constant, θ', $\phi' =$ constant.

The Lorentz transformations relating observers in uniform relative motion are such that $t'^2 - r'^2/c^2$, $dt'^2 - (dx'^2+dy'^2+dz'^2)/c^2$ are the invariants under the group. Comparing these expressions with (99) and (100), we see that the group in the general model is the Lorentz group only when $\rho(t', r') \equiv r'$, *i.e.* when $K = 1$, as can easily be verified from (98) and (96). In all other cases, the transformations in three spatial dimensions do not belong to the Lorentz group. It is interesting to observe, however, that the transformations in a one-dimensional continuum are always the Lorentz transformations, for, writing $\theta' = 0$, the invariants (99) and (100) become $t'^2 - r'^2/c^2$, $dt'^2 - dr'^2/c^2$. This is to be expected, for Milne has given* a general derivation of the Lorentz transformations in the case of a one-dimensional continuum.

When the condition $K = -1$ is not satisfied, the transformations leaving (99) and (100) invariant can be found explicitly but are very complicated. It is probable that more convenient descriptions of the various types of models are given by other conventions for assigning distances and epochs to distant events, in which case the relative motion will not, in general, turn out to be uniform.

19. *Milne's system.*

It can now be shown that the general solution with $K = -1$ is exactly the solution derived by Milne, for the fundamental particles are in uniform relative motion and the transformation from one observer to any other is of the Lorentz form. To express the distribution in the form given by Milne, we find the number of fundamental particles at time t' with coordinates x', y', z' in the range x', $x'+dx'$, etc. The transformation (98) now takes the form

$$t' = t(1+r^2)^{\frac{1}{2}}, \quad r' = ctr, \quad \theta' = \theta, \quad \phi' = \phi. \tag{101}$$

Keeping t' constant, we find $dr' = ct(1+r^2)^{-1}dr$, whence (91) becomes

$$dN = \frac{Bt'}{c^3 t^4} r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi' = \frac{Bt' \, dx' \, dy' \, dz'}{c^3(t'^2 - r'^2/c^2)^2}, \tag{102}$$

thus agreeing with Milne's expression† for the distribution. Finally we get Milne's equations of motion‡ by transforming to coordinates t', x', y', z' in (89) and (92) by means of (101), for we get equations of the form

$$\frac{dx'}{dt'} = v', \qquad \frac{dv'}{dt'} = (x' - t'v') \frac{Y}{X} \bar{G}(\xi), \tag{103}$$

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* W.S., ch. II.

† W.S., § 92.

‡ W.S., § 101.

where

$$X = t'^2 - r'^2/c^2, \quad Y = 1 - (\Sigma v'^2)/c^2, \quad Z = t' - (\Sigma x' v')/c^2,$$

and

$$\xi = \frac{Z^2}{XY} = \frac{1}{\Omega}.$$

The function $\bar{G}(\xi)$ is derived from the function $G(\Omega)$ by means of the relation

$$\bar{G}(\xi) = -1 - \{1 + G(\Omega)\}/\Omega \quad (\Omega = 1/\xi). \tag{104}$$

20. *Statistical systems.*

We have up to the present considered a simple kinematic system of fundamental particles, but the results obtained hold for a statistical system of elementary particles of the form considered by Milne. To construct such a system, we refer to observers constrained to move as in the simple system, and then consider a distribution of elementary particles such that the cosmological principle and the principle of symmetry are again satisfied. It follows at once that the paths of these elementary particles are similar to the free paths given by equations of the form (89). These equations are satisfied by the curves $x^i = \text{constant}$, whence the fundamental observers may be considered to be attached to freely moving particles, *i.e.* the constraints introduced at first are superfluous.

Since the distribution of elementary particles must be invariant under the transformation from one observer to any other, it can be verified that the number of particles at time t in the coordinate range $(x^i, x^i + dx^i)$ and the velocity range $(p^i, p^i + dp^i)$ must be given by an expression of the form

$$dN = \psi(t, \Omega)\, g\, dx^1 dx^2 dx^3 dp^1 dp^2 dp^3 \quad (g = |g_{ij}|). \tag{105}$$

Neglecting collisions, the distribution function $\psi(t, \Omega)$ will be related to the function $G(t, \Omega)$ occurring in the path equations in order that particle number shall be conserved. From (88) and (89) it can be verified that ψ and $G(t, \Omega)$ must satisfy

$$\frac{\partial \psi}{\partial t} - 2(1-\Omega)\left\{G + \frac{1}{R}\frac{dR}{dt}\right\}\frac{\partial \psi}{\partial \Omega} + \left\{3G - 2(1-\Omega)\frac{\partial G}{\partial \Omega}\right\}\psi = 0. \tag{106}$$

Milne's dimensional hypothesis will restrict the form of $\psi(t, \Omega)$, since dN in (105) is a pure number. Referring to coordinates r, θ, ϕ as in § 16, and remembering that these coordinates have no dimensions, we see that

ψ must have dimension T^3. Hence $\psi(t, \Omega)$ must be of the form

$$\psi(t, \Omega) \equiv t^3 \psi(\Omega). \tag{107}$$

Substituting from (92) and (107) in (106), this equation can now be integrated, and we find

$$\psi(\Omega) = -\frac{C}{(1-\Omega)^{\frac{3}{2}}\{1+G(\Omega)\}}, \tag{108}$$

where C is a pure constant. It can be verified that, when $K=-1$, the distribution given by this function substituted in (105) is identically the distribution obtained by Milne.

We observe from (108) that there is, in general, a singularity in $\psi(\Omega)$ at $\Omega = 1$. From (88) we see that $\Omega = 1$ is satisfied only by the paths of the fundamental particle observers, whence these particles are at the centres of highly dense clusters of elementary particles. These clusters therefore behave as the fundamental particles of a simple system.

The logical significance of the relation (108) is that the function $G(\Omega)$, specifying the paths of elementary particles, is determined to within a single constant by the distribution of these particles. There is no corresponding relation for the simple kinematical system, but since the free paths in the system are similar to the paths of elementary particles in the statistical system, it would seem that we can find a statistical system having exactly the properties of any given simple system. For if we know the function $G(\Omega)$ occurring in the free path equations for a simple system, the relation (108) will, in general, determine a distribution function $\psi(\Omega)$. Milne has considered this point, and has suggested* that the function $G(\Omega)$ for a simple system should be of such a form that the associated distribution function $\psi(\Omega)$ is indeterminate. This would give

$$G(\Omega) \equiv -1 \tag{109}$$

for a simple system, corresponding to Milne's result $\bar{G}(\xi) = -1$ in (103). This argument is by no means conclusive, but Milne has given other arguments leading to the same result. There is strong observational evidence that (109) holds when $\Omega = 1$, for it can be shown from a calculation of local accelerations that the Newtonian constant of gravitation is given by

$$\gamma = \frac{2+G(1)}{\frac{4}{3}\pi\rho t^2},$$

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* *Proc. Royal Soc.* (A), 154 (1936), 25.

where ρ is the local density and t is given by the age of the universe. When $G(1) = -1$, this relation is in good agreement with observed values.

21. *Conclusion.*

We have now two models before us : the simple kinematical system, involving pure constants B, K, and the statistical system, involving pure constants B, C, K and a function $\psi(\Omega)$ of a dimensionless argument. It is hoped that it will be possible to test these models as descriptions of the universe when more is known about the extra galactic nebulae, and when the local consequences of these models have been worked out along the lines now being pursued by Milne.

I wish to express my sincere thanks and to acknowledge my indebtedness to Professor E. A. Milne for the help and stimulation derived from his book, and for the many helpful and encouraging conversations on the subject of this paper.

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EIN MITTELWERT EPSTEINSCHER ZETAFUNKTIONEN

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[Received 1 June, 1936.—Read 18 June, 1936.]

I. Es sei $Q(x, y) = ax^2 + bxy + cy^2$ eine positiv-definite reelle Form, für $s = \sigma + it$, $\sigma > 1$ sei

$$Z_Q(s) = \sum_{m,n}' \{Q(m, n)\}^{-s} = \sum_{\substack{m=-\infty \\ m^2+n^2>0}}^{\infty} \sum_{n=-\infty}^{\infty} \{Q(m, n)\}^{-s} = \sum_{\nu=1}^{\infty} \frac{r_Q(\lambda_\nu)}{\lambda_\nu^{\,s}},$$

wobei λ_ν alle durch Q darstellbaren Zahlen durchläuft, $r_Q(\lambda_\nu)$ die Anzahl der Darstellungen von λ_ν durch Q bedeutet. Dann ist für $\sigma > \frac{1}{2}$ der Mittelwert[*]

$$\frac{1}{T}\int_1^T |Z_Q(\sigma+it)|^2\, dt \sim \sum_{\nu=1}^{\infty} r_Q^{\,2}(\lambda_\nu)\lambda_\nu^{-2\sigma} \quad \text{für} \quad T \to \infty,$$

mit Hilfe der Funktionalgleichung folgt hieraus leicht für $\sigma < \frac{1}{2}$

$$\int_1^T |Z_Q(\sigma+it)|^2\, dt \sim \frac{(2\pi)^{4\sigma-2}\Delta^{1-2\sigma}}{3-4\sigma} \sum_{\nu=1}^{\infty} \frac{r_Q^{\,2}(\lambda_\nu)}{\lambda_\nu^{2-2\sigma}}\, T^{3-4\sigma} \quad \text{für} \quad T \to \infty,$$

wobei $\Delta = 4ac - b^2$ ist. Hier soll nun der Mittelwert auf der kritischen Geraden $\sigma = \frac{1}{2}$ untersucht werden, und ich erhalte folgende Ergebnisse:

Es seien a, b, c ganze Zahlen, $d = b^2 - 4ac$ eine Fundamentaldiscriminante; dann ist für $T \to \infty$

$$\int_0^T |Z_Q(\tfrac{1}{2}+it)|^2\, dt = O(T \log^2 T).$$

Es sei die Klassenzahl $h = 1$ oder $d = -15, -20, -24$ ($h = 2$), a_d eine nur von d abhängige, von Q sonst unabhängige Konstante, die später angegeben

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[*] Carlson, "Contribution à la théorie des séries de Dirichlet", *Arkiv. för Math. Astr. o. Fys.*, 19 (1926), N. 25, 12.

werden wird. Dann ist genauer

$$\text{(A)} \qquad \int_0^T |Z_Q(\tfrac{1}{2}+it)|^2\,dt \sim a_d\,T\log^2 T,$$

$$\text{(B)} \qquad \Re\left(\int_0^T Z_{Q_1}(\tfrac{1}{2}+it)\,Z_{Q_2}(\tfrac{1}{2}-it)\,dt\right) = o(T\log^2 T),$$

wenn Q_1 und Q_2 Formen von verschiedener Klasse in den Fällen $d=-15$, -20, -24 bedeuten. Wahrscheinlich gilt wohl die Gleichung (A) und somit der Satz, dass die Mittelwerte für alle Klassen derselben Discriminante $d<0$ asymptotisch gleich sind, ganz allgemein. Für den Beweisgang wird der Mittelwert der Dedekindschen Zetafunktion des Körpers $P(\sqrt{d})$ gebraucht und hergeleitet; es ergibt sich für ihn eine Gleichung von genau derselben Art wie (A).

Herrn H. Heilbronn, Cambridge, schulde ich für seine Hilfe besonderen Dank. Ferner danke ich K. Silberberg, Breslau, für die Mühe des Durchsehens und Durchrechnens der Arbeit.

Im Folgenden soll stets die Gesamtmenge aller Primzahlen zerlegt sein in die drei Teilmengen der Primzahlen P, p, q, welche durch die Bedingungen bestimmt sind:

$$\left(\frac{d}{P}\right)=1 \quad \text{bezw.} \quad \left(\frac{d}{p}\right)=-1 \quad \text{bezw.} \quad \left(\frac{d}{q}\right)=0, \quad \text{also} \quad q\,|\,(-d).$$

$\left(\frac{a}{b}\right)$ bedeute das Jacobische Symbol, in den nicht erfassten Fällen das Kroneckersche.

II. Zunächst seien noch a, b, c *beliebig reell*;

$$a>0, \quad \rho=\sqrt{(-d)}=\sqrt{(4ac-b^2)}>0;$$

es ist, da für $\sigma>0$

$$\int_0^\infty e^{-(2\pi/\rho)Qx}\,x^{s-1}\,dx = \Gamma(s)\left(\frac{2\pi}{\rho}\,Q\right)^{-s},$$

$$Z_Q(s)\,\Gamma(s)\left(\frac{\rho}{2\pi}\right)^s = \int_0^\infty \sum_{m,n}{}' \, e^{-(2\pi/\rho)Q(m,n)x}\,x^{s-1}\,dx$$

für $\sigma>1$, also nach Mellins Umkehrungsformel für $\sigma>1$

$$\sum_{m,n}{}' \, e^{-(2\pi/\rho)Q(m,n)x} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} Z_Q(s)\,\Gamma(s)\left(\frac{2\pi x}{\rho}\right)^{-s} ds.$$

Durch Verlegung des Integrationsweges folgt, da

$$(1,11)\quad \begin{cases} Z_Q(s) \sim \dfrac{2\pi}{\rho(s-1)} & \text{für} \quad s \to 1, \\ |\Gamma(s)| = O(|t|^{\sigma-\frac{1}{2}}\, e^{-\frac{1}{2}\pi|t|}) & \text{für} \quad |t| \to \infty, \quad \sigma_0 \leqslant \sigma \leqslant \sigma_1, \end{cases}$$

$$(1,1)\qquad \sum_{m,\,n} e^{-(2\pi/\rho)Q(m,\,n)x} - 1 - \frac{1}{x} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Z_Q(s)\,\Gamma(s)\left(\frac{2\pi x}{\rho}\right)^{-s} ds,$$

gültig für $\Re(x) > 0$, also $|\arg x| < \frac{1}{2}\pi$ mit Rücksicht auf die Abschätzung von $|\Gamma(s)|$ und auf $Z_Q(\frac{1}{2}+it) = O(|t|^A)$ für $|t| \to \infty$. Es sei jetzt

$$0 < \epsilon < \tfrac{1}{2}\pi, \quad \phi = \tfrac{1}{2}\pi - \epsilon, \quad x = e^{y+i\phi}, \quad y \gtreqless 0,$$

$$F(y) = e^{\frac{1}{2}y}\Big(\sum_{m,\,n} e^{-(2\pi/\rho)Qe^{y+i\phi}} - 1 - e^{-y-i\phi}\Big);$$

$$G(t) = \frac{\rho^{\frac{1}{2}}}{2\pi}\, Z_Q(\tfrac{1}{2}+it)\,\Gamma(\tfrac{1}{2}+it)\left(\frac{\rho}{2\pi}\right)^{it} e^{\phi t - i\frac{1}{2}\phi}.$$

Dann folgt aus (1, 1)

$$(1,2)\quad \sum_{m,\,n} e^{-(2\pi/\rho)Qe^{y+i\phi}} - 1 - e^{-y-i\phi}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_Q(\tfrac{1}{2}+it)\,\Gamma(\tfrac{1}{2}+it)\left(\frac{2\pi}{\rho}\, e^{y+i\phi}\right)^{-\frac{1}{2}-it} dt,$$

$$F(y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} G(t)\, e^{-iyt}\, dt;$$

daher ist nach der Parsevalschen Formel

$$\int_{-\infty}^{\infty} |F(y)|^2\, dy = \int_{-\infty}^{\infty} |G(t)|^2\, dt.$$

Setzt man $e^y = x$, so folgt schliesslich

$$\int_0^{\infty}\left|\sum_{m,\,n} e^{-(2\pi/\rho)Qxe^{i\phi}} - 1 - \frac{e^{-i\phi}}{x}\right|^2 dx = \frac{\rho}{4\pi^2}\int_{-\infty}^{\infty} |Z_Q(\tfrac{1}{2}+it)\,\Gamma(\tfrac{1}{2}+it)|^2\, e^{2\phi t}\, dt.$$

Zerlegt man das Integral der linken Seite in $\int_0^1 + \int_1^{\infty}$, setzt man im ersten Teilintegral $x = \tilde{x}^{-1}$, und benutzt man die bekannte Transformation der Θ-Reihe

$$\sum_{m,\,n} e^{-(2\pi/\rho)Q(m,\,n)u} = \frac{1}{u}\sum_{m,\,n} e^{-(2\pi/\rho)Q(m,\,n)(1/u)}$$

so zeigt sich die Gleichheit der beiden Teilintegrale, also ist

$$(1)\quad \frac{\rho}{8\pi^2}\int_{-\infty}^{\infty}|\mathrm{Z}_Q(\tfrac{1}{2}+it)\,\Gamma(\tfrac{1}{2}+it)|^2\,e^{2\phi t}\,dt=\int_1^{\infty}\left|\sum_{m,\,n}{}'\,e^{-(2\pi/\rho)Qxe^{i\phi}}-\frac{e^{-i\phi}}{x}\right|^2dx$$

$$=\int_1^{\infty}\left|\sum_{\nu=1}^{\infty}r_Q(\lambda_\nu)\,e^{-(2\pi/\rho)\lambda_\nu xe^{i\phi}}-\frac{e^{-i\phi}}{x}\right|^2dx.$$

III. Das letzte Integral ist gleich

$$\int_1^{\infty}\left|\sum_{\nu=1}^{\infty}r_Q(\lambda_\nu)\,e^{-(2\pi/\rho)\lambda_\nu xe^{i\phi}}\right|^2dx+1-2\Re\left(e^{i\phi}\int_1^{\infty}\sum_{\nu=1}^{\infty}r_Q(\lambda_\nu)\,e^{-(2\pi/\rho)\lambda_\nu xe^{i\phi}}\,\frac{dx}{x}\right)$$

$$=F_1(\epsilon)+O(\epsilon^{-\delta})$$

für $\epsilon\to 0$, jedes feste $\delta>0$, wenn $F_1(\epsilon)$ das erste Integral hier bedeutet. Denn nach dem 2. Mittelwertsatze ist mit $1<M\leqslant\infty$

$$\left|\int_1^{\infty}\Re\left(\sum_{\nu=1}^{\infty}r_Q(\lambda_\nu)\,e^{-(2\pi/\rho)\lambda_\nu xe^{i\phi}}\right)\frac{dx}{x}\right|$$

$$=\left|\int_1^{M}\Re\left(\sum_{m,\,n}{}'\,e^{-(2\pi/\rho)Q(m,\,n)\,xe^{i\phi}}\right)dx\right|$$

$$=\frac{\rho}{2\pi}\left|\Re\left[\sum_{m,\,n}{}'\,\frac{e^{-(2\pi/\rho)Q(m,\,n)\,xe^{i\phi}}}{Q(m,\,n)\,e^{i\phi}}\right]_M^1\right|\leqslant\frac{\rho}{\pi}\sum_{m,\,n}{}'\,\frac{e^{-(2\pi/\rho)Q(m,\,n)\sin\epsilon}}{Q(m,\,n)}$$

$$=\frac{1}{2\pi i}\,\frac{\rho}{\pi}\int_{\delta-i\infty}^{\delta+i\infty}\Gamma(s)\,\mathrm{Z}_Q(s+1)\left(\frac{2\pi}{\rho}\sin\epsilon\right)^{-s}ds=O(\epsilon^{-\delta}),$$

mit Rücksicht auf $Z(1+\delta+it)=O(1)$ für $|t|\to\infty$ und die Abschätzung von $\Gamma(s)$ in (1, 11); ebenso der imaginäre Teil. Es ist ferner

$$F_1(\epsilon)=\int_1^{\infty}\sum_{\nu=1}^{\infty}r_Q{}^2(\lambda_\nu)\,e^{-(4\pi/\rho)\lambda_\nu x\sin\epsilon}\,dx$$

$$+\int_1^{\infty}\sum_{\substack{\mu\gtrless\nu\\ \mu=1\ldots\infty\\ \nu=1\ldots\infty}}r_Q(\lambda_\mu)\,r_Q(\lambda_\nu)\,e^{-(2\pi/\rho)x(\lambda_\mu e^{i\phi}+\lambda_\nu e^{-i\phi})}\,dx$$

$$=\frac{\rho}{4\pi\sin\epsilon}\sum_{\nu=1}^{\infty}\frac{r_Q{}^2(\lambda_\nu)}{\lambda_\nu}\,e^{-(4\pi/\rho)\lambda_\nu\sin\epsilon}+\frac{\rho}{2\pi}\sum_{\substack{\mu\gtrless\nu\\ 1\ldots\infty}}r_Q(\lambda_\mu)\,r_Q(\lambda_\nu)\,\frac{e^{-(2\pi/\rho)(\lambda_\mu e^{i\phi}+\lambda_\nu e^{-i\phi})}}{\lambda_\mu e^{i\phi}+\lambda_\nu e^{-i\phi}}.$$

Die linke Seite von (1) ist, mit Rücksicht auf

$$\mathrm{Z}_Q(\tfrac{1}{2}+it)=O(|t|^{\frac{1}{2}(1+\delta)}),\qquad|\Gamma(\tfrac{1}{2}+it)|=(2\pi)^{\frac{1}{2}}\,e^{-\frac{1}{2}\pi|t|}\left\{1+O\left(\frac{1}{t^2}\right)\right\},$$

gleich

$$\frac{\rho}{4\pi}\int_0^{\infty}|\mathrm{Z}_Q(\tfrac{1}{2}+it)|^2\,e^{-2t\epsilon}\,dt+O(\epsilon^{-\delta}),$$

also ist für $\epsilon \to 0$

$$(2)\quad \int_0^\infty |Z_Q(\tfrac{1}{2}+it)|^2 e^{-2t\epsilon}\,dt = \frac{1}{\sin\epsilon} \sum_{\nu=1}^{\infty} \frac{r_Q{}^2(\lambda_\nu)}{\lambda_\nu} e^{-(4\pi/\rho)\lambda_\nu \sin\epsilon}$$

$$+2 \sum_{\substack{\mu=1\ldots\infty \\ \nu=1\ldots\infty}}^{\mu\neq\nu} r_Q(\lambda_\mu) r_Q(\lambda_\nu) \frac{e^{-(2\pi/\rho)(\lambda_\mu e^{i\phi}+\lambda_\nu e^{-i\phi})}}{\lambda_\mu e^{i\phi}+\lambda_\nu e^{-i\phi}} + O(\epsilon^{-\delta}).$$

IV. Es seien fortan a, b, c ganz rational, $d = -\rho^2$ eine Fundamental-Discriminante. Für $\sigma > 0$ sei

$$L_d(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s}.$$

Wenn Q ein System von Repräsentanten aller primitiven Klassen der Discriminante d durchläuft, w den Wert 6, 4 oder 2 bedeutet, je nachdem $d = -3$, -4 oder < -4 ist, wenn ferner $r(n)$ die Gesamtzahl der Darstellungen von n durch alle repräsentierenden Formen bedeutet, so ist bekanntlich

$$(3)\qquad \sum_{n=1}^{\infty} \frac{r(n)}{n^s} = \sum_{Q} Z_Q(s) = w\zeta(s)\,L_d(s).$$

Man wähle in (2) $\lambda_\nu = \nu$ und also, falls ν nicht durch Q darstellbar ist, $r_Q(\nu) = 0$; es ist $r_Q(n) \leqslant r(n)$.

Jetzt soll zunächst der Mittelwert von $\zeta(s)\,L_d(s)$, der ζ-Funktion des Körpers $P(\sqrt{d})$, auf der kritischen Geraden asymptotisch berechnet werden. Aus (1, 1) folgt, wenn Q alle repräsentierenden Formen durchläuft,

$$\sum_{Q} \left(\sum_{m,\,n} e^{-(2\pi/\rho)Q(m,\,n)x} - 1 - \frac{1}{x} \right) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} w\zeta(s)\,L_d(s)\,\Gamma(s) \left(\frac{2\pi x}{\rho}\right)^{-s} ds,$$

und schliesslich durch dieselben Überlegungen, die zu (2) geführt haben:

$$(2')\quad \int_0^\infty |\zeta(\tfrac{1}{2}+it)\,L_d(\tfrac{1}{2}+it)|^2 e^{-2t\epsilon}\,dt$$

$$= \frac{1}{w^2 \sin\epsilon} \sum_{\nu=1}^{\infty} \frac{r^2(\nu)}{\nu} e^{-(4\pi/\rho)\nu\sin\epsilon}$$

$$+\frac{2}{w^2} \sum_{\substack{\mu=1\ldots\infty \\ \nu=1\ldots\infty}}^{\mu\neq\nu} r(\mu)\,r(\nu) \frac{e^{-(2\pi/\rho)(\mu e^{i\phi}+\nu e^{-i\phi})}}{\mu e^{i\phi}+\nu e^{-i\phi}} + O(\epsilon^{-\delta}).$$

Es soll nun gezeigt werden, dass die erste Reihe der rechten Seite gleich $C\epsilon^{-1}\log^2(\epsilon) + O(\epsilon^{-1}\log\epsilon^{-1})$, die zweite $O(\epsilon^{-1}\log\epsilon^{-1})$ ist; hierzu beweise ich, dass die Funktion

$$F(s) = \sum_{1}^{\infty} r^2(n)\, n^{-s}, \tag{4, 1}$$

zunächst in der Halbebene $\sigma > 1$ definiert, über die Gerade $\sigma = 1$ fortgesetzt werden kann und im Punkte 1 einen Doppelpol hat.

V. Es sei (P, p, q siehe I)

$$n = P_1^{a_1} P_2^{a_2} \dots p_1^{b_1} p_2^{b_2} \dots q_1^{c_1} q_2^{c_2} \dots; \quad \tau = P_1^{\alpha_1} P_2^{\alpha_2} \dots p_1^{\beta_1} p_2^{\beta_2} \dots q_1^{\gamma_1} q_2^{\gamma_2} \dots;$$

$$0 \leqslant \alpha_i \leqslant a_i, \quad 0 \leqslant \beta_i \leqslant b_i, \quad 0 \leqslant \gamma_i \leqslant c_i,$$

also τ ein Teiler von n. Nun ist bekanntlich

$$\frac{1}{w}\, r(n) = \sum_{\tau \mid n} \left(\frac{d}{\tau}\right), \tag{3'}$$

also gleich

$$\sum_{\substack{0 \leqslant \alpha_i \leqslant a_i \\ 0 \leqslant \beta_i \leqslant b_i}} \prod_{P_i \mid n} \left(\frac{d}{P_i}\right)^{\alpha_i} \prod_{p_i \mid n} \left(\frac{d}{p_i}\right)^{\beta_i} = \prod_{P_i \mid n} \sum_{\alpha_i = 0}^{a_i} \left(\frac{d}{P_i}\right)^{\alpha_i} \prod_{p_i \mid n} \sum_{\beta_i = 0}^{b_i} \left(\frac{d}{p_i}\right)^{\beta_i} = \prod_{P_i \mid n} (a_i + 1),$$

falls alle b_i gerade sind, sonst gleich 0. Folglich ist

$$\frac{1}{w^2} F(s) = \sum_{\substack{a_i,\, b_i,\, c_i \\ 0 \dots \infty}} \frac{(a_1+1)^2 (a_2+1)^2 \dots}{P_1^{a_1 s} P_2^{a_2 s} \dots p_1^{2b_1 s} p_2^{2b_2 s} \dots q_1^{c_1 s} q_2^{c_2 s} \dots}$$

$$= \prod_{P,\, p,\, q} \sum_{a=0}^{\infty} \frac{(a+1)^2}{P^{as}} \sum_{b=0}^{\infty} \frac{1}{p^{2bs}} \sum_{c=0}^{\infty} \frac{1}{q^{cs}} = \prod_{P,\, p,\, q} \frac{1+P^{-s}}{(1-P^{-s})^3} \frac{1}{1-p^{-2s}} \frac{1}{1-q^{-s}},$$

wobei $P_1, P_2, \dots$ sämtliche P sind, $p_1, p_2, \dots$ und $q_1, q_2, \dots$ sämtliche p bezw. q. Da aber

$$\zeta(s) = \prod_{P,\, p,\, q} \frac{1}{1-P^{-s}} \frac{1}{1-p^{-s}} \frac{1}{1-q^{-s}}, \quad L_d(s) = \prod_{P,\, p} \frac{1}{1-P^{-s}} \frac{1-p^{-s}}{1-p^{-2s}},$$

so folgt*, wenn man $g(s) = \prod\limits_{q} (1-q^{-s})^{-1}$ setzt,

$$\frac{1}{w^2} F(s) = \frac{\zeta^2(s)\, L_d^2(s)\, g(2s)}{\zeta(2s)\, g(s)}; \quad \frac{F(s)}{w^2} (s-1)^2 \sim \frac{6}{\pi^2} L_d^2(1) \prod_{q \mid -d} \frac{q}{q+1} \tag{4}$$

für $s \to 1$. Also ist $F(s)$ für $\sigma > \frac{1}{2}$ regulär bis auf den Doppelpol, für

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* Landau, *Vorlesungen über Zahlentheorie* (Leipzig, 1927), 2, 251 u.f. im Falle $d = -4$.

$\sigma \geqslant \frac{1}{2}+\delta$, $|t| \to \infty$ ist $F(s)=O(|t|^A)$. So ist das 1. Glied der rechten Seite von (2′) für irgend ein $\sigma_0 > 0$ und $0 < \delta < \frac{1}{2}$ gleich

$$\frac{1}{w^2 \sin \epsilon}\,\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s)\,F(s+1)\left(\frac{4\pi}{\rho}\sin\epsilon\right)^{-s} ds$$

$$=\frac{1}{w^2\sin\epsilon}\left(\operatorname*{Res}_{s=0}+\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\right)$$

$$=\frac{3}{\pi^2}\,L_d{}^2(1)\prod_{q|-d}\frac{q}{q+1}\,\frac{\log^2(\sin\epsilon)}{\sin\epsilon}+O\left(\frac{|\log\sin\epsilon|}{\sin\epsilon}\right)+O\left(\frac{\epsilon^\delta}{\sin\epsilon}\right)$$

$$=\frac{1}{\epsilon}\log^2\epsilon\,\frac{3L_d{}^2(1)}{\pi^2}\prod_{q|-d}\frac{q}{q+1}+O\left(\frac{\log\epsilon^{-1}}{\epsilon}\right), \tag{5, 1}$$

da
$$\int_{-\delta-i\infty}^{-\delta+i\infty}=O\left(\int_1^\infty t^A\,e^{-\frac{1}{2}\pi t}\,t^{-\delta-\frac{1}{2}}\,\epsilon^\delta\,dt\right)+O(\epsilon^\delta)=O(\epsilon^\delta), \quad (\epsilon\to 0).$$

Zur Abschätzung der letzten Reihe* in (2′) fasse man dort je zwei konjugiert komplexe Glieder zusammen; man erhält

$$\sum_{\mu \gtrless \nu} = 2\sum_{1\leqslant\nu<\mu\leqslant\infty}\frac{(\mu+\nu)\sin\epsilon\cos\{(2\pi/\rho)(\mu-\nu)\cos\epsilon\}}{(\mu+\nu)^2\sin^2\epsilon+(\mu-\nu)^2\cos^2\epsilon}\,e^{-(2\pi/\rho)(\mu+\nu)\sin\epsilon}\,r(\mu)\,r(\nu)$$

$$+2\sum_{1\leqslant\nu<\mu\leqslant\infty}\frac{(\mu-\nu)\cos\epsilon\sin\{(2\pi/\rho)(\mu-\nu)\cos\epsilon\}}{(\mu+\nu)^2\sin^2\epsilon+(\mu-\nu)^2\cos^2\epsilon}\,e^{-(2\pi/\rho)(\mu+\nu)\sin\epsilon}\,r(\mu)\,r(\nu)$$

$$=2\Sigma_1+2\Sigma_2.$$

$$|\Sigma_1|\leqslant\epsilon\sum_{1\leqslant\nu<\mu\leqslant\infty}\frac{(\mu+\nu)\,e^{-(2\pi/\rho)(\mu+\nu)\sin\epsilon}}{(\mu-\nu)^2}\,r(\mu)\,r(\nu)$$

$$<\epsilon\sum_{\substack{k=1\ldots\infty\\ \mu=k+1\ldots\infty}}\frac{2\mu}{k^2}\,e^{-(2\pi/\rho)(2\mu-k)\sin\epsilon}\,r(\mu)\,r(\mu-k)$$

$$=2\epsilon\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{\mu=k+1}^{\infty}\mu\,r(\mu)\,e^{-(2\pi/\rho)\mu\sin\epsilon}\,r(\mu-k)\,e^{-(2\pi/\rho)(\mu-k)\sin\epsilon}$$

$$\leqslant 2\epsilon\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sum_{\mu=k+1}^{\infty}\mu^2 r^2(\mu)\,e^{-(4\pi/\rho)\mu\sin\epsilon}\sum_{m=1}^{\infty}r^2(m)\,e^{-(4\pi/\rho)m\sin\epsilon}\right)^{\frac{1}{2}}$$

$$<2\epsilon\,\frac{\pi^2}{6}\left(\sum_{\mu=1}^{\infty}\mu^2 r^2(\mu)\,e^{-(4\pi/\rho)\mu\sin\epsilon}\sum_{m=1}^{\infty}r^2(m)\,e^{-(4\pi/\rho)m\sin\epsilon}\right)^{\frac{1}{2}}$$

$$=\frac{\pi^2}{3}\,\epsilon(\Sigma_{11}\,\Sigma_{12})^{\frac{1}{2}}.$$

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* E. C. Titchmarsh, " The mean value of the zeta-function on the critical line ", *Proc. Lond. Math. Soc.* (2), 27 (1928), 144 u. 145; hier ähnlich.

Aber auf Grund einer entsprechenden Überlegung wie bei (5, 1) ist $\Sigma_{11} = O(\epsilon^{-3}\log\epsilon^{-1})$, $\Sigma_{12} = O(\epsilon^{-1}\log\epsilon^{-1})$, also ist $\Sigma_1 = O(\epsilon^{-1}\log\epsilon^{-1})$.

$$\frac{2}{\cos\epsilon}\Sigma_2 = \sum_{1\leqslant\nu<\mu\leqslant\infty} 2\sin\left(\frac{2\pi}{\rho}(\mu-\nu)\cos\epsilon\right) e^{-(2\pi/\rho)(\mu+\nu)\sin\epsilon}\frac{r(\mu)\,r(\nu)}{\mu-\nu}$$
$$+O\left\{\epsilon^2 \sum_{1\leqslant\nu<\mu} \frac{\mu\,\nu\, r(\mu)\, r(\nu)\, e^{-(2\pi/\rho)(\mu+\nu)\sin\epsilon}}{(\mu-\nu)^3}\right\};$$

das letzte Glied ist* gleich

$$O\left\{\epsilon^2 \sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{\mu=k+1}^{\infty} \mu r(\mu)\, e^{-(2\pi/\rho)\mu\sin\epsilon}(\mu-k)\, r(\mu-k)\, e^{-(2\pi/\rho)(\mu-k)\sin\epsilon}\right\}$$
$$= O\left(\frac{\log\epsilon^{-1}}{\epsilon}\right),$$

ähnlich wie bei Σ_1. Das 1. Glied der rechten Seite aber ist absolut

$$= \left| \tfrac{1}{2}\int_{-(2\pi/\rho)\cos\epsilon}^{(2\pi/\rho)\cos\epsilon} \left|\sum_{\mu=1}^{\infty} r(\mu)\, e^{-(2\pi/\rho)\mu\sin\epsilon + i\mu\vartheta}\right|^2 d\vartheta - \frac{2\pi}{\rho}\cos\epsilon \sum_{\mu=1}^{\infty} r^2(\mu)\, e^{-(4\pi/\rho)\mu\sin\epsilon}\right|$$
$$\leqslant \tfrac{1}{2}\int_{-2\pi}^{2\pi} \left|\sum_{\mu=1}^{\infty} r(\mu)\, e^{-(2\pi/\rho)\mu\sin\epsilon + i\mu\vartheta}\right|^2 d\vartheta + \frac{2\pi}{\rho}\cos\epsilon \sum_{\mu=1}^{\infty} r^2(\mu)\, e^{-(4\pi/\rho)\mu\sin\epsilon}$$
$$= 2\pi\left(1+\frac{\cos\epsilon}{\rho}\right)\sum_{\mu=1}^{\infty} r^2(\mu)\, e^{-(4\pi/\rho)\mu\sin\epsilon} = O(\epsilon^{-1}\log\epsilon^{-1})^*,$$

zum Schlusse durch entsprechende Überlegung wie für (5, 1). So folgt schliesslich aus (2′) für $\epsilon\to 0$

$$(5) \qquad \int_0^{\infty} |\zeta(\tfrac{1}{2}+it)\, L_d(\tfrac{1}{2}+it)|^2 e^{-2t\epsilon}\, dt = \frac{3}{\pi^2} L_d^2(1) \prod_{q|-d}\frac{q}{q+1}\,\frac{\log^2\epsilon}{\epsilon} + O\left(\frac{\log\epsilon^{-1}}{\epsilon}\right),$$

$$(5') \qquad \int_0^{T} |\zeta(\tfrac{1}{2}+it)\, L_d(\tfrac{1}{2}+it)|^2\, dt \sim \frac{6}{\pi^2} L_d^2(1) \prod_{q|-d}\frac{q}{q+1}\, T\log^2 T$$

für $T\to\infty$. Daraus folgt sofort die gewünschte Mittelwertformel für $Z_Q(s)$ in den Fällen der Klassenzahl $h=1$ ($d=-3, -4, -7, -8, -11, -19$, u.s.w.) durch Multiplikation mit w^2.

VI. Da $r_Q(n) \leqslant r(n)$ ist, folgt, dass die rechte Seite von (2) absolut nicht grösser ist als

$$\frac{1}{\sin\epsilon}\sum_{\nu=1}^{\infty}\frac{r^2(\nu)}{\nu} e^{-(4\pi/\rho)\nu\sin\epsilon} + 4\sum_{1\leqslant\nu<\mu\leqslant\infty} r(\mu)\, r(\nu)\,\frac{e^{-(2\pi/\rho)(\mu+\nu)\sin\epsilon}}{\mu-\nu} + O(\epsilon^{-\delta})$$
$$= O\left(\frac{\log^2\epsilon}{\epsilon}\right) + O\left(\frac{\log^2\epsilon}{\epsilon}\right) + O(\epsilon^{-\delta}) = O(\epsilon^{-1}\log^2\epsilon),$$

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* Hier ist die Überlegung ähnlich wie bei E. C. Titchmarsh, *loc. cit.*, 149 u. 150.

die 1. Reihe auf Grund von (5, 1), die zweite durch ähnliche Überlegung wie Σ_1, da sie* kleiner ist als

$$\sum_{k=1}^{\infty} \frac{e^{-(\pi/\rho) k \sin \epsilon}}{k} \sum_{\mu=k+1}^{\infty} r(\mu)\, e^{-(\pi/\rho)\mu \sin \epsilon}\, r(\mu-k)\, e^{-(\pi/\rho)(\mu-k) \sin \epsilon}.$$

So ist für $\epsilon \to 0$ bezw. für $T \to \infty$

$$(6) \quad \int_0^{\infty} |Z_Q(\tfrac{1}{2}+it)|^2 e^{-2\epsilon t}\, dt = O\left(\frac{\log^2 \epsilon}{\epsilon}\right), \quad \int_0^{T} |Z_Q(\tfrac{1}{2}+it)|^2\, dt = O(T \log^2 T).$$

VII. Es sollen nun in einigen Fällen der Klassenzahl 2 die genauen Mittelwertformeln bestimmt werden. Im Falle

$$d = -15$$

sind die beiden reduzierten Formen

$$Q_1(x, y) = x^2+xy+4y^2, \quad Q_2(x, y) = 2x^2+xy+2y^2.$$

Es ist $q_1 = 3$, $q_2 = 5$, ferner für jede Primzahl r:

$$\left(\frac{-15}{r}\right) = \left(\frac{r}{15}\right) = \left(\frac{r}{3}\right)\left(\frac{r}{5}\right),$$

also
$$\left(\frac{P}{5}\right) = \left(\frac{P}{3}\right), \quad \left(\frac{p}{5}\right) = -\left(\frac{p}{3}\right).$$

Da eine Zahl n durch Formen von der Discriminante $-d$ dann und nur dann darstellbar ist, wenn sie nur gerade Potenzen der Primteiler p enthält, sei

$$n = \prod_{i=1}^{k} P_i^{a_i} \prod_{j=1}^{k'} p_j^{2b_j}\, 3^l\, 5^m = n'\, 3^l\, 5^m.$$

Es sei
$$\psi(n) = (-1)^{l+m} \left(\frac{n'}{5}\right),$$

dann ist n durch Q_1 darstellbar, wenn $\psi(n) = +1$, durch Q_2, wenn $\psi(n) = -1$ ist. Das soll kurz gezeigt werden: es sei n_1 durch Q_1, n_2 durch Q_2 darstellbar, also

$$(7) \quad 4n_1 = 4Q_1 = (2x_1+y_1)^2+15y_1^2, \quad 8n_2 = 8Q_2 = (4x_2+y_2)^2+15y_2^2;$$

es sei $l = m = 0$, dann ist also $4n_1$ Rest mod 15, also auch mod 5, also n_1 Rest mod 5, $\psi(n_1) = 1$; entsprechend ist $8n_2$ Rest mod 5, n_2 Nichtrest mod 5, $\psi(n_2) = -1$. Es sei jetzt $l > 0$ und gerade. Dann ist wegen (7) $3 \mid (2x_1+y_1)$, also $3^2 \mid 15y_1^2$, $3 \mid y_1$, also $3^3 \mid (2x_1+y_1)^2$, $3^2 \mid (2x_1+y_1)$, u.s.f.,

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* Ähnlich wie die Abschätzung von Σ_2, E. C. Titchmarsh, *loc. cit.*, 145.

$$3^{\frac{1}{2}l} \mid (2x_1+y_1),$$

also ergibt sich für die Zahl $4n_1 3^{-l}$ bezw. $8n_2 3^{-l}$ die Gestalt

$$4n_1 3^{-l} = \xi_1{}^2 + 15\eta_1{}^2; \quad 8n_2 3^{-l} = \xi_2{}^2 + 15\eta_2{}^2. \tag{7, 1}$$

Es sei jetzt m auch gerade, so erhält man entsprechend weiter

$$4n_1' = 4n_1 3^{-l} 5^{-m} = \xi_1'^2 + 15\eta_1'^2; \quad 8n_2' = 8n_2 3^{-l} 5^{-m} = \xi_2'^2 + 15\eta_2'^2. \tag{7, 2}$$

Hieraus folgt unter Benutzung des Ergebnisses im Falle $l = m = 0$, dass n_1' Rest, n_2' Nichtrest mod 5 ist, also $\psi(n_1) = 1$, $\psi(n_2) = -1$ ist.

Es sei jetzt l ungerade > 0. Dann folgt, ähnlich wie oben,

$$3^l \mid 15y_1{}^2, \quad 3^{\frac{1}{2}(l-1)} \mid y_1, \quad 3^{\frac{1}{2}(l+1)} \mid (2x_1+y_1), \quad 3^{\frac{1}{2}(l+1)} \mid (4x_2+y_2),$$

$$4n_1 3^{-l} = 3\xi_1{}^2 + 5\eta_1{}^2; \quad 8n_2 3^{-l} = 3\xi_2{}^2 + 5\eta_2{}^2. \tag{7, 3}$$

Nimmt man m jetzt gerade an, so folgt entsprechend weiter

$$4n_1' = 4n_1 3^{-l} 5^{-m} = 3\xi_1'^2 + 5y_1'^2; \quad 8n_2' = 8n_2 3^{-l} 5^{-m} = 3\xi_2'^2 + 5\eta_2'^2. \tag{7, 4}$$

$4n_1'$ und $8n_2'$ sind daher Nichtreste mod 5, also ist

$$\psi(n_1) = (-1)\left(\frac{n_1'}{5}\right) = 1, \qquad \psi(n_2) = (-1)\left(\frac{n_2'}{5}\right) = (-1)(+1) = -1.$$

Darstellungen genau derselben Art wie (7, 4) erhält man, wenn l gerade, m ungerade ist, wie (7, 2), wenn l *und* m ungerade sind.

Die Zahl n ist daher entweder durch Q_1 oder durch Q_2 darstellbar, also ist $r_1{}^2(n) - r_2{}^2(n)$ gleich plus oder minus $r^2(n)$,

$$r_1{}^2(n) - r_2{}^2(n) = (-1)^{l+m}\left(\frac{n'}{5}\right) r^2(n) = (-1)^{l+m} r^2(n) \prod_{P_i \mid n} \left(\frac{P_i}{5}\right)^{a_i},$$

$$G(s) = \tfrac{1}{4} \sum_{n=1}^{\infty} \frac{r_1{}^2(n) - r_2{}^2(n)}{n^s} = \sum_{\substack{a_i,\, b_i,\, l,\, m \\ 0 \ldots \infty}} \prod_{P_i} \frac{(a_i+1)^2}{P_i^{a_i s}} \left(\frac{P_i}{5}\right)^{a_i} \prod_{p_i} \frac{1}{p_i^{2b_i s}} \frac{(-1)^l}{3^{ls}} \frac{(-1)^m}{5^{ms}}$$

$$= \prod_{P,\, p} \sum_{a=0}^{\infty} \frac{(a+1)^2}{P^{as}} \left(\frac{P}{5}\right)^a \sum_{b=0}^{\infty} \frac{1}{p^{2bs}} \sum_{l=0}^{\infty} \frac{(-1)^l}{3^{ls}} \sum_{m=0}^{\infty} \frac{(-1)^m}{5^{ms}}$$

$$= (1+3^{-s})^{-1} (1+5^{-s})^{-1} \prod_p (1-p^{-2s})^{-1} \prod_P \frac{1 + \left(\frac{P}{5}\right) P^{-s}}{\left\{1 - \left(\frac{P}{5}\right) P^{-s}\right\}^3}.$$

Nun ist

$$\prod_P \frac{1}{1-\left(\frac{P}{5}\right)P^{-s}} \prod_p \frac{1}{1-\left(\frac{p}{5}\right)p^{-s}} = \prod_{P,\,p} \frac{1}{1-\left(\frac{5}{P}\right)P^{-s}} \frac{1}{1-\left(\frac{5}{p}\right)p^{-s}}$$

$$= L_5(s)(1+3^{-s}),$$

$$\prod_{P,\,p} \frac{1-\left(\frac{p}{5}\right)p^{-s}}{1-\left(\frac{P}{5}\right)P^{-s}} = \prod_{P,\,p} \frac{1-p^{-2s}}{\left\{1-\left(\frac{P}{3}\right)P^{-s}\right\}\left\{1-\left(\frac{p}{3}\right)p^{-s}\right\}}$$

$$= \prod_{P,\,p} \frac{1-p^{-2s}}{\left\{1-\left(\frac{-3}{P}\right)P^{-s}\right\}\left\{1-\left(\frac{-3}{p}\right)p^{-s}\right\}}$$

$$= L_{-3}(s)(1+5^{-s}) \prod_p (1-p^{-2s}),$$

daher ist

$$G(s) = \frac{L_5{}^2(s)\, L_{-3}^2(s)}{\zeta(2s)(1-3^{-s})(1-5^{-s})}, \tag{8}$$

regulär für $\sigma > \frac{1}{2}$ und gleich $O(|t|^A)$ für $\sigma \geqslant \frac{1}{2}+\delta$; also ist

$$\sum_1^\infty \frac{r_1{}^2(n)}{n^s} \sim \frac{12}{\pi^2} L_{-15}^2(1) \prod_{q\,|\,15} \frac{q}{q+1} \frac{1}{(s-1)^2} \sim \sum_1^\infty \frac{r_2{}^2(n)}{n^s}$$

für $s \to 1$ und sonst regulär für $\sigma > \frac{1}{2}$ und ist gleich $O(|t|^A)$ für $\sigma \geqslant \frac{1}{2}+\delta$, auf Grund von $\Sigma\{r_1{}^2(n)+r_2{}^2(n)\}n^{-s} = F(s)$ und (4), (8). So erhält man für $\epsilon \to 0$ bezw. für $T \to \infty$,

$$\int_0^\infty |Z_{Q_1}(\tfrac{1}{2}+it)|^2 e^{-2\epsilon t}\,dt = \frac{6}{\pi^2} L_{-15}^2(1) \prod_{q\,|\,15} \frac{q}{q+1} \frac{\log^2 \epsilon}{\epsilon} + O\left(\frac{\log \epsilon^{-1}}{\epsilon}\right) \tag{9}$$

$$= \frac{\log^2 \epsilon}{\epsilon} + O\left(\frac{\log \epsilon^{-1}}{\epsilon}\right)$$

unter Heranziehung derselben Überlegungen wie für (5) und mit Rücksicht auf die Gleichung*

$$L_d(1) = -\frac{\pi}{|d|^{\frac{3}{2}}} \sum_{r=1}^{|d|-1} \left(\frac{d}{r}\right) r, \quad \text{also} \quad L_{-15}(1) = \frac{2\pi}{\sqrt{15}}.$$

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* Landau, *loc. cit.*, 1, 177.

Genau das gleiche Ergebnis folgt für Z_{Q_2}, somit ist für $T\to\infty$

$$(9') \qquad \int_0^T |Z_{Q_1}(\tfrac{1}{2}+it)|^2 dt \sim \int_0^T |Z_{Q_2}(\tfrac{1}{2}+it)|^2 dt \sim 2T\log^2 T.$$

Es ist ferner

$$\int_0^\infty |Z_{Q_1}+Z_{Q_2}|^2 e^{-2t\epsilon} dt$$

$$= \sum_{r=1}^{2} \int_0^\infty |Z_{Q_r}|^2 e^{-2t\epsilon} dt + 2\Re\left(\int_0^\infty Z_{Q_1}(\tfrac{1}{2}+it)\, Z_{Q_2}(\tfrac{1}{2}-it)\, e^{-2t\epsilon} dt\right),$$

$$(5) \quad \int_0^\infty |2\zeta(\tfrac{1}{2}+it)\, L_{-15}(\tfrac{1}{2}+it)|^2 e^{-2t\epsilon} dt$$

$$= \frac{12}{\pi^2} L^2_{-15}(1) \prod_{q\mid 15} \frac{q}{q+1} \frac{\log^2 \epsilon}{\epsilon} + O\left(\frac{\log \epsilon^{-1}}{\epsilon}\right),$$

durch Vergleich folgt mit Rücksicht auf (3) und (9)

$$(10) \qquad \Re\left(\int_0^\infty Z_{Q_1}(\tfrac{1}{2}+it)\, Z_{Q_2}(\tfrac{1}{2}-it)\, e^{-2t\epsilon} dt\right) = O\left(\frac{\log \epsilon^{-1}}{\epsilon}\right),$$

$$(10') \qquad \Re\left(\int_0^T Z_{Q_1}(\tfrac{1}{2}+it)\, Z_{Q_2}(\tfrac{1}{2}-it)\, dt\right) = o(T\log^2 T),$$

(10′) aus (9′) und (5′).

VIII. $d=-20, \quad Q_1(x,y)=x^2+5y^2, \quad Q_2(x,y)=2x^2+2xy+3y^2.$

$$q_1=2, \quad q_2=5; \qquad \left(\frac{P}{5}\right)=\left(\frac{-4}{P}\right), \quad \left(\frac{p}{5}\right)=-\left(\frac{-4}{p}\right).$$

n sei eine durch Formen von der Discriminante -20 darstellbare Zahl, also

$$n = P_1^{a_1} P_2^{a_2} \ldots p_1^{2b_1} p_2^{2b_2} \ldots 2^l 5^m = n'\, 2^l 5^m,$$

dann ist n entweder durch Q_1 oder durch Q_2 darstellbar, je nachdem

$$\psi(n) = (-1)^l \left(\frac{n'}{5}\right)$$

plus 1 oder minus 1 ist. Also ist, entsprechend wie im Falle -15,

$$r_1^2(n)-r_2^2(n)=(-1)^l\left(\frac{n'}{5}\right)r^2(n)=4(-1)^l\prod_{P_i\mid n}(a_i+1)^2\left(\frac{P_i}{5}\right)^{a_i},$$

$$G(s)=\tfrac{1}{4}\sum_{n=1}^{\infty}\frac{r_1^2(n)-r_2^2(n)}{n^s}$$

$$=\Big((1+2^{-s})(1-5^{-s})\prod_p(1-p^{-2s})\Big)^{-1}\prod_P\frac{1+\left(\frac{P}{5}\right)P^{-s}}{\left\{1-\left(\frac{P}{5}\right)P^{-s}\right\}^3},$$

$$\prod_{P,\,p}\frac{1}{1-\left(\frac{P}{5}\right)P^{-s}}\;\frac{1}{1-\left(\frac{p}{5}\right)p^{-s}}=\prod_{P,\,p}\frac{1}{1-\left(\frac{5}{P}\right)P^{-s}}\;\frac{1}{1-\left(\frac{5}{p}\right)p^{-s}}$$

$$=L_5(s)(1+2^{-s}),$$

$$\prod_{P,\,p}\frac{1-\left(\frac{p}{5}\right)p^{-s}}{1-\left(\frac{P}{5}\right)P^{-s}}=\prod_{P,\,p}\frac{1}{1-\left(\frac{-4}{P}\right)P^{-s}}\;\frac{1-p^{-2s}}{1-\left(\frac{-4}{p}\right)p^{-s}}$$

$$=L_{-4}(s)(1-5^{-s})\prod_p(1-p^{-2s}),$$

$$G(s)=\frac{L_5{}^2(s)\,L_{-4}^2(s)}{\zeta(2s)(1-2^{-s})(1+5^{-s})},$$

$$\sum_{n=1}^{\infty}\frac{r_l^2(n)}{n^s}\sim\frac{12}{\pi^2}\,L_{-20}^2(1)\prod_{q\mid 20}\frac{q}{q+1}\,\frac{1}{(s-1)^2}=\frac{4}{3(s-1)^2}\quad\text{für}\quad s\to 1,\ l=1,\ 2.$$

(11) $$\int_0^{\infty}|Z_{Q_l}(\tfrac{1}{2}+it)|^2e^{-2\epsilon t}\,dt=\tfrac{2}{3}\,\frac{\log^2\epsilon}{\epsilon}+O\left(\frac{\log\epsilon^{-1}}{\epsilon}\right)\quad(l=1\ \text{od.}\ 2).$$

(11′) $$\int_0^{T}|Z_{Q_1}(\tfrac{1}{2}+it)|^2\,dt\sim\int_0^{T}|Z_{Q_2}(\tfrac{1}{2}+it)|^2\,dt\sim\tfrac{4}{3}\,T\log^2T.$$

(12) $$\Re\left(\int_0^{\infty}Z_{Q_1}(\tfrac{1}{2}+it)\,Z_{Q_2}(\tfrac{1}{2}-it)\,e^{-2t\epsilon}\,dt\right)=O\left(\frac{\log\epsilon^{-1}}{\epsilon}\right),$$

(12′) $$\Re\left(\int_0^{T}Z_{Q_1}(\tfrac{1}{2}+it)\,Z_{Q_2}(\tfrac{1}{2}-it)\,dt\right)=o(T\log^2T).$$

IX. $d=-24$, $Q_1(x,y)=x^2+6y^2$, $Q_2(x,y)=2x^2+3y^2$.

$$q_1=2,\quad q_2=3;\qquad\left(\frac{P}{3}\right)=\left(\frac{2}{P}\right),\quad\left(\frac{p}{3}\right)=-\left(\frac{2}{p}\right).$$

$$n=P_1^{a_1}P_2^{a_2}\ldots p_1^{2b_1}p_2^{2b_2}\ldots 2^l\,3^m=n'\,2^l\,3^m$$

ist entweder durch Q_1 oder durch Q_2 darstellbar, je nachdem die Funktion

$$\psi(n) = (-1)^{l+m}\left(\frac{n'}{3}\right)$$

den Wert plus oder minus 1 hat. Das ist ganz ähnlich wie im Falle $d=-15$ einzusehen. Entsprechend ist weiter

$$r_1^2(n) - r_2^2(n) = (-1)^{l+m}\left(\frac{n'}{3}\right) r^2(n) = 4(-1)^{l+m} \prod_{P_i \mid n} (a_i+1)^2 \left(\frac{P_i}{3}\right)^{a_i},$$

$$G(s) = \tfrac{1}{4}\sum_1^\infty \frac{r_1^2(n)-r_2^2(n)}{n^s}$$

$$= \Big((1+2^{-s})(1+3^{-s})\prod_p (1-p^{-2s})\Big)^{-1} \prod_P \frac{1+\left(\frac{P}{3}\right)P^{-s}}{\left\{1-\left(\frac{P}{3}\right)P^{-s}\right\}^3},$$

$$\prod_{P,\,p} \frac{1}{1-\left(\frac{P}{3}\right)P^{-s}} \frac{1}{1-\left(\frac{p}{3}\right)p^{-s}} = \prod_{P,\,p} \frac{1}{1-\left(\frac{-3}{P}\right)P^{-s}} \frac{1}{1-\left(\frac{-3}{p}\right)p^{-s}}$$

$$= L_{-3}(s)(1+2^{-s}),$$

$$\prod_{P,\,p} \frac{1-\left(\frac{p}{3}\right)p^{-s}}{1-\left(\frac{P}{3}\right)P^{-s}} = \prod_{P,\,p} \frac{1}{1-\left(\frac{2}{P}\right)P^{-s}} \frac{1-p^{-2s}}{1-\left(\frac{2}{p}\right)p^{-s}} = L_2(s)(1+3^{-s})\prod_p(1-p^{-2s}),$$

$$G(s) = \frac{L_2^2(s)\,L_{-3}^2(s)}{\zeta(2s)(1-2^{-s})(1-3^{-s})}$$

$$\sum_{n=1}^\infty \frac{r_l^2(n)}{n^s} \sim \frac{12}{\pi^2} L_{-24}^2(1) \prod_{q \mid 24} \frac{q}{q+1} \frac{1}{(s-1)^2} = \frac{1}{(s-1)^2} \quad \text{für} \quad s \to 1 \ (l=1 \text{ od. } 2),$$

$$\int_0^\infty |\mathrm{Z}_{Q_l}(\tfrac{1}{2}+it)|^2 e^{-2\epsilon t}\,dt = \tfrac{1}{2}\frac{\log^2 \epsilon}{\epsilon} + O\left(\frac{1}{\epsilon}\log\frac{1}{\epsilon}\right) \quad (l=1 \text{ od. } 2). \tag{13}$$

$$\int_0^T |\mathrm{Z}_{Q_1}(\tfrac{1}{2}+it)|^2\,dt \sim \int_0^T |\mathrm{Z}_{Q_2}(\tfrac{1}{2}+it)|^2\,dt \sim T\log^2 T. \tag{13'}$$

Ebenso ergeben sich die (12) und (12′) entsprechenden Gleichungen.

9 Frobenstr., Breslau.

ON THE PROJECTIVE CLASSIFICATION OF SURFACES

By Leonard Roth.

[Received 2 May, 1936.—Read 14 May, 1936.]

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The present paper gives an account, as far as possible systematic, of the classification of algebraic surfaces by their projective as well as their invariant characters; it includes, among new results, theorems which have appeared in previous notes*, not invariably in the form in which they are presented here. Acknowledgment* must be made, as in the previous work, to Comessatti for the theorems §§ 2.42, 3.11, and 5.31, which are in substance due to him.

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* Roth, 1, 2, 3. In cases where the proofs of these theorems were merely outlined, the demonstrations are now given in full.

We confine our attention throughout to non-singular surfaces, which are not scrolls, situated in space S_r $(r > 3)$. If n is the order and π the sectional genus of such a surface F, then, for given values of n and r, π can assume only a finite number of values, the greatest being the maximum genus of an irreducible curve of order n in S_{r-1}. The first step in attempting to classify this series of surfaces is to examine what effect a specified set of characters n, π, and r has upon the values of the invariants. We begin, naturally, with the question of regularity.

The characteristic property of a regular surface is that any complete linear system of curves contained in it has its characteristic series complete. Accordingly we begin by supposing that the prime sections $|C|$ of F are normal curves (so that their characteristic series is complete), and inquire in what circumstances we can assert that F is regular. Using a method due to Castelnuovo, we show, by Picard's theorem, that F is regular when the characteristic series of $|C|$ is non-special (§ 2.21); in the case where this series is special but the system $|C|$ is itself non-special, we give a sufficient condition for regularity (§ 2.31). Next, we remove the latter restriction and show that, when $|C|$ coincides with the canonical system and π lies within certain limits, F is in general regular (§ 2.41); an analogous result is obtained for the case in which a given multiple of $|C|$ forms the canonical system of F (§ 2.43).

In § 3 we discuss the regularity of a surface which is known to lie on a given threefold. We first obtain sufficient conditions for F to lie on a primal of given order, in particular, a primal of S_4. We then show that, if F lies on a quadric of S_4, it is regular, and that, if it lies on a cubic primal of S_4, it is either regular or else has geometrical genus zero; a similar result holds for a whole class of rational threefolds (§ 3.33). Next, we consider the rational normal threefolds whose curve sections are rational; we show that, if F lies on such a threefold, it is either regular or referable to a scroll. Thence follows the principal result of this section (§ 3.6); taken in conjunction with the theorem of Castelnuovo-Enriques (§ 4.12), it states that

For sufficiently small values of π, F belongs to the class of referable (including rational) surfaces; and for sufficiently large values of π and n, F belongs to the class of regular surfaces.

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* I should like also to thank Dr. Du Val for helpful criticism at several points.

In § 4 we show how the regularity of a surface is affected by the existence on it of a linear system of curves of specified type. We have first to recall the classical results of Noether and Enriques concerning the surfaces which contain a pencil of rational curves, and Castelnuovo's theorems on the surfaces containing a net of elliptic or hyperelliptic curves. We then consider the surfaces which contain a net of curves, subject to certain restrictions, of genus 3 or 4; and then the surfaces containing a rational pencil of elliptic curves. From the theorems of this and the preceding sections, it follows that, if a surface F contains a rational pencil of quartics or quintics, it is regular or referable.

Many of the propositions used to establish these results depend implicitly on the concept of continuous systems of curves; in § 5 we use the same concept to examine further the surfaces F whose prime sections $|C|$ are normal. If C defines a continuous system $\{C\}$, the generic curve of the system must be reducible; in § 5.1 we consider such systems in detail. Now, if F is irregular and C is virtually effective and of sufficiently high genus, C is normal and the generic curve of $\{C\}$ must be reducible; this result has an important application to the classification of surfaces, which is illustrated briefly in § 6. The theorem § 5.23 of this section replaces a result given by Comessatti (and hitherto used) in an incomplete form.

The paper concludes by applying the preceding results to the surfaces of S_4, up to the tenth order; we give also a summary of the classification of non-singular surfaces by their sectional genera, for values of $\pi \leqslant 6$, taken from previous work. From this analysis we obtain the minimum orders of the non-singular referable, and irregular non-referable, surfaces. This part of the work has been dealt with before*, but it is now considered in more detail, leading to greatly improved results. Incidentally, it will be seen how the Riemann-Roch theorem may be used to furnish upper and lower limits for the arithmetic genus of a surface. We thus succeed in establishing not only the regularity or referability of the surfaces under discussion, but also, in many cases, the values that can be assumed by their genera.

1. *Introduction.*

1.1. We denote by F the surface under consideration and by $|C|$ the system of its prime sections. Unless otherwise stated, we shall assume that F is non-singular† and non-ruled, and that it is situated in S_r $(r \geqslant 4)$. If its order and sectional genus are respectively n and π, the dimension ρ of its normal space is given by

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* Roth, 1.

† If situated in S_4, it may, of course, have a finite number of improper nodes.

$$\rho = p_a + n - \pi + 1 - i + \sigma,$$

where i is the index of speciality and σ is the redundance of $|C|$. If $|C|$ is a *regular* system, then $i = \sigma = 0$.

The *virtual dimension* of $|C|$ is defined as

$$p_a + n - \pi + 1 - i,$$

and if this expression is non-negative the system $|C|$ is termed *virtually effective*.

1.2. If F is a regular surface, $p_a = p_g \geqslant 0$; and if it is referable to a scroll, then $p_g = 0$, $p_a \leqslant -1$, the scroll being of genus $-p_a$. It is important to note that the condition $p_a < -1$ characterizes all the referable surfaces save those which are elliptic*. If, however, $p_a = -1$, F may be an elliptic non-referable surface having $p_g \geqslant 0$. The non-referable surfaces with the invariants $p_g = 0$, $p_a = -1$ (for which we shall reserve the term "elliptic") are characterized† by the fact that they contain an elliptic pencil of curves of genus $\pi_1 > 0$, and a rational pencil of elliptic curves, without base-points, which includes a number of multiple elliptic components. Thus, in conclusion, it follows that, if F is irregular and non-referable, $p_a \geqslant -1$.

1.3. *Criteria of regularity.*

In a classification of algebraic surfaces which takes their projective characters into account, the general propositions concerning their invariants are utilized with the help of assumptions about the order and sectional genus. Examples of these general propositions, which we shall employ later, are as follows.

1.31. *The system of curves adjoint to the prime sections of a surface is regular.* (Picard's theorem.)

1.32. *A surface of S_r which is the complete intersection of $r-2$ general primals is regular.*

---

* Castelnuovo, 1.

† Enriques, 1.

1.33. *On a regular threefold, a surface contained in a linear system, the variable part of whose characteristic system is not compounded of an irrational pencil, is regular**.

1.34. *A surface of* S_3, *whose only singularities are a number of isolated multiple points in arbitrary position, is either regular or else has* $p_g = 0$.

1.35. *A double plane with an irreducible, non-singular branch-curve represents a regular surface.*

In addition to these principles there are results obtained by enumeration of actual possibilities; thus, Campedelli† has shown, by examining the various types, that a double plane with branch-curve of order not exceeding 8 represents a regular or referable surface, and that a double plane with decimic branch-curve represents a regular, referable, or elliptic surface. Other instances are the classification of surfaces by sectional genus, which we shall describe later, in § 6.

1.4. *On curves of maximum genus.*

We shall require later the following results concerning curves with genus near to the maximum; the first two of these are due to Castelnuovo‡ and the others to Fano§.

1.41. *The maximum genus of an irreducible curve* Γ *of order* n *in* S_r *is*

$$\pi_r(n) = \chi\{n-r-\tfrac{1}{2}(\chi-1)(r-1)\},$$

where

$$\frac{n-1}{r-1}-1 \leqslant \chi < \frac{n-1}{r-1}.$$

1.42. *The curve* Γ *of maximum genus lies on* $\binom{r-1}{2}$ *linearly independent quadrics. If* $n < 2r$, Γ *is normal, non-special, and of genus* $n-r$; *and if* $n = 2r$, Γ *is the canonical curve of genus* $r+1$.

It follows that, if $n > 2r$, Γ lies on a surface of order $r-1$, common to all the quadrics. If $r \neq 5$, this is a rational normal scroll, whose generators meet Γ in $\chi+1$ points in general, and, alternatively, in $\chi+2$ points, when $n-r$ is a multiple of $r-1$. If $r = 5$ and n is even, there is another type of Γ lying on a Veronese surface.

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* Severi, 1.

† Campedelli, 1, 2.

‡ Castelnuovo, 2.

§ Fano, 1.

If $n \leqslant 2r$, there are types of Γ, in general sub-species, lying on surfaces of order $r-1$. From § 1.41 we observe that, if $n = 2r$, then $\chi = 2$, so that, if the surface is a scroll, its generators are trisecant to Γ. If $n < 2r$, $\chi = 1$, and the generators are bisecant or trisecant to Γ, the latter case arising only when $n-r$ is a multiple of $r-1$. If $r = 5$ and n is even, Γ may also lie on a Veronese surface.

Similar results hold for all curves of sufficiently high genus. Thus we have

1.43. *If* $n > 2r$, *and*

$$\pi > \chi_1 \left\{ n - \frac{r+1}{2} - \chi_1 \frac{r-1}{2} \right\} - \chi_1 + 3,$$

where

$$\frac{n-r-1}{r-1} \leqslant \chi_1 < \frac{n-r-1}{r-1} + 1,$$

then Γ *lies on* $\binom{r-1}{2}$ *linearly independent quadrics.*

1.44. *If* $n \geqslant (k+2)(r-1)+2$, *and* $\pi = \pi_r(n) - k$, *then* Γ *lies on* $\binom{r-1}{2}$ *linearly independent quadrics.*

2. *Applications of Picard's theorem.*

2.1. If $|C'|$ is the system of curves adjoint to the system $|C|$, and n', π' are respectively the grade and genus of $|C'|$, it follows from §§ 1.1, 1.31 that the dimension r' of $|C'|$ is given by

$$r' = p_a + n' - \pi' + 1.$$

But the grade and genus of $|C'|$ satisfy the relation*

$$n' - \pi' = \pi - 2,$$

so that

$$r' = p_a + \pi - 1.$$

2.2. *On surfaces whose prime sections are normal curves.*

If F is a regular surface, the characteristic series of any complete linear system contained in it is complete; in particular, if F is normal, its prime sections $|C|$ are normal curves. Conversely, if the sections $|C|$ of a surface F are normal, it is natural to inquire in what circumstances F is regular*. A first answer to this question was provided by Castelnuovo†, in the following theorem.

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* Enriques, **2**, 95.

2.21. *If the characteristic series of* $|C|$ *is complete and non-special, then* F *is regular, with geometrical genus zero.*

The proof of the theorem follows the lines of § 2.3. As a corollary we have

2.211. *If* $n = 2\pi - 2$, *and* C *is normal in* $S_{\pi-2}$, *then* $p_a = p_g = 0$.

2.3. We now obtain an extension of the previous result to surfaces for which $p_g > 0$. If $|C|$ is a complete, regular system, it follows from the Riemann-Roch theorem that

$$r = n - \pi + p_a + 1.$$

If, moreover, the characteristic series of $|C|$ is complete, it has index of speciality p_a, so that, in the case where F is regular, its index of speciality is equal to p_g. We are thus led to consider a system $|C|$ whose characteristic series is complete, having index of speciality p_g; and we obtain the following result.

2.31. *If the characteristic series of* $|C|$ *is complete, with index of speciality* p_g, *where* $2p_g \leqslant \pi$, *then* F *is regular.*

The dimension r of the normal space of F is, by hypothesis,

$$r = n - \pi + p_g + 1. \tag{1}$$

Projecting F from $r-3$ arbitrary points of itself, we obtain in S_3 a normal surface f of order

$$m = \pi - p_g + 2, \tag{2}$$

having only ordinary singularities.

We now proceed to evaluate the dimension of the system of adjoint surfaces $|\phi^{m-3}|$ of order $m-3$. The restriction placed on the value of p_g is such that there exists at most one adjoint surface ϕ^{m-4} containing a given line l; if $p_g - 1 < m - 3$, *i.e.* if $2p_g < \pi$, we can construct a single adjoint ϕ^{m-4} to pass through $p_g - 1$ given points of l without containing l. It follows that there are at least ∞^1 adjoints ϕ^{m-3} passing through l and having

---

[*] In point of fact, there is no known example of a non-singular, irregular surface whose prime sections are normal.

[†] Castelnuovo, 3.

nodes at p_g-1 specified points on l, namely those consisting of ϕ^{m-4} and a plane through l. Since a node on l imposes two conditions on a surface already containing l, we have

$$p_a+\pi-1 \geqslant (m-2)+2(p_g-1)+1,$$

so that, by (2), $p_a \geqslant p_g$, whence $p_a = p_g$.

If $2p_g = \pi$, a single adjoint ϕ^{m-4} can be made to contain l, so that there exist at least ∞^1 adjoints ϕ^{m-3} having l as double line. Now the postulation of a double line on ϕ^{m-3} is $3(m-3)+1$; thus it follows that

$$p_a+\pi-1 \geqslant 3(\pi-p_g-1)+2,$$

i.e.

$$p_a \geqslant 2\pi-3p_g \geqslant p_g.$$

Some idea of the restriction imposed on the value of p_g may be gained from the fact that, in any case, if the canonical system of F is irreducible and without base-points, then* $2p_g \leqslant p^{(1)}+3$, where $p^{(1)}$ is the linear genus of F.

From the above theorem we deduce

2.311. *If* $n = 2\pi-2$ *and* C *is normal in* $S_{\pi-1}$, *then* $p_a = p_g = 1$. For the characteristic series of $|C|$ coincides with the canonical series, and thus $p_g = 1$.

If $n = 2\pi-2$, the system $|C|$ being free from exceptional curves, and F is irregular, it must be non-referable and normal in $S_{\pi-2}$, with genus $p_g \leqslant 1$. If $p_g = 1$, F is hyperelliptic, having $p_a = -1$; and if $p_g = 0$, F is elliptic, containing an elliptic pencil of elliptic curves. Any complete linear system of curves on F, free from fundamental curves, having genus π, is of dimension $\pi-2$.

From the above corollary and §2.211, we have

2.312. *If* $n = 2\pi-2$ *and* F *is normal in* S_π *or* $S_{\pi-1}$, *it is regular, with* $p_g \leqslant 1$.

For, if F is normal in S_π, then C is normal in $S_{\pi-1}$, so that $p_a = p_g = 1$. And if F is normal in $S_{\pi-1}$ and irregular, then $p_g = 1$, $p_a = 0$, which is impossible†.

---

* Enriques, 2, 390.

† Enriques, 5.

2.4. *On regular canonical surfaces.*

Suppose now that $|C|$ is special of index unity, and that, in fact, it coincides with the canonical system; this implies that $\pi = n+1$, and that $p_g = r+1$. Then the system $|C'|$, or one contained in it, is cut out by the quadrics of S_r. If F does not lie on a quadric, we have

$$p_a + \pi - 1 \geqslant \tfrac{1}{2}r(r+3).$$

Then, if $\pi \leqslant \frac{1}{2}r(r+1)$, it follows that

$$p_a \geqslant \tfrac{1}{2}r(r+3) - \tfrac{1}{2}r(r+1) + 1 \geqslant r+1 \geqslant p_g.$$

Hence

2.41. *If $|C|$ represents the complete canonical system, and F does not lie on a quadric, then, provided that $\pi \leqslant \frac{1}{2}r(r+1)$, F is regular.*

From the condition $\pi \leqslant \frac{1}{2}r(r+1)$, it follows that $p_g \geqslant 2\pi/r$; and we may note incidentally that in any case an upper limit for p_g is given by* $p_g \leqslant \frac{1}{3}(\pi+6)$.

A somewhat similar result is the following.

2.42. *If F is normal in S_r and does not lie on a quadric, and $\pi = n+1 = \frac{1}{2}r(r+1)$, then F is a regular canonical surface.*

For a prime section C of F cannot lie on a quadric of S_{r-1}, else F would lie on a quadric of S_r (§ 3.11). Hence the quadrics of S_r cut out on C a series of order $2\pi-2$ and dimension $\frac{1}{2}(r-1)(r+2) = \pi-1$, by hypothesis; thus the system $|2C|$ cuts out the complete canonical series on C, so that F is regular; and, since $|2C|$ is adjoint to $|C|$, it follows that F is a canonical surface.

Denoting by $N_r(h)$ the dimension of primals of order h in S_r, we obtain in a precisely similar manner the theorem:

2.43. *If $|hC|$ represents the complete canonical system, and F does not lie on a primal of order $h+1$, then if $\pi \leqslant N_r(h+1) - N_r(h)$, F is regular.*

The hypotheses of the theorem imply that

$$2\pi - 2 = (h+1)n, \quad p_g = N_r(h)+1.$$

---

* Enriques, 2, 394.

3. *Surfaces contained in given varieties.*

3.1. *Conditions for a surface to lie on a primal.*

The postulation formula of a surface for a primal of order l is derived from the Riemann-Roch theorem, and does not apply unless l exceeds a certain minimum value which it is, in general, difficult to specify. In this section we obtain some criteria for a surface F to lie on a given primal V_{r-1}^l when it is known that its section C lies on a V_{r-2}^l. We have first the theorem

3.11. *If F is normal in S_r and C lies on a quadric V_{r-2}^2, then F lies on a quadric V_{r-1}^2.*

Take any section C of F and let V_{r-2}^2 be a quadric containing it. The quadrics of S_r which pass through V_{r-2}^2 form a system of dimension $r+1$, and cut out on F a residual system contained in $|C|$. Since, by hypothesis, F is normal, $|C|$ is a complete system of dimension r, so that there is at least one quadric containing F. In the case $r=4$, we have the corollary

3.111. *If C has maximum genus in S_3, then F lies on a quadric, and is either a complete intersection or else the intersection residual to a plane.*

For, if $\pi = \pi_3(n)$, C lies on a quadric surface, and F is normal in S_4. Hence F lies on a quadric primal Q. If Q is not a cone, and is therefore without nodes, F must be the complete intersection of Q with another primal*. Thus, if n is odd, Q must be a cone; in this case it is easily shown that F is the intersection of Q with a primal of order $\frac{1}{2}(n+1)$, residual to a plane.

3.2. The theorem § 3.11 can obviously be generalized in two different ways. One of these is

3.21. *If C lies on a V_{r-2}^l of S_{r-1} and the system $|(l-1)C|$ has dimension $N_r(l-1)$, then F lies on a primal of order l.*

The proof is as above. The second analogue of § 3.11 is

3.22. *If F is normal in S_r and C lies on a V_{r-2}^l of S_{r-1}, then, provided that the varieties V_{r-2}^l have the intersections appropriate to the prime sections of a V_{r-1}^l, F lies on a primal of order l.*

Take $l-1$ arbitrary sections C and let $V^{(1)}, V^{(2)}, \ldots, V^{(l-1)}$ be the varieties V_{r-2}^l containing them. Since, by hypothesis, these can form the prime sections of a V_{r-1}^{l}, the primals of order l which contain them form a system* of dimension $r+1$. These cut out on F a residual system contained in $|C|$, and, since F is normal, there is at least one primal V_{r-1}^{l} containing F.

---

* Severi, 2.

This theorem has useful applications to surfaces of S_4. An immediate deduction is

3.221. *If F is normal in S_4 and C lies on a surface of order l of S_3, where $l^2 < n$, then F lies on a primal of order l.*

For, since $l^2 < n$, only one curve of order l in a plane can contain the n points of F lying in it; hence any two surfaces V_2^l, each containing a section C, must meet in the same curve of order l, and any three such surfaces must meet in l points; thus the previous theorem applies.

3.222. *If F is normal in S_4 and C lies on a pencil of surfaces of order l in S_3, where $\frac{1}{2}l(l+3) \leqslant n+1$, then F lies on a pencil of primals of order l.*

Let $C^{(1)}$ and $C^{(2)}$ be any two prime sections of F, situated in primes $S^{(1)}$ and $S^{(2)}$ respectively, the latter meeting in the plane ϖ. We can, in a single infinity of ways, choose a surface $V^{(1)}$, of order l, in $S^{(1)}$ containing $C^{(1)}$; and, by the restriction placed on the value of l, there is one and only one surface $V^{(2)}$, of order l, in $S^{(2)}$ which contains $C^{(2)}$ and meets ϖ in the same curve as $V^{(1)}$. We thus construct a set of $l-1$ surfaces $V^{(1)}$, $V^{(2)}$, ... lying in prime sections and satisfying the conditions of § 3.22. Hence F lies on at least one primal of order l passing through these surfaces. In a similar manner we can construct a second such primal, and therefore a pencil of primals, containing F.

3.3. *Surfaces lying on quadric and cubic primals S_4.*

It is a familiar fact that the general surface of S_3 is regular; an analogous results holds for surfaces on a quadric V_3^2, namely

3.31. *A non-singular surface which lies on a quadric of S_4 is regular.*

If the quadric has no node, F must be a complete intersection and therefore, by § 1.32, is regular. If the quadric has a node† (*i.e.* is a cone), it may be represented on S_3 by means of quadric surfaces passing through a line l and two arbitrary points P, Q whose join corresponds to the vertex of the cone. Thus F is represented by a surface f of order n, having a ν-ple line l and two general $(n-\nu)$-ple points P and Q. Moreover, since F has at most an improper node at the vertex of the cone,

---

* Bertini, **1**, 191.

† Since a general surface of S_4 imposes at most a finite number of nodes on primals which contain it, we restrict ourselves to primals of this type.

$$-2 \leqslant 2\nu - n \leqslant 2.$$

It is evident that f is regular provided that $p_g > 0$; and, if $p_g = 0$, we may show that F is rational. For the two systems of generating planes of the cone meet F in curves having at most a node at the vertex; if one system consists of rational curves, F is rational; if both systems consist of elliptic curves, *i.e.* cubics, F is regular, having $p_g = 1$; and if the systems consist of curves of higher genus, we can construct canonical curves on F so that $p_g > 0$, necessarily*. Hence in all cases F is regular, and, if $p_g = 0$, F is rational.

We can prove rather less than this for surfaces on a cubic primal; we have, in fact,

3.32. *If F lies on a cubic primal of S_4, it is regular or else $p_g = 0$.*

Since a cubic primal is a regular threefold, it follows from § 1.33 that, if F belongs to a linear system, the variable part of whose characteristic system is irreducible or, if reducible, is not compounded of an irrational pencil, then F is regular. But this criterion is not easy to apply; we therefore fall back on other considerations.

If the primal V in question has fewer than six nodes and contains no plane, it has been shown by Fano† that V contains only complete intersections, so that F is regular. In order to discuss the remaining cases we observe that, if F lies on a primal V having ν nodes, the same surface, or a projectively specialized form of it, will lie on a primal with $\nu' > \nu$ nodes. Since we are dealing with invariantive properties, we therefore suppose F to lie a ten-nodal cubic primal V, which may be represented on S_3 by means of quadrics having five base-points, whose joins in pairs correspond to the nodes of V. In this representation F corresponds to a surface f of order α having five β_i-ple points of general type, in arbitrary position $(i = 1, 2, \ldots, 5)$, where

$$-2 \leqslant \alpha - (\beta_i + \beta_j) \leqslant 2 \quad (i \neq j),$$

since F has at most improper nodes at the nodes of V. It is easy to see that

---

* *Cf.* Enriques, 3, 73.

† Fano, 2.

f is either regular (when $p_a \geqslant 0$) or else $p_g = 0$. This result can obviously be generalized; we have

3.33. *If F lies on a rational threefold whose prime sections are represented on S_3 by surfaces with only isolated base-points, then either F is regular or else $p_g = 0$.*

3.4. *Surfaces containing a rational pencil of plane curves.*

If F lies on a rational threefold generated by planes, and $r > 4$, the planes meet it in a system of non-singular curves which, if the threefold is not a cone, are without base-points*. We shall consider such surfaces in this and the following sections.

We first establish the following lemma:

3.41. *If F contains a rational pencil of plane curves, it cannot be elliptic.*

For, if it were, it would follow by §1.2 that the curves could not be cubics; hence F would contain an elliptic pencil of curves and a rational pencil of curves of genus greater than unity, so that $p_g > 0$, necessarily†; this contradicts the hypothesis.

We shall now show that

3.42. *If F contains a rational pencil of plane curves of order ν, without base-points, it is regular if $4\pi \geqslant (n-1)(\nu-1)$, and referable otherwise.*

Suppose that F is normal in S_r; then the planes generate a rational normal V_3^{r-2} which may, in particular, be a cone. This may be represented‡ on S_3 by means of scrolls of order $r-2$ with a common $(r-3)$-ple line l, and $r-3$ simple lines l' meeting l.

Since a prime section of F is a curve C of order n and genus π on a rational scroll of order $r-2$, whose generators are ν-secant to C, we have

$$\pi = (\nu-1)(n-1) - \binom{\nu}{2}(r-2). \tag{1}$$

In the representation of V_3^{r-2} on S_3, F corresponds to a surface f of order n, for which l is $(n-\nu)$-ple and the lines l' are ν-ple. The arithmetic genus

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* Since F is non-singular, the curves can have at most one base-point, at the vertex of the cone.

† Enriques, 3, 73.

‡ An alternative method is indicated in § 3.31.

p_a of F is found by calculating that of f, which may be shown, by Noether's formulae*, to be

$$p_a = \binom{\nu-1}{2}(n-1) - 2\binom{\nu}{3}(r-2). \tag{2}$$

This, by (1), is equivalent to

$$p_a = \frac{(\nu-2)}{6}\{4\pi - (\nu-1)(n-1)\}. \tag{3}$$

Evidently f is a regular surface except when p_a is negative, in which case $p_g = 0$. In this case f is referable, except possibly when $p_a = -1$, when it may be elliptic; this possibility is, however, excluded by § 3.41. Hence the result stated follows from (3).

By the same formulae, the linear genus of F is found to be

$$p^{(1)} = 1 + 3(\nu-3)(\nu-1)n - 4\nu(\nu-3) - 2(r-2)\nu(\nu-1)(\nu-3). \tag{4}$$

The following are two particular cases of the above theorem.

3.421. *If C is hyperelliptic, then F is rational.*

For the g_2^1 on C is cut out by a rational pencil of conics, so that F is rational. This result is due to Castelnuovo†.

3.422. *If C contains a g_3^1, F is regular or referable to an elliptic scroll.*

Unless F is the complete intersection of a quadric and a cubic primal of S_4, the g_3^1 is cut out by a rational pencil of cubics; if these are rational, then F is rational, and if they are elliptic, F is regular or referable to an elliptic scroll, according as $2\pi \gtrless n-1$.

This completes results given by Bronowski‡.

3.5. *Surfaces lying on a rational cone.*

Suppose that F lies on a rational normal cone K_3^{r-2} $(r > 4)$ which, if $r \neq 6$, is generated by planes passing through the vertex and which, if $r = 6$, may be the cone projecting a Veronese surface from an external point. If the cone K is generated by planes, an arbitrary prime through the vertex O meets K in a cone containing a section C of F. The generators of the cone meet C in a number, $\chi+1$ say, of variable points; and since, when $r>4$, F has at most a simple point at O,

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* Noether, 1.

† Castelnuovo, 4.

‡ Bronowski, 1.

$$(r-2)(\chi+1)=n \quad \text{or} \quad (r-2)(\chi+1)=n-1.$$

Thus, by § 1.41, it follows that C has maximum genus $\pi_{r-1}(n)$. In the first case F is the complete intersection of K with a primal of order $\chi+1$, not passing through O; in the second it is the intersection of K with a primal V of order $\chi+2$, passing through O, residual to $r-3$ planes of K. Since O is a simple point of F, it must be simple for V, so that the $r-3$ planes are coprimal. Hence they meet a fixed directrix plane of K through O in lines.

The first case is included in the preceding theorem, from which it follows that F is regular. In the second case, the representation of K on S_3 is obtained from that given in § 3.4 by supposing the $r-3$ lines l' to be coplanar and concurrent at a point P of l; the plane ϖ of the lines l' represents the directrix plane of K, and P represents the vertex O. Then the intersection of K with a primal of order $\chi+2$ passing through O, residual to $r-3$ generating planes, corresponds in S_3 to a surface f of order

$$(\chi+2)(r-2)-(r-3),$$

for which the lines l' are $(\chi+2)$-ple, l is $(\chi+1)(r-3)$-ple, and the point P is multiple of order $(\chi+1)(r-3)+1$; here, it will be noted, the order of the primal is such that it contains the $r-3$ generating planes without, in consequence, containing their directrix. In this case it is seen that f is regular.

If $r=6$, the cone K_3^4 may be generated by planes or, alternatively, be the cone projecting a Veronese surface from an external point O. In the latter case it may be represented on S_3 by means of the quadrics touching a fixed plane at a fixed point*. Now the only non-singular surfaces which K contains are the complete intersections of K with primals not passing through O; such surfaces are represented on S_3 by surfaces of order 4ν $(\nu=1, 2, \ldots)$ whose only singularity is a point of contact of ν sheets; and these surfaces are regular. Moreover, by § 1.4, it follows that they have maximum sectional genus. Hence, in conclusion, we obtain the result:

3.51. *If F lies on a rational normal cone, it is regular and has maximum sectional genus.*

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* Roth, 2.

3.6. *On surfaces of maximum sectional genus.*

We now prove that

3.61. *If F has maximum sectional genus, it is regular.*

The case $r = 4$ has already been dealt with in §§ 3.111, 3.31. Suppose then that $r > 4$ and that $\pi = \pi_{r-1}(n)$. By § 1.42, if $n < 2r-2$, C is normal and non-special, whence, by § 2.21, F is regular with genus zero; also, by § 4.12, it is rational. If $n = 2r-2$, C is the canonical curve of genus π, so that, by § 2.311, F is regular. If $n > 2r-2$ and $r \neq 6$, C lies on a rational normal scroll of order $r-2$, whose generators are $(\chi+1)$- or $(\chi+2)$-secant to C, where χ is determined by § 1.41. The locus of these generators is a rational normal V_3^{r-2} generated by planes (in particular, a cone), so that, by the preceding sections, F is regular or referable; that it is regular follows from § 3.42 by virtue of the fact that in the present case $\nu = \chi+1$ or $\chi+2$. If $r = 6$ and C lies on a quartic scroll, the same reasoning holds. If, however, C lies on a Veronese surface, it is easily shown* that F lies on a cone projecting such a surface from an external point and hence, by § 3.51, that it is regular.

This result extends to any surface for which C lies on $\binom{r-2}{2}$ linearly independent quadrics of S_{r-1}. Thus, from § 1.4, we have

3.62. *If* $\pi = \pi_{r-1}(n)-k$, *where* $k \leqslant (n-2r+2)/(r-2)$, *then F is regular.*

3.63. *If* $n > 2r-2$, *and* $\pi > \frac{1}{2}\chi_1\{2n-r-\chi_1(r-2)\}$, *where*

$$\frac{n-r}{r-2} \leqslant \chi_1 < \frac{n-r}{r-2}+1,$$

then F is regular.

These theorems may be summarized thus:

If, for a given value of r, n and π are sufficiently large, then F is regular.

In the above results we have assumed that $n > 2r-2$; if $n = 2r-2$, we have the theorem:

3.64. *If* $n = 2r-2$ *and* $\pi = \pi_{r-1}(n)-1$, *then F is regular, with genus zero.*

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* Roth, 2.

For, by § 1.41, if $n = 2r-2$, $\pi_{r-1}(n) = r$ and $\pi_r(n) = r-2$. Hence, if $\pi = \pi_{r-1}(n)-1$, the characteristic series of $|C|$ must be complete (since C is necessarily normal) and non-special, so that F is regular and of genus zero (§ 2.21).

4. *On surfaces containing a given system of curves.*

4.1. In this section we show how the regularity or referability of a surface may be deduced from the hypothesis that it contains a pencil or a net of curves of specified type. The fundamental result in this field is a theorem due to Enriques*, itself a generalization of a theorem of Noether's:

4.11. *If F contains a pencil, of genus p, of rational curves, it is referable to a scroll of genus p.*

By means of this theorem it may be shown† that

4.12. *If $n > 2\pi-2$, or $r > \pi$, F is rational or referable.*

Here, it is understood, the characters n, π, and r may be the grade, genus, and dimension of any linear system on F.

4.2. We suppose first that there is given on F a net of curves of specified genus. In the case where these are elliptic or hyperelliptic, by reasoning from first principles Castelnuovo‡ has established the following two theorems:

4.21. *If F contains a net of elliptic curves, it is rational or referable to an elliptic scroll; in the latter case the curves may be represented by a net of directrices of the scroll.*

4.22. *If F contains a net of hyperelliptic curves of genus $p > 1$, and the characteristic series of the net is not compounded of the g_2^1 of the generic curve, then F is rational or referable. If the characteristic series is compounded of $\nu \geqslant p$ pairs of the g_2^1, then F is rational or referable.*

4.3. We now wish to see how far it is possible to extend these results. We begin by supposing that F contains a net of curves of genus 3, free from base-points and exceptional curves; if F is not rational, the latter condition implies that the net does not contain an infinity of reducible curves. If the grade ν of the net exceeds 4, F is rational or referable (§ 4 . 12), so there remain to be considered the cases $\nu \leqslant 4$.

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* Enriques, **4**.

† Castelnuovo-Enriques, **1**.

‡ Castelnuovo, **5**.

If $\nu = 2$, F may be represented on a double plane having for branch-curve an irreducible, non-singular octavic: this follows by virtue of the assumptions made with regard to the nature of the net. Hence F is regular, with $p_g = 3$, and the net of curves is the canonical system. If $\nu = 3$, F may be represented on a triple plane having for branch-curve an irreducible decimic, whose only singularities are cusps, whence it follows that F is regular*.

If $\nu = 4$, the characteristic series of the net cannot be compounded of a g_2^1, since the representation of F on the double plane, obtained on this assumption, would imply the existence of an infinity of reducible curves in the net. If F is regular, $p_a = p_g \leqslant 1$, and if it is irregular, $p_g \leqslant 1, p_a = -1$. In the latter case, however, a system of grade 4 and genus 3 would have dimension unity (§ 2 . 311). Hence, in conclusion, we have

4 . 31. *If F contains a net of curves of genus* 3, *free from base-points and exceptional curves, it is regular or referable.*

Suppose now that the curves of the net are of genus 4, and subject to the same restrictions as before. If $\nu > 6$, F is rational or referable. If $\nu = 2$, F may be represented on a double plane having for branch-curve a general decimic, so that F is regular, with $p_g = 6$. If $\nu = 3$, F is represented on a triple plane having for branch-curve an irreducible curve of order 12, of general character, so that F is regular or referable†.

If $\nu = 4$, the characteristic series of the net must be simple, for the same reason as before; since the grade of the net equals its genus, F is regular or else‡ $p_g = 0$.

If $\nu = 5$, and the characteristic series is non-special, then $p_g = 0$. If it is special, the residual with respect to the canonical series is a point. This must vary with the curves of the net, so that its locus is either a curve unisecant to the curves of the net, or else fills the surface F. In the first case, however, the net contains (partially) an exceptional curve, contrary to hypothesis; and in the second it is evident that $p_g = 0$.

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* Du Val, 1.

† Du Val, 1.

‡ Du Val, 2. Dr. Du Val remarks that his theorem here quoted is subject to exception, but that it holds in the present case.

If $\nu = 6$ and F is irregular, it may be either elliptic or hyperelliptic (§2.311). Thus we have

4.32. *If F contains a net of curves of genus* 4, *free from base-points and exceptional curves, it is either regular or else has* $p_g = 0$, *except possibly when the grade of the net equals* 6, *in which case F may be hyperelliptic.*

4.4. *Surfaces containing a rational pencil of curves.*

We have seen in §3.4 that if F contains a rational pencil of plane curves it is regular or referable; we consider now some other types of pencils, beginning with the case of elliptic curves; these we may suppose to be without base-points, since otherwise F would certainly be rational or referable (§4.12).

It has been shown by Enriques* that, if F contains a pencil of elliptic curves, then $p^{(1)} = 1$, and the canonical system is compounded of the pencil; also that, in general, the genus of the pencil is $p_g - p_a$, the exceptional cases being the elliptic surfaces and the surfaces referable to an elliptic scroll. Hence we have

4.41. *If F contains a rational pencil of elliptic curves, it is regular, referable, or elliptic.*

Now, if F is elliptic, a certain number of its elliptic curves must reduce to multiples of other elliptic curves (§1.2); and this cannot happen if the curves are quartics or of prime order. Thus we have

4.42. *If F contains a rational pencil, without base-points, of elliptic curves of order* 4 *or of order p* (*where p is any prime*), *then it is regular or referable to an elliptic scroll.*

From this result follows†

4.43. *If C contains a g_4^1, F is regular or referable to a scroll.*

For, unless F lies on a quadric of S_4 (in which case it is regular, by §3.31) the g_4^1 is cut out by a rational pencil of quartics† which may be rational, elliptic, or plane curves; in the first case F is rational (§4.11), in the second it is regular or referable (§4.42), and in the third it is regular or referable (§3.42).

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* Enriques, 5.

† Bronowski, 2.

4.5. If F contains a rational pencil of curves of genus 2, it obviously contains an involution, set up by the g_2^1 on each curve, by which it can be represented on a double plane. By this means, Bronowski* has established the result:

4.51. *If F contains a rational pencil of curves of genus 2, it is regular or referable.*

It now follows that

4.52. *If F contains a rational pencil of quintics, it is regular or referable.*

For, if the quintics are rational, F is rational (§ 4.11); if they are elliptic, F is regular or referable (§ 4.42); so also if they are of genus 2 (§ 4.52), and if they are plane curves (§ 3.42).

5. *On continuous systems of curves.*

5.1. *Continuous systems of reducible curves.*

If F is an irregular surface and C is its prime section, then, if C is virtually effective, it defines† on F a continuous system $\{C\}$ composed of $\infty^{p_g-p_a}$ non-equivalent linear systems; the system $\{C\}$ is unique, complete, and irreducible as an aggregate of linear systems, though not necessarily as an aggregate of curves‡. In fact, when the curves $|C|$ are normal, the only irreducible curves of $\{C\}$ are those contained in the system of prime sections.

Suppose, then, that C is a normal curve; all the linear systems contained in $\{C\}$, with the exception of $|C|$, are composed of reducible curves. It is a familiar fact that such systems may be of two kinds; the first kind is such that the generic curve consists of a single variable (irreducible) part of a linear system, together with one or more fixed parts, while in the second kind, the variable part of the generic curve is compounded of a pencil. If $|\Gamma|$ is a linear system of $\{C\}$, not coincident with $|C|$, the fixed parts of $|\Gamma|$ must be the same for all linear systems of $\{C\}$ other than $|C|$; otherwise the curve Γ cannot, by varying continuously, coincide with a curve of $\{C\}$ not lying in $|\Gamma|$. Moreover, since Γ can move into coincidence with certain curves of $|C|$, it follows that these fixed parts must also lie in a linear system ∞^s ($s > 0$) of primes. If we assume that F is non-singular,

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[*] See Bronowski, 2.

[†] Severi, 1.

[‡] Albanese, 1.

the variable part of the curve Γ has no fixed points. Now suppose that $|\Gamma|$ is a system of the first kind; then all linear systems of $\{C\}$, other than $|C|$, must be of the first kind, since an irreducible curve cannot, by varying continuously, coincide with a reducible one consisting of a number of unconnected parts. Similarly, it is clear that, if $|\Gamma|$ is of the second kind, then all linear systems of $\{C\}$, other than $|C|$, are of the second kind. Also, in this case, the variable part of $|\Gamma|$ cannot be compounded of a *rational* pencil, since $\{C\}$ would then be composed of equivalent linear systems. Thus, in conclusion, we have the result:

5.11. *If C is virtually effective and the generic curve of the system $\{C\}$ which it defines is reducible, it consists either of a single, variable, irreducible curve with a number of fixed parts, or else a number of curves of an irrational pencil, with a number of fixed curves. The aggregate of fixed curves lies in a linear system of primes, of dimension greater than zero.*

5.2. If the genus of C exceeds the maximum genus of an irreducible curve of the same order in S_r, the characteristic series of $|C|$ is complete; and if F is irregular and C virtually effective, the continuous system $\{C\}$ defined by C is such that its generic curve is reducible. Thus

5.21. *If C is virtually effective and $\pi > \pi_r(n)$, then either F is regular or else the generic curve of $\{C\}$ is reducible.*

In order to apply this result, we may require to obtain sufficient conditions for C to be virtually effective. We now show that

5.22. *If F is non-referable and $\pi < n+1$, then $|C|$ is non-special and virtually effective.*

For, if F is non-referable, $p_a \geqslant -1$, and if $\pi < n+1$, it follows that $2\pi - 2 - n < n$, so that $|C|$ is non-special. Thus the virtual dimension of $|C|$ is $n-\pi+p_a+1 \geqslant 0$.

Combining these two results, we have the theorem

5.23. *If $\pi_r(n) < \pi < n+1$, and F is irregular and non-referable, then the generic curve of $\{C\}$ is reducible.*

The usefulness of this criterion lies in the theorem § 5.11 which describes how the curves of $\{C\}$ are distributed. Thus, the variable part of C is either a single curve of order less than n, or else a number of curves of an irrational pencil, of total order less than n; and the fixed part lies in a linear system of primes. The above theorems have a particularly simple application to surfaces of S_4, as will be seen later. In this connection it is interesting to note that no irregular surface, referable or otherwise, is known with characters which satisfy the conditions of § 5.23; a referable surface of S_5, for which $\pi = \pi_5(n)$, is the octavic surface of sectional genus 3, which represents the congruence of chords of an elliptic quartic of S_3.

In cases where $|C|$ is non-special it is often possible to calculate its virtual dimension, for the purpose of using § 5.21, by means of the following lemma which is derived from a result due to Castelnuovo*.

5.24. *If the systems $|2C|$, $|3C|$, ... cut out non-special series on C, with deficiencies δ_2, δ_3, ..., then the redundance of $|C|$ is equal to*

$$\delta_2 + \delta_3 + \ldots + \delta_k,$$

where $|kC|$ is the first multiple of $|C|$ such that $|hC|$ is regular for $h \geqslant k$.

For, if we denote by σ_h the redundance of $|hC|$, it has been shown by Castelnuovo that

$$\sigma_h = \sigma_{h+1} + \delta_{h+1}.$$

Now for all values of h exceeding a certain number $k-1$, the systems $|hC|$ are regular. Hence, writing down the above relation and summing for values of h from 1 to $k-1$ inclusive, we obtain the required result. This may be used to determine an upper limit for the redundance of $|C|$ and thence a lower limit for p_a, as will be seen later. Incidentally, *if F is normal in S_r and $|C|$ is non-special, an upper limit for p_a is given by*

$$p_a \leqslant r - n + \pi - 1,$$

since, if $p_a > r - n + \pi - 1$, the system $|C|$ is special†.

5.3. Suppose that F is of sectional genus $\pi = \frac{1}{2}(n+3)$, and that $r \geqslant n - \pi + 1$. By § 5.22, if F is irregular and non-referable, C is virtually effective and therefore defines a continuous system $\{C\}$, the generic curve of which may, possibly, be reducible. Let $|C_1|$ be a system contained in $\{C\}$; on any curve C_1, $|C|$ cuts out a series $g_n^{\,r}$ which, by hypothesis, is special. Since $2\pi - 2 = n + 1$, the sets of $g_n^{\,r}$ are residual to a single point; we may thus fix rationally a point P of C_1. As C_1 varies in a pencil ϖ of $|C_1|$ $\big($which has dimension at least $n - \pi = \frac{1}{2}(n-3) \geqslant 3\big)$, the point P describes a rational curve Γ which meets any curve of ϖ in one variable point and which is clearly not a fixed part of $\{C\}$. If Γ varies with ϖ, F is referable; if, however, Γ remains fixed, it cannot pass through the base-points of ϖ and is unisecant to any curve of $\{C\}$, including the system $|C|$. Now $|C+\Gamma|$ is adjoint to the curves of $|C_1|$, so that, if, when C_1 varies within $\{C\}$, Γ remains fixed, the system $|C+\Gamma|$ is adjoint to any system contained in $\{C\}$, which is absurd. Hence

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* Castelnuovo, 6, 309.

† Castelnuovo, 6, 300.

5.31. *If* $\pi=\frac{1}{2}(n+3)$ *and* $r \geqslant \frac{1}{2}(n-1)$, *then* F *is regular or referable.*

Since $2\pi-2-n=1$, it follows that $p_g \leqslant 1$.

Precisely analogous theorems may be established for surfaces of sectional genus $\pi=\frac{1}{2}(n+i)$, where $i=4$, 5, 6, and where $r \geqslant n-\pi+1$. Consider, for example, the case $i=4$; with the above notation, if F is irregular and non-referable, the systems $|C|$ cuts out on C_1 a special $g_n{}^r$, the sets of which are residual to a $g_2{}^1$ or to a pair of points. If $i=5$, the residual may be a $g_3{}^1$, or a $g_2{}^1$ and a point, or three points; if $i=6$, it may be a $g_4{}^1$ or a $g_3{}^1$ and a point, and so on. The case in which the curves of $|C_1|$ are reducible makes only a slight modification of the argument necessary. Then, using the results of §§ 3.421, 3.422, 4.43, we deduce

5.32. *If* $\pi=\frac{1}{2}(n+i)$ $(i=4, 5, 6)$, *and* $r \geqslant \frac{1}{2}(n-i+2)$, *then* F *is regular or referable.*

When $i=4$, we have $p_g \leqslant 1$, and when $i=5$, 6, it is clear that $p_g \leqslant 2$.

6. *The classification of surfaces.*

6.1. We now apply the preceding theorems to the examination of non-singular surfaces in S_4; in the more detailed analysis the classification of skew curves, made by Noether*, is indispensable.

In the first place, those surfaces for which $\pi < \frac{1}{2}(n+2)$ are rational or referable (§ 4.12), and in the second, those for which

$$\pi_4(n) < \pi < n+1$$

are regular or referable, or else the generic curve of $\{C\}$ is reducible (§ 5.23). In the latter case, it follows from § 5.11 that the fixed part of C must be a line or a plane curve, so that the various possibilities are easily investigated. This crude form of § 5.21 can, as before remarked, often be improved upon, as examples will show.

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* Noether, 2.

6.2. We observe that, if $n < 6$, the surface F has rational, elliptic, or hyperelliptic sections, and is therefore rational. Since, by § 1.4, $\pi_3(6) = 4$, it follows that, when $n = 6$ and $\pi \leqslant 3$, F is rational or referable (§ 4.12); when $\pi = 4$, F is regular, with $p_g = 1$ (§ 2.312).

For $n = 7$, $\pi_3(n) = 6$; when $\pi \leqslant 4$, F is rational or referable (§ 4.12), and when $\pi = 5$, regular or referable (§ 5.31). When $\pi = 6$, F is regular § 3.61); and in all cases $p_g \leqslant 2$.

6.3. For $n = 8$, we have $\pi_3(n) = 9$, $\pi_4(n) = 5$. When $\pi \leqslant 4$, F is rational or referable (§ 4.12), and when $\pi = 5$, it is regular (§ 2.312). When $\pi = 8$, 9, F is regular, by §§ 3.61, 3.62. When $\pi = 6$, 7, and $|C|$ is virtually effective, F is regular or else the generic curve of $\{C\}$ is reducible (§ 5.21). Since in both cases $\pi < n$, $|C|$ is certainly non-special (§ 5.22); we shall show, by means of § 5.24, that $|C|$ is also virtually effective. When $\pi = 6$, C cannot lie on a quadric, so that the system $|2C|$ cuts out on C a (non-special) series of order 16 and dimension 9 at least; thus $\delta_2 \leqslant 1$; and the series cut out on C by $|hC|$ $(h > 2)$ have deficiency zero. It follows that $\sigma \leqslant 1$; and since F is normal in S_4 and thus $p_a + \sigma = 1$, we have $p_a \geqslant 0$, so that $|C|$ is virtually effective. Also, by the remark made in § 5.2, $p_a \leqslant 1$. Using the same reasoning, we find that when $\pi = 7$, $\sigma = 0$, so that $p_a = 1$, and hence $|C|$ is again virtually effective.

In order to prove that, for $\pi = 6$, 7, F is regular, it remains to show that, on the supposition that it is irregular, the generic curve of $\{C\}$ cannot be reducible. Consider first the case $\pi = 6$; if a curve Γ of $\{C\}$, situated in S_4, is reducible, suppose that its fixed part is a line l, so that the variable part, if irreducible, is a septimic at most of genus 3, meeting l in at least four points. If it is of genus 3, F is regular (§ 4.31); and if it is of genus 2, F is referable (§ 4.22), while it cannot be elliptic since it would meet l in six points, leading to a contradiction. The fixed part cannot be a conic, since the variable part would be a sextic, at most of genus 2, meeting the conic in at least five points. The same reasoning applies *a fortiori* to the case $\pi = 7$; and in both cases it is evident, when we consider the possible degenerations of C, that $\{C\}$ cannot be compounded of an irrational pencil. Hence, when $n = 8$, F is regular or referable, and $p_g \leqslant 5$.

6.4. For $n = 9$, it follows from § 1.4 that $\pi_3(n) = 12$, $\pi_4(n) = 7$. When $\pi \leqslant 5$, F is rational or referable (§ 4.12); when $\pi = 6$, 7, it is regular or referable (§§ 5.31, 5.32). The type $\pi = 11$ does not exist*; and when $\pi = 12$, F is regular (§ 3.61).

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* Noether, 2.

We have finally to consider the cases $\pi = 8, 9, 10$. In the first two of these $|C|$ is non-special (§5.22), and we shall show that it is virtually effective. When $\pi = 8$, the series cut out by $|2C|$ and $|3C|$ on C are non-special; from these we have $\delta_2 \leqslant 1$, $\delta_3 \leqslant 1$ ($\delta_3 = 0$ unless, exceptionally, C can lie on a cubic surface), and $\delta_h = 0$ $(h > 3)$. It follows that $\sigma \leqslant 2$ and, since F is normal, that $p_a \geqslant 1$, while $p_a \leqslant 2$.

When $\pi = 9$, the same reasoning shows that $\sigma = 0$ and that, in consequence, $p_a = 3$; thus in both cases $|C|$ is virtually effective and therefore, if F is irregular, the generic curve of $\{C\}$ is reducible. If $\pi = 8$, the most favourable hypothesis is that the fixed part of the curve is a line l and that the variable part is an octavic $C_8{}^5$ meeting l in four points; but this is impossible since, even if $C_8{}^5$ is the sub-species which lies on a cubic scroll, it cannot have a quadrisecant. Other degenerations of C, such as a conic and a sextic, are obviously impossible; and, *a fortiori*, the same is true of the case $\pi = 9$.

There remains to be considered the case $\pi = 10$; this is included in §2.42, from which it follows that, if $n = 9$, $\pi = 10$, F is a regular canonical surface having $p_g = 5$. Hence, in conclusion, if $n = 9$, F is regular or referable, and $p_g \leqslant 8$.

6.5. We come now to the surfaces of order 10. When $\pi \leqslant 5$, F is rational or referable (§4.12); when $\pi = 6$, F is regular provided that it is not normal in S_4 (§2.312); and when $\pi = 7, 8$, F is regular or referable (§5.32). We have $\pi_3(10) = 16$, and from §§3.61, 3.62 it follows that F is regular when $\pi = 15, 16$; the types $\pi = 13, 14$ are non-existent*.

Next consider the case $\pi = 12$; since $\pi_4(10) = 9$, F must be normal, and since C lies on a cubic surface† of S_3, F lies on a cubic primal V (§3.221). We may, for present purposes, suppose this to be ten-nodal‡; from the representation of V on S_3 we readily show that F is the intersection of V with a quartic primal, residual to a quadric, and hence that it is regular, having $p_g = 6$.

There remain to be discussed the cases $\pi = 9, 10, 11$. When $\pi = 9$, we show, with the previous notation, that $\delta_2 \leqslant 2$, $\delta_3 \leqslant 2$, and that $\delta_h = 0$ $(h > 3)$, assuming that F is normal in S_4, since otherwise it is certainly regular (§3.61). Thus $\sigma \leqslant 4$, and $p_a \geqslant -2$, so that $|C|$ is virtually effective. When $\pi = 10$ we find, similarly, that $\sigma = 2$, $p_a \geqslant 1$, and that $|C|$ is virtually effective. In the first case $p_a \leqslant 2$, and in the second, $p_a \leqslant 3$.

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* Noether, 2.

† C may, exceptionally, lie on a quadric, but it is easily seen that F cannot.

‡ *Cf.* §3.32.

It follows that when $\pi=10$, F is regular or else the generic curve of $\{C\}$ is reducible. In the latter case, suppose the curve to consist of a fixed line l and a variable curve C_9 (the maximum genus of which is 7) quadrisecant to l. Then the composite curve must lie on the ∞^2 quadrics containing C_9, so that l and C_9 must lie on a cubic scroll; but this is impossible since l would then be trisecant to C_9, whether it were a generator or the directrix. Other degenerations of C, including the case in which $\{C\}$ is compounded of an irrational pencil, are easily shown to be impossible.

When $\pi=11$, there are two cases to be considered. If $|C|$ is special, it must have index of speciality unity and coincide with the canonical system. Then the system $|2C|$ is adjoint to $|C|$; this system, or one contained in it, is cut out by the quadrics of S_4, so that, since F does not lie on a quadric (for C does not), we have $p_a+\pi-1\geqslant 14$, or $p_a\geqslant 4$, whence it follows that $|C|$ is virtually effective. Then, as above, F must be regular. If $|C|$ is non-special, we find that $\delta_h=0$ $(h=2,\ 3,\ \ldots)$, so that $\sigma=0$ and $p_a=4$. Thus $|C|$ is virtually effective and we may argue as before.

When $\pi=9$ and F is normal in S_4, we have seen that $|C|$ is virtually effective; thus, if F is irregular, C defines a continuous system $\{C\}$ of curves of order and grade 10 and genus 9. If the generic curve C_1 of the system is irreducible, it lies on ∞^2 quadrics (§1.4); thus C_1 imposes twelve conditions on a quadric which contains it. Since every curve of $\{C\}$ meets C_1 in ten points, it follows that there is a quadric containing it and C_1, and hence that the curves of $\{C\}$, other than C_1, are contained in a linear system of dimension 2, which is absurd. If the possibility that C_1 is reducible be considered, it is found that the same conclusion holds.

From the above analysis it follows that, *when* $n\leqslant 10$, *F is regular or referable except possibly when* $n=10$, $\pi=6$, *and F is normal in* S_4. In this case F may be hyperelliptic ($p_g=1$, $p_a=-1$) or elliptic; it has, in fact, been shown[*] that surfaces of the first category exist, while those of the second do not. It may be added that the minimum order for a non-singular elliptic surface, without exceptional curves, is 18.

6.6. *Classification by sectional genus.*

The previous classification of non-singular surfaces of order $n\leqslant 10$ can be completed in detail[†] by considering the surfaces according to their sectional genera. For small values of π, nearly all the surfaces that occur are rational or referable; such surfaces have the property that the process of successive adjunction, applied to any linear system of curves lying on them, terminates after a finite number of stages. This means that, after a certain stage, the genera of the successive adjoints form a decreasing series. The surfaces which are most readily classified by this method are obviously those for which $\pi' < \pi$; in that case, if $|C'|$ is irreducible and of dimension $r' > 2$, F can be represented on a surface F' whose prime sections are of genus π', if $|C'|$ is simple; if, however, $|C'|$ is compound of order ν, F and F' are in $(1, \nu)$ correspondence.

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[*] Roth, 7.

[†] It is perhaps superfluous to add that we consider only general types, and not subspecies.

For $\pi = 2$, the classification has been performed by Castelnuovo*, and for $\pi = 3$ by Castelnuovo† and Scorza‡, from whose work it follows that

6.61. *If* $\pi \leqslant 3$, *F is rational or referable.*

The one case in which F is referable is $n = 8$, $\pi = 3$; F then lies on a quadric of S_5, and represents the congruence of chords of an elliptic quartic. This is the non-singular irregular surface of minimum order.

We now come to the case $\pi > 3$. Since the curves C' are of order $2\pi - 2$, it follows that, when $\pi = 4$ and C' is irreducible, we have $\pi' \leqslant 4$.

When $\pi = 4$ and $n > 6$, F is rational or referable (§ 4.12). In the case $n = 6$, C is the canonical curve of genus 4, and F is therefore a regular surface of genus 1 (§ 2.311). In the other cases, when C' is irreducible, it follows that $\pi' < \pi$, $r' = 3$, so that the classification can be carried out by using the results for surfaces having $\pi < 4$. In those cases where $|C'|$ is reducible, it is seen to be compounded of a pencil of conics§.

Generally, for the classification of rational surfaces we use the following theorem, due to Castelnuovo‖:

6.62. *If F is rational, C′ is irreducible, and* $n > 2\pi - 1$, *then* $\pi' \leqslant \pi - 2$.

When C' is reducible, the curves C are hyperelliptic, so that the exceptional case presents no difficulty. Hence, when $\pi = 5$ and F is rational, it may be classified provided that $n > 9$; moreover, since the curves C' are octavics, if they do not lie in prime sections (in which case $\pi' \leqslant \pi$) and are irreducible, then $\pi' \leqslant 5$ (§ 1.41). It is found¶, however, that there are no rational or referable surfaces with $\pi' = 5$; and when $n = 9$ and F is rational, then $r = 5$ and hence $\pi' \leqslant 4$.

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[*] Castelnuovo, 4.

[†] Castelnuovo, 7.

[‡] Scorza, 1.

[§] Roth, 4.

[‖] Castelnuovo, 3.

[¶] Roth, 5.

When $\pi = 6$, it may be shown* that, if F is referable, $\pi' \leqslant 4$. By §6.62, we can classify all rational surfaces with $\pi = 6$ for which $n > 11$. When $n = 11$, $r = 6$, so that, if C' does not lie in a prime section, we must have $\pi' \leqslant 4$ (§1.41). Thus the only cases in which a difficulty may arise are $n = 9$ ($\pi' = 6, 7, 8, 9$) and $n = 10$ ($\pi' = 6$). But it is known† that the types having $\pi' = 6$ cannot occur, so that there remain the possibilities $n = 9$, $\pi' = 7, 8, 9$, $r = 4$. If in these cases we impose an additional nodal base-point on the system of plane curves which represent the prime sections, we obtain a pencil of curves of genus 5, having four (and only four) simple base-points; and using the method of analysis given by Du Val‡ for plane curves of genera 3 and 4, we find that such types are non-existent. In conclusion, therefore, when $\pi = 6$ and F is rational or referable, we have $\pi' \leqslant 5$; thus the results obtained from the classification of rational and referable surfaces having $\pi < 6$ suffice for the classification of those having $\pi = 6$.

6.7. Surfaces which are not rational or referable begin to appear for values of $\pi > 3$; these cannot be classified by any systematic process and frequently involve considerations of a difficult nature. Those for which $\pi = 4, 5, 6$ have, however, been classified§; from these investigations the following results are obtained:

6.71. *If* $n < 8$, *F is regular.*

6.72. *If* $\pi < 6$ *and F is irregular, it is referable to an elliptic scroll.*

6.73. *If* $n = 8$ *and F is irregular, it is generated by an elliptic pencil of conics.*

Other results provide information about the orders of surfaces containing an irrational pencil of curves of specified type. Thus it is found that the minimum orders of the non-singular surfaces containing an elliptic pencil of rational cubics and a pencil, of genus 2, of conics are 9 and 11 respectively. It is thus suggested that, if $n(\nu, \pi, p)$ denotes the minimum order for a non-singular surface containing a pencil, of genus p, of curves of order ν and genus π, then $n(\nu, \pi, p)$ is an increasing function of each of its three arguments.

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* Roth, 6.

† Du Val, 4.

‡ Du Val, 3.

§ Roth, 4, 5, 6. In the last of these papers, the classification of the octavic surfaces is completed.

References.

Albanese, 1, *Annali di Mat.* (3), 24 (1915), 159.
Bertini, 1, *Geometria proiettiva degli iperspazi*, Messina (1923).
Bronowski, 1, *Proc. Camb. Phil. Soc.*, 30 (1934), 170.
———, 2, *Journal London Math. Soc.*, 11 (1936), 107.
Campedelli, 1, *Rend. Lincei* (6), 15 (1932), 203.
———, 2, *ibid.* (6), 15 (1932), 358.
Castelnuovo, 1, *Rend. Palermo*, 20 (1905), 55.
———, 2, *Atti Acc. Torino*, 24 (1889), 346.
———, 3, *Math. Annalen*, 44 (1894), 125.
———, 4, *Rend. Palermo*, 4 (1890), 73.
———, 5, *Rend. Lincei* (5), 3 $(1894)_1$, 473.
———, 6, *Annali di Mat.* (2), 25 (1897), 235.
———, 7, *Atti Acc. Torino*, 25 (1890), 695.
Castelnuovo-Enriques, 1, *Annali di Mat.* (3), 6 (1901), 165.
Du Val, 1, *Journal London Math. Soc.*, 8 (1933), 199.
———, 2, *ibid.*, 10 (1935), 12.
———, 3, *ibid.*, 10 (1935), 7.
———, 4, *Proc. London Math. Soc.* (2), 35 (1933), 1.
Enriques, 1, *Rend. Palermo*, 20 (1905), 1.
———, 2, *Lezioni sulla teoria delle superficie algebriche*, Padova (1932).
———, 3, *Rend. Seminario Mat. Roma* (3), 1 (1934), 7.
———, 4, *Math. Annalen*, 52 (1899), 449.
———, 5, *Rend. Acc. Bologna*, 9 (1906), 11.
Fano, 1, *Mem. Acc. Torino* (2), 44 (1894), 335.
———, 2, *Atti Acc. Torino*, 39 (1903), 597.
Noether, 1, *Annali di Mat.* (2), 5 (1871), 164.
———, 2, *Abhandlungen Akad. Wiss. Berlin* (1883), 104.
Roth, 1, *Proc. Camb. Phil. Soc.*, 30 (1934), 4.
———, 2, *ibid.*, 30 (1934), 271.
———, 3, *Rend. Lincei* (6), 21 (1935), 547.
———, 4, *Proc. Camb. Phil. Soc.*, 29 (1933), 184.
———, 5, *ibid.*, 30 (1934,) 123.
———, 6, *ibid.*, 32 (1936), 355.
———, 7, *ibid.*, 31 (1935), 159.
Scorza, 1, *Annali di Mat.* (3), 16 (1909), 255.
Severi, 1, *Atti Ist. Veneto*, 65 (1906), 625.
———, 2, *Rend. Lincei* (5), 15 (1906), 691.

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ON A WARING'S PROBLEM WITH SQUARES OF LINEAR FORMS

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[Received 9 June, 1936.—Read 18 June, 1936.]

1. Let
$$f(x, y) = ax^2 + 2hxy + by^2,$$
where a, h, b are given integers, be a positive definite form, *i.e.* $ab > h^2$, $a > 0$; or a positive semi-definite form, *i.e.* $ab = h^2$, $a \geqslant 0$. Prof. L. J. Mordell suggested the problem of finding the least value of n, say N, such that, for all admissible a, h, b, integers
$$A_r,\ B_r \quad (r = 1,\ 2,\ \ldots,\ n)$$
exist for which

$$f(x, y) = \sum_{r=1}^{n} (A_r x + B_r y)^2 \tag{1}$$

identically in x, y, and proved* that $N = 5$. For a given $n < 5$, Prof. Mordell gave necessary and sufficient conditions under which a representation in the form (1) exists†.

Dr. P. Erdös has now suggested the following problem. Suppose that $\epsilon_r = \pm 1$ is given and that $f(x, y)$ is a quadratic form not necessarily definite. Do integers A_r, B_r exist such that

$$f(x, y) = \sum_{r=1}^{n} \epsilon_r (A_r x + B_r y)^2 \tag{2}$$

identically in x, y for $n = 4$?

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* Mordell, " A new Waring's problem with squares of linear forms ", *Quart. J. of Math.* 1 (1930), 276–288. In " Über die Zerlegung definiter Funktion in Quadrate", *Archiv der Math. und Physik* (3), 7 (1904), 21, Landau proved that rational solutions always exist when $n = 5$.

† Mordell, " A new Waring's problem with squares of linear forms ", *Quart. J. of Math.* 1 (1930), 276–288; " On binary quadratic forms expressed as a sum of three linear squares with integer coefficients ", *Journal für Math.*, 167 (1931), 12–19; " On the representation of a binary quadratic form as a sum of squares of linear forms ", *Math. Zeitschrift*, 35 (1932), 1–15.

I prove that the answer is in the negative by investigating necessary and sufficient conditions under which (2) exists identically in x, y for $n \leqslant 4$. From these, I deduce

THEOREM 1. *For* $n = 5$, *and*

(i*a*) $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 1$, *a representation* (2) *exists for all positive definite and semi-definite forms*;

(i*b*) $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = -1$, *a representation* (2) *exists for all negative definite and semi-definite forms*;

(ii*a*) $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 1$, $\epsilon_5 = -1$, *a representation* (2) *exists except for negative definite forms*;

(ii*b*) $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = -1$, $\epsilon_5 = 1$, *a representation* (2) *exists except for positive definite forms*;

(iii) $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, $\epsilon_4 = \epsilon_5 = -1$ *or* $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$, $\epsilon_4 = \epsilon_5 = 1$, *a representation* (2) *exists for all quadratic forms.*

The proof is given in § 4.

For $n = 4$, Prof. Mordell proved that a representation (1) exists if and only if $ab - h^2 \neq 4^{\rho}(8\sigma - 1)$, where $\rho \geqslant 0$, $\sigma \geqslant 0$ are integers. In § 2, by using a method suggested by his paper on a similar problem for cubic forms*, I prove the analogous theorem:

THEOREM 2. *For* $n = 4$, *and* $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = \epsilon_4 = -1$, (2) *is impossible if*

$$a \equiv 2 \pmod 4, \quad h \equiv 1 \pmod 2, \quad b \equiv \pmod 4,$$

and is possible in all other cases.

In § 3, I prove

THEOREM 3. *Suppose that* $n = 4$ *and* $\epsilon_1 = \epsilon_2 = \epsilon_3 = \pm 1$, $\epsilon_4 = \mp 1$. *Then a representation* (2) *exists if* $d = h^2 - ab$ *is a perfect square. If* d *is not a square, write* $d = \Delta_1^2 \Delta_2$, $a = a_1^2 a_2$, *where* $|a_2| \geqslant 1$, $|\Delta_2| > 1$ *are square free integers. Let* ω *be the product of all those prime factors of* $a_2/[a_2, \Delta_2]$ *which are congruent to* $\epsilon_4 \pmod 4$. *Then a representation* (2) *exists if and only if*

1. $\epsilon_4 f(x, y)$ *is not a positive definite form,*

2. Δ_2 *is a quadratic residue of* ω,

3. *if* $d = 4^{\rho}(8\sigma - 1)$, *then* $a \neq 2^{\rho_1}(4\sigma_1 + \epsilon_4)$, *where* ρ, ρ_1, σ, σ_1 *are integers.*

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* Mordell, " On the representation of a binary cubic form as a sum of cubes ", *Journal London Math. Soc.*, 11 (1936), 204–208.

In § 5, I deal with the different possible cases for $n < 4$.

It may be of interest to note that it is shown that a decomposition

$$f(x, y) = \sum_{r=1}^{n} (A_r x + B_r y)^2 - (A_{n+1} x + B_{n+1} y)^2 \quad (n = 2, 3, 4)$$

is possible if and only if $f(x, y)$ represents at least one positive integer which is the sum of $n-1$ squares. Here a, A, etc., are integers.

By a slight modification, the method applies to decompositions

$$f(x, y) = \sum_{r=1}^{n} C_r (A_r x + B_r y)^2 + (A_{n+1} x + B_{n+1} y)^2 - (A_{n+2} x + B_{n+2} y)^2 \quad (n = 1, 2, 3)$$

for certain integers C_r, *e.g.* if C_1, C_2 are integers such that the integer obtained by removing a square factor from any integer of the form $C_1 X^2 + C_2 Y^2$ is of the same form, *i.e.* each genus of quadratic form of determinant $-C_1 C_2$ contains only one class of forms. Then

$$f(x, y) = \sum_{r=1}^{2} C_r (A_r x + B_r y)^2 + (A_3 x + B_3 y)^2 - (A_4 x + B_4 y)^2$$

holds if and only if $f(x, y)$ represents an integer of the form $C_1 X^2 + C_2 Y^2$.

2. In order to prove Theorem 2, we consider the following cases:

I. At least one of a, b is odd, and h is unrestricted.

II. a, b are both even, and h is odd. Then we consider four sub-cases:

II (1) either $a \equiv 0 \pmod 8$ or $b \equiv 0 \pmod 8$,

II (2) both a and $b \equiv 4 \pmod 8$,

II (3) one of $a, b \equiv 4 \pmod 8$ and the other $\equiv 2 \pmod 4$,

II (4) both a and $b \equiv 2 \pmod 4$.

It will be proved that II (4) is the only impossible case among the cases I, II.

III. a, b, h all even.

Since

$$2(X^2 + Y^2 - Z^2 - T^2) = (X+Y)^2 + (X-Y)^2 - (Z+T)^2 - (Z-T)^2,$$

the decomposition is possible for $2f(x, y)$, if it is possible for $f(x, y)$. Hence, if we assume the result just stated, a decomposition is possible in all cases except perhaps when

$$f(x, y) = 2^\lambda\{(4a_1+2)x^2+2(2h_1+1)xy+(4b_1+2)y^2\}.$$

But a decomposition is now possible if $\lambda \geqslant 1$, for, from the identity above, it suffices to prove the result when $\lambda = 1$. The result is obvious by putting $2x = x_1$, since then it is reduced to the case I.

We now prove that II (4) is the only impossible case among the cases I and II.

I. Either a or b is odd, say $a = 2a_1+1$. Write

$$g(x, y) = \{(a_1+1)x+b_1y\}^2 - (a_1x+b_2y)^2$$
$$= ax^2+2\{b_1(a_1+1)-b_2a_1\}xy+(b_1^2-b_2^2)y^2.$$

Let $b_1 = h+a_1k$, $b_2 = h+(a_1+1)k$; then we have

$$g(x, y) = ax^2+2hxy-\{2k(h+a_1k)+k^2\}y^2.$$

Hence $$f(x, y)-g(x, y) = \{b+2k(h+a_1k)+k^2\}y^2.$$

By suitable choice of k, we can make the coefficients of y^2 odd, say $2b_1+1$.

Then $$f(x, y)-g(x, y) = (b_1+1)^2y^2-(b_1y)^2,$$

and so

$$f(x, y) = \{(a_1+1)x+b_1y\}^2+\{(b_1+1)y\}^2-(a_1x+b_2y)^2-(b_1y)^2.$$

II (1). Either a or $b \equiv 0 \pmod 8$, say $a = 8a_1$, and h is odd. Then

$$\{(2a_1+1)x+b_1y\}^2-\{(2a_1-1)x+b_2y\}^2$$
$$= ax^2+2\{b_1(2a_1+1)-b_2(2a_1-1)\}xy+(b_1^2-b_2^2)y^2.$$

Since $2a_1+1$ and $2a_1-1$ are, obviously, relatively prime, we can choose b_1 and b_2 such that

$$b_1(2a_1+1)-b_2(2a_1-1) = h.$$

Since h is odd, one of b_1, b_2 must be odd and the other even. Since b is even, it follows that $b-b_1^2+b_2^2$ is odd and can be written as $b_3^2-b_4^2$. Hence

$$f(x, y) = \{(2a_1+1)x+b_1y\}^2+(b_3y)^2-\{(2a_1-1)x+b_2y\}^2-(b_4y)^2.$$

II(2). Both a and b are $\equiv 4 \pmod 8$, say $a = 4a_1$, $b = 4b_1$, where a_1, b_1 are odd and h is odd. When we make a linear transformation

$$x = px_1+ry_1, \quad y = qx_1+sy_1$$

with determinant unity and integral coefficients, the new coefficient of x_1^2 is

$$4a_1p^2+2hpq+4b_1q^2.$$

Choose $p=2$, $q=1$; then this is congruent to $4(h+b_1)\equiv 0 \pmod 8$, since both h and b_1 are odd. This reduces the present case to the case II(1).

II(3). Either $a\equiv 2 \pmod 4$, $b\equiv 0 \pmod 4$, or $a\equiv 0 \pmod 4$, $b\equiv 2 \pmod 4$; say $a=2(2a_1+1)$, $b=2\,.\,2b_1$, and h is odd. Take a_2, b_2 such that

$$a_2+b_2=\tfrac{1}{2}(h+2a_1+2b_1-1),$$

which is possible since h is odd, and then determine a_3, b_3 from

$$2a_1=a_2+a_3, \quad 2b_1=b_2-b_3.$$

Then
$$h=a_2+b_2+b_3-a_3+1,$$

and

$$\begin{aligned}\{(a_2+1)x+(b_2+1)y\}^2+\{(a_3+1)x+b_3y\}^2-(a_2x+b_2y)^2-\{a_3x+(b_3+1)y\}^2\\ =2(a_2+a_3+1)x^2+2\{(a_2+1)(b_2+1)-a_2b_2+b_3(a_3+1)-a_3(b_3+1)\}xy\\ +2(b_2-b_3)y^2\\ =ax^2+2(a_2+b_2+1+b_3-a_3)xy+by^2\\ =ax^2+2hxy+by^2.\end{aligned}$$

II(4). $a\equiv 2 \pmod 4$, $b\equiv 2 \pmod 4$, and $h\equiv 1 \pmod 2$. Since $a=A_1^2+A_2^2-A_3^2-A_4^2\equiv 2 \pmod 4$ and $A_1^2\equiv 0, 1 \pmod 4$, we have either

$$A_1\equiv A_2\equiv 1 \pmod 2, \quad A_3\equiv A_4\equiv 0 \pmod 2;$$

or
$$A_1\equiv A_2\equiv 0 \pmod 2, \quad A_3\equiv A_4\equiv 1 \pmod 2.$$

Similarly, $b\equiv 2 \pmod 4$ implies either

$$B_1\equiv B_2\equiv 1 \pmod 2, \quad B_3\equiv B_4\equiv 0 \pmod 2;$$

or
$$B_1\equiv B_2\equiv 0 \pmod 2, \quad B_3\equiv B_4\equiv 1 \pmod 2.$$

In any case we have

$$h=A_1B_1+A_2B_2-A_3B_3-A_4B_4\equiv 0 \pmod 2,$$

which contradicts $h\equiv 1 \pmod 2$. This completes the proof of Theorem 2*.

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* I should like to thank Prof. Mordell and Dr. Todd for correcting a flaw in my proof of this theorem.

3. LEMMA 1. *The representation*

$$f(x, y) = \sum_{r=1}^{3} (A_r x + B_r y)^2 - (A_4 x + B_4 y)^2 \tag{3}$$

is possible if $d = \Delta^2$, a perfect square. If d is not a perfect square, it is possible if and only if $f(x, y)$ represents at least one positive integer of the form $X^2 + Y^2$.

1. If $d = \Delta^2$,

$$f(x, y) = (a_1 x + b_1 y)(a_2 x + b_2 y).$$

By a linear transformation with determinant unity, we can, since the coefficient of xy must always be even, change $f(x, y)$ into

$$\begin{aligned} f_1(x_1, y_1) &= a_3 x_1 (a_4 x_1 + 2b_4 y_1) \\ &= (a_5 x_1)^2 + x_1 \{(a_3 a_4 - a_5^2) x_1 + 2a_3 b_4 y_1\}, \end{aligned}$$

where a_5 is any integer chosen so that $a_3 a_4 - a_5^2$ is odd. Then

$$f_1(x_1, y_1) = (a_5 x_1)^2 + \{\tfrac{1}{2}(a_3 a_4 - a_5^2 + 1) x_1 + a_3 b_4 y_1\}^2 \\ - \{\tfrac{1}{2}(a_3 a_4 - a_5^2 - 1) x_1 + a_3 b_4 y_1\}^2.$$

Hence $f_1(x_1, y_1)$, and so also $f(x, y)$, can be expressed as the sum of two positive and one negative squares of linear forms.

2. If $d \neq \Delta^2$ and (3) exists, we can find ξ_1, η_1 such that

$$A_1 \xi_1 + B_1 \eta_1 = A_4 \xi_1 + B_4 \eta_1,$$

e.g.

$$\xi_1 = B_1 - B_4, \quad \eta_1 = A_4 - A_1.$$

If

$$\xi_1 = \eta_1 = 0, \quad f(x, y) = \sum_{r=2}^{3} (A_r x + B_r y)^2.$$

If ξ_1, η_1 are not both zero, we have

$$f(\xi_1, \eta_1) = (A_2 \xi_1 + B_2 \eta_1)^2 + (A_3 \xi_1 + B_3 \eta_1)^2,$$

which cannot be zero unless $A_2 B_3 - A_3 B_2 = 0$. If $f(\xi_1, \eta_1) = 0$, we can replace ξ_1, η_1 by either

$$\xi_2 = B_2 - B_4, \quad \eta_2 = A_4 - A_2;$$

or

$$\xi_3 = B_3 - B_4, \quad \eta_3 = A_4 - A_3.$$

If $f(\xi_1, \eta_1) = f(\xi_2, \eta_2) = f(\xi_3, \eta_3) = 0$, then

$$A_1 B_2 - A_2 B_1 = A_2 B_3 - A_3 B_2 = A_3 B_1 - A_1 B_3 = 0,$$

and hence $f(x, y)$ can be written in the form

$$f(x, y) = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(A_5 x + B_5 y)^2 - (A_4 x + B_4 y)^2.$$

Let
$$\xi_4 = \alpha_3 B_5 - B_4, \quad \eta_4 = A_4 - \alpha_3 A_5,$$

so that the term in α_3 cancels the negative term and

$$f(\xi_4, \eta_4) = (\alpha_1^2 + \alpha_2^2)(A_5 B_4 - B_5 A_4)^2.$$

Since $d \neq \Delta^2$, $\alpha_1^2 + \alpha_2^2 \neq 0$. Hence $f(\xi_4, \eta_4) = 0$ only if

$$A_5 B_4 - B_5 A_4 = 0;$$

but then $f(x, y)$ is a multiple of a perfect square, which contradicts $d \neq \Delta^2$. Hence it is necessary that $f(x, y)$ can represent a number of the form $X^2 + Y^2 > 0$.

Conversely, let
$$f(\xi_1, \eta_1) = X_1^2 + Y_1^2,$$

where ξ_1, η_1 are not both zero. Let $[\xi_1, \eta_1] = m$. Then

$$f\left(\frac{\xi_1}{m}, \frac{\eta_1}{m}\right) = \frac{1}{m^2}(X_1^2 + Y_1^2) = X_2^2 + Y_2^2,$$

since the number obtained by removing any square factor from a number of the form $X^2 + Y^2$ is of the same form. Hence we can suppose that $[\xi_1, \eta_1] = 1$. By the linear transformation

$$x = \xi_1 x_1 + \zeta y_1, \quad y = \eta_1 x_1 + \tau y_1,$$

with $\xi_1 \tau - \eta_1 \zeta = 1$, we obtain

$$f_1(x_1, y_1) = (X_1^2 + Y_1^2) x_1^2 + 2H x_1 y_1 + B y_1^2.$$

Then

$$f_1(x_1, y_1) - (X_1 x_1 + b_1 y_1)^2 - (Y_1 x_1)^2 = 2(H - X_1 b_1) x_1 y_1 + (B - b_1^2) y_1^2.$$

Choose b_1 so that $B - b_1^2$ is odd, and then we can write

$$f_1(x_1, y_1) = (X_1 x_1 + b_1 y_1)^2 + (Y_1 x_1)^2 + \{(H - X_1 b_1) x_1 + \tfrac{1}{2}(B - b_1^2 + 1) y_1\}^2$$
$$- \{(H - X_1 b_1) x_1 + \tfrac{1}{2}(B - b_1^2 - 1) y_1\}^2.$$

Hence the lemma.

We require another lemma.

Suppose that $ad \neq 0$. Write $d = \Delta_1^2 \Delta_2$, $a = a_1^2 a_2$, where a_2, Δ_2 are square free integers, and denote by ω_1 the product of those prime factors of $a_2/[a_2, \Delta_2]$ which are congruent to 3 (mod 4). Then we have

LEMMA 2. *The equation*

$$f(x, y) = ax^2 + 2hxy + by^2 = z^2 + t^2 \tag{4}$$

has a solution in integers other than $x = y = z = t = 0$, *if and only if*

1. $f(x, y)$ *is not a negative definite form,*
2. Δ_2 *is a quadratic residue of* ω_1,
3. $a \neq 2^{\rho_1}(4\sigma_1 - 1)$ *when* $d = 4^{\rho}(8\sigma - 1)$.

Proof. By the correspondences

$$x = x_1 - hy_1, \quad y = ay_1, \quad z = az_1, \quad t = at_1,$$

and

$$x_1 = ax + hy, \quad y_1 = y, \quad z_1 = z, \quad t_1 = t,$$

the equation (4) is solvable if and only if

$$x_1^2 - dy_1^2 - az_1^2 - at_1^2 = 0 \tag{5}$$

is solvable. By means of the correspondences

$$x_2 = x_1, \quad y_2 = \Delta_1 y_1, \quad z_2 = a_1 z_1, \quad t_2 = a_1 t_1,$$

and

$$x_1 = \Delta_1 a_1^2 x_2, \quad y_1 = a_1^2 y_2, \quad z_1 = \Delta_1 a_1 z_2, \quad t_1 = \Delta_1 a_1 t_2,$$

(5) is solvable if and only if

$$x_2^2 - \Delta_2 y_2^2 - a_2 z_2^2 - a_2 t_2^2 = 0 \tag{6}$$

is solvable. If $[\Delta_2, a_2] = w$, then $x_2 \equiv 0 \pmod{w}$, *i.e.* $x_2 = wx_3$, and (6) becomes

$$wx_3^2 - \Delta_3 y_3^2 - a_3 z_3^2 - a_3 t_3^2 = 0, \tag{7}$$

where $\Delta_2 = \Delta_3 w$, $a_2 = a_3 w$.

Now necessary and sufficient conditions for integer solutions of

$$ax^2 + by^2 + cz^2 + dt^2 = 0 \tag{8}$$

have been given by Prof. Mordell in the following form*.

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* Mordell, "The condition for integer solutions of $ax^2+by^2+cz^2+dt^2 = 0$", *Journal für Math.*, 164 (1931), 40–49; "Note on the diophantine equation $ax^2+by^2+cz^2+dt^2 = 0$", *Bull. American Math. Soc.*, 38 (1932), 277–282.

Suppose that a, b, c, d are integers none of which is zero, and that

I. a, b, c, d have no squared factors;

II. no three of a, b, c, d have a common factor.

Then the equation (8) has integer solutions not all zero, if and only if

III. a, b, c, d are not all of the same sign;

IV. every odd prime factor p_{ab} of $[a, b]$ for which

$$\left(\frac{-cd}{p_{ab}}\right) = -1$$

also satisfies
$$\left(\frac{-ab/p_{ab}^2}{p_{ab}}\right) = 1,$$

together with five corresponding conditions derived by permuting the letters.

Either V(1) $\qquad abcd \equiv 2, 3, 5, 6, 7 \pmod 8$,

or

V(2) $\qquad abcd \equiv 1 \pmod 8, \quad a+b+c+d \equiv 0 \pmod 8$.

This means that we must exclude

(9) $\qquad a \equiv b \pmod 8, \quad c \equiv d \pmod 8, \quad a \equiv c \pmod 4$,

and the corresponding sets derived by permuting the letters; or

V(3). Two of a, b, c, d are even, say a, b, in which case we must have either

V(3.1) $\qquad \tfrac{1}{4}abcd \equiv 3, 5, 7 \pmod 8$,

or

V(3.2) $\tfrac{1}{4}abcd \equiv 1 \pmod 8$, and $\tfrac{1}{2}a+\tfrac{1}{2}b+c+d \equiv \tfrac{1}{2}(c^2d^2-1) \pmod 8$;

this means that we must exclude the sets

(10a) $\qquad \tfrac{1}{2}a \equiv \tfrac{3}{2}b \pmod 8, \quad c \equiv 3d \pmod 8$,

or (10b) $\qquad \tfrac{1}{2}a \equiv \tfrac{1}{2}b \pmod 8, \quad c \equiv d \pmod 8, \quad \tfrac{1}{2}a \equiv c \pmod 4$,

or (10c) $\qquad \tfrac{1}{2}a \equiv \tfrac{5}{2}b \pmod 8, \quad c \equiv 5d \pmod 8, \quad \tfrac{1}{2}a \equiv -c \pmod 4$.

Now the coefficients in (7) satisfy the conditions I and II. It is clear that the condition III is equivalent to the fact that $f(x, y)$ is not a negative definite form.

Since $[\Delta_3, a_3] = 1$, $[a_3, a_3] = a_3$, condition IV means that every odd prime factor p_i of a_3 for which

$$\left(\frac{w\Delta_3}{p_i}\right) = \left(\frac{\Delta_2}{p_i}\right) = -1$$

must satisfy

$$\left(\frac{-a_3^2/p_i^2}{p_i}\right) = \left(\frac{a_3^2/p_i^2}{p_i}\right)\left(\frac{-1}{p_i}\right) = \left(\frac{-1}{p_i}\right) = (-1)^{\frac{1}{2}(p_i-1)} = 1,$$

i.e. if p_k divides a_3 and $p_k \equiv 3 \pmod 4$, we must have $(\Delta_2/p_k) = 1$. This condition is exactly the condition 2 of the lemma.

From the congruences (9) and all the corresponding sets derived by permuting the letters, we get

$$w \equiv -\Delta_3 \pmod 8, \quad w \equiv -a_3 \pmod 4. \tag{11}$$

Since wa_3, $w\Delta_3$ are square free numbers and $[a_3, \Delta_3] = 1$, the only possibility in which two of the coefficients in (7) are even is that a_3 is even. The congruences (10*a*) and (10*c*) evidently cannot be satisfied. From (10*b*) we get

$$w \equiv -\Delta_3 \pmod 8, \quad w \equiv -\tfrac{1}{2}a_3 \pmod 4. \tag{12}$$

From (11) we obtain

$$d = \Delta_1^2 \Delta_3 w = \Delta_1^2 \Delta_3 (8m - \Delta_3) = \Delta_1^2(8m\Delta_3 - \Delta_3^2),$$

$$a = a_1^2 a_3 w = a_1^2 a_3 (4n - a_3) = a_1^2(4n\, a_3 - a_3^2).$$

From the second congruence of (12), we obtain

$$a = a_1^2 a_3 w = a_1^2 a_3 (4n - \tfrac{1}{2}a_3) = 2a_1^2(2na_3 - \tfrac{1}{4}a_3^2).$$

Combining these cases and writing $\frac{1}{2}a_3 = a_4$, we conclude that (7) gives no solution, other than $x_3 = y_3 = z_3 = t_3 = 0$, when

$$d = \Delta_1^2(8m\Delta_3 - \Delta_3^2), \quad a = a_1^2(4na_3 - a_3^2) \text{ or } 2a_1^2(4na_4 - a_4^2).$$

Since Δ_3, a_3, a_4 are odd, and a square of an integer is always of the form

$$4^{\alpha}(8\beta+1) \quad \text{or} \quad 4^{\alpha}(4\beta+1),$$

we have
$$d = 4^{\rho}(8\sigma - 1), \quad a = 2^{\rho_1}(4\sigma_1 - 1).$$

Conversely, if d and a are of these forms, we have

$$w\Delta_3 = 8\sigma_2 - 1, \quad wa_3 = 4\sigma_3 - 1, \quad \text{or} \quad 2(4\sigma_4 - 1),$$

and so

$$w\Delta_3^2 = 8\sigma_2\Delta_3 - \Delta_3, \quad wa_3^2 = 4\sigma_3 a_3 - a_3, \quad \text{or} \quad w\left(\frac{a_3}{2}\right)^2 = 4\sigma_4\frac{a_3}{2} - \frac{a_3}{2}.$$

Then either (11) or (12) follows.

Hence we can state the condition V in our case as condition 3 of the lemma. Hence the lemma is proved.

It is clear that Theorem 3 follows immediately from these two lemmas when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, $\epsilon_4 = -1$. Similarly we can prove the other part of Theorem 3 by proving the analogous lemmas.

4. (i*a*) of Theorem 1 has been proved by Prof. Mordell and (i*b*) is an immediate consequence.

The proof of (iii) of Theorem 1 is immediate, since, by suitable choice of A_1, we can make $a \pm A_1^2$ odd, and hence, from Theorem 2, we have

$$f(x, y) \pm (A_1 x)^2$$
$$= (A_2x+B_2y)^2+(A_3x+B_3y)^2-(A_4x+B_4y)^2-(A_5x+B_5y)^2.$$

In order to prove (ii*a*) of Theorem 1, we need the following lemmas.

LEMMA 3. *Given integers a, h, b with $d = h^2 - ab \neq 0$, we can choose integers p, q such that $[p, q] = 1$ and*

$$f(p, q) = ap^2 + 2hpq + bq^2 \neq 4^\rho(8\sigma - 1),$$

i.e. if $f(p, q) > 0$, it is a sum of three integer squares.

Proof. If there exist integers p, q, where $[p, q] = m$, such that

$$f(p, q) \neq 4^\rho(8\sigma - 1),$$

then
$$f(p/m, q/m) = \frac{1}{m^2} f(p, q) \neq 4^\rho(8\sigma - 1),$$

since m^2 is of the form $4^{\rho_1}(8\sigma_1 + 1)$. Hence we need only to show that p, q, not both zero, exist such that $f(p, q) \neq 4^\rho(8\sigma - 1)$.

If $a \neq 4^\rho(8\sigma - 1)$, we can take $p = 1$, $q = 0$.

Suppose then that $a = 4^{\rho_2}(8\sigma_2 - 1) = 4^{\rho_2}a_1$, say. Take $q = 4^{\rho_2}$. Since $af(p, q) = (ap + hq)^2 - dq^2$, we have

$$a_1 f(p, q) = 4^{\rho_2}\{(a_1 p + h)^2 - d\} = 4^{\rho_2} f_1.$$

Since $a_1 \equiv -1 \pmod{8}$, it suffices to show that we can choose p such that $f_1 \neq 4^\rho(8\sigma + 1)$, or, since a_1 is odd, that x can be chosen so that

$$x^2 - d \neq 4^\rho(8\sigma + 1).$$

Here we need only to consider the values of $d \not\equiv 0 \pmod 4$, since the factor 4 can be removed by putting $x = 2x_1$.

If $d \equiv \pm 1, \pm 2, -3 \pmod 8$, we take $x \equiv 2 \pmod 4$, so that $x^2 - d \equiv \pm 3, \pm 2, -1 \pmod 8$.

If $d \equiv +3 \pmod 8$, we take $x \equiv \pm 1 \pmod 4$, so that

$$x^2 - d \equiv -2 \pmod 8.$$

Hence the result.

LEMMA 4. *If*

$$f(x, y) = \sum_{r=1}^{4} (A_r x + B_r y)^2 - (A_5 x + B_5 y)^2,$$

then $f(x. y)$ *cannot be a negative definite form.*

This is obvious, because

$$f(B_5, -A_5) = \sum_{r=1}^{4} (A_r B_5 - B_r A_5)^2 \geqslant 0.$$

From Lemma 3, if $f(x, y)$ is not a negative definite form, we can change $f(x, y)$ into

$$f_1 = (\xi_1^2 + \xi_2^2 + \xi_3^2) x^2 + 2h_1 xy + b_1 y^2$$

by a unimodular integral linear transformation. Hence, by proceeding as in the last part of the proof of Lemma 1, we can prove the sufficiency of (ii*a*) of Theorem 1. The necessity of (ii*a*) of Theorem 1 follows from Lemma 4.

Similarly, we can prove the analogous lemmas to verify (ii*b*) of Theorem 1.

5. Write

$$m(ax^2 + 2hxy + by^2) = \sum_{r=1}^{n} (A_r x + B_r y)^2, \tag{13}$$

$$[a, h, b] = 1, \quad \gamma = [a, 2h, b], \quad m = m_1^2 m_2, \quad D = ab - h^2 = D_1^2 D_2,$$

where D_2, m_2 are free from square factors greater than 1.

Prof. Mordell gave the necessary and sufficient conditions that (13) is possible for $n < 4$ as follows.

$n = 2$. D is a perfect square and m_2 must not be divisible by positive primes $\equiv 3 \pmod 4$.

$n = 3$. If m is even, 2^γ being the highest power of 2 dividing m, and the form (a, h, b) is improperly primitive, so that $[m, \gamma] \equiv 0 \pmod 2$, then, if

η is odd, (13) is impossible; while, if η is even, (13) is possible if and only if (13) is possible when m is replaced by $m/2^{\eta}$. This disposes of the case $m \equiv \gamma \equiv 0 \pmod{2}$.

We assume then that $[m, \gamma] \not\equiv 0 \pmod{2}$. Necessary and sufficient conditions for the possibility of (13) are as follows. Let P be any prime factor of D_2; then, according as $P^{2\xi}$ or $P^{2\xi+1}$, where $\xi \geqslant 0$ is an integer, is the highest power of P dividing ma, either $-ma/P^{2\xi}$ is a quadratic residue of P or $-maD_2/P^{2\xi+2}$ is a quadratic residue of P. Further, if Q is any prime factor of m_2, we must have $-D_2$ is a quadratic residue of Q if Q is not a divisor of D_2.

I now prove the analogous theorems for sums of squares with given signs.

For $n = 2$ we have

THEOREM 4. *Necessary and sufficient conditions for the representability of* $f(x, y)$ *as*

$$f(x, y) = (A_1x+B_1y)^2-(A_2x+B_2y)^2$$

are (1) $d = \Delta^2$, *a perfect square, and* (2) *either* a *or* b *is odd or* $a \equiv b \equiv 0 \pmod{4}$, $h \equiv 0 \pmod{2}$.

Proof. Since $f(x, y)$ can be factorized, the condition (1) is necessary.

Since $a = A_1^2-A_2^2$, it is clear that $a \equiv \pm 1$ or $0 \pmod{4}$. When $a \equiv b \equiv 0 \pmod{4}$, we must have $h \equiv 0 \pmod{2}$; for $f(x, y) \not\equiv 2 \pmod{4}$ and $f(1, 1) \equiv 2 \pmod{4}$, if $h \equiv 1 \pmod{2}$. Hence the condition 2 is also necessary.

Conversely, suppose that $f(x, y)$ satisfies conditions 1 and 2.

Suppose first that a or b is odd, say that a is odd. Then

$$f(x, y) = (a_1x+b_1y)(a_2x+b_2y),$$

where the a's and b's are integers and a_1, a_2 are odd, since a is odd. Since

$$a_1b_2+a_2b_1 = 2h \equiv 0 \pmod{2},$$

and a_1, a_2 are odd, we have $b_1+b_2 \equiv 0 \pmod{2}$. Hence

$$f(x, y) = \{\tfrac{1}{2}(a_1+a_2)x+\tfrac{1}{2}(b_1+b_2)y\}^2-\{\tfrac{1}{2}(a_1-a_2)x+\tfrac{1}{2}(b_1-b_2)y\}^2.$$

Suppose next that $a \equiv b \equiv 0 \pmod{4}$, $h \equiv 0 \pmod{2}$; then we can write

$$f(x, y) = 4(a_1x^2+h_1xy+b_1y^2) = 4l_1l_2,$$

where the l's are linear forms. Hence $f(x, y) = (l_1+l_2)^2-(l_1-l_2)^2$. The conditions are therefore also sufficient.

For $n = 3$ we have

THEOREM 5. *The representation*

$$(14)\qquad f(x, y) = ax^2+2hxy+by^2 = \epsilon\{(A_1x+B_1y)^2+(A_2x+B_2y)^2-(A_3x+B_3y)^2\},$$

where $\epsilon = \pm 1$, *is possible when* $d = \Delta^2$, *a perfect square. If* d *is not a perfect square, write* $a = a_1^2 a_2$, $d = \Delta_1^2 \Delta_2$, *where* $|a_2| \geqslant 1$, $|\Delta_2| > 1$ *are square free integers. Then* (14) *is possible if and only if*

1. $\epsilon f(x, y)$ *is not a negative definite form*;

2. *if* $[a_2, \Delta_2] = 1$, ϵa_2 *and* Δ_2 *are quadratic residues of* Δ_2 *and* a_2 *respectively*;

3. *if* $[a_2, \Delta_2] = m$, *say* $a_2 = ma_3$, $\Delta = m\Delta_3$, *then* Δ_2, ϵa_2, $-\epsilon a_3\Delta_3$ *are quadratic residues of* a_3, Δ_3 *and* m *respectively.*

It is to be noted that, if $[a, d] = 1$, the conditions 2 and 3 are simply that ϵa_2 is a quadratic residue of Δ_2, for $d = h^2-ab$ is already a quadratic residue of a.

The first part of this theorem, *i.e.* when $d = \Delta^2$, has been proved in part 1 of Lemma 1.

For the second part of this theorem, *i.e.* when d is not a perfect square, we need only prove one of the cases $\epsilon = \pm 1$, say $\epsilon = 1$, and this follows immediately from the following two lemmas.

LEMMA 5. *If* d *is not a perfect square,* (14) *is possible for* $\epsilon = 1$, *if and only if* $f(x, y)$ *represents an integer* $z^2 > 0$.

The proof is precisely the same as the proof of part 2 of Lemma 1.

LEMMA 6. *The equation*

$$(15)\qquad f(x, y) = ax^2+2hxy+by^2 = z^2 \quad (ad \neq 0)$$

is solvable with x, y, z *not all zero, if and only if the following conditions are satisfied*:

1. $f(x, y)$ *is not a negative definite form.*

Let $a = a_1^2 a_2$, $d = \Delta_1^2 \Delta_2$, *where* $|a_2| \geqslant 1$, $|\Delta_2| > 1$ *are square free integers. Then*

2. *If* $[a_2, \Delta_2] = 1$, a_2 *and* Δ_2 *are quadratic residues of* Δ_2 *and* a_2 *respectively.*

3. *If* $[a_2, \Delta_2] = m > 1$, *say* $a_2 = ma_3$, $\Delta_2 = m\Delta_3$, *then* Δ_2, a_2, $-a_3\Delta_3$ *are quadratic residues of* a_3, Δ_3, *and* m *respectively.*

Proof. Write (15) as

$$(ax+by)^2-dy^2-az^2=0. \tag{16}$$

Then, by the correspondences

$$x_1=ax+by,\quad y_1=y,\quad z_1=z,$$

and
$$x=x_1-hy_1,\quad y=ay_1,\quad z=az_1,$$

the equation (16) is solvable if and only if

$$x_1^2-dy_1^2-az_1^2=0 \tag{17}$$

is solvable. By the correspondences

$$x_2=x_1,\quad y_2=\Delta_1 y_1,\quad z_2=a, z_1,$$

and
$$x_1=\Delta_1 a_1 x_2,\quad y_1=a_1 y_2,\quad z_1=\Delta_1 z_2,$$

the equation (17) is solvable if and only if

$$x_2^2-\Delta_2 y_2^2-a_2 z_2^2=0 \tag{18}$$

is solvable.

Necessary and sufficient conditions in order that (18) has solutions other than $x_2=y_2=z_2=0$ are classical* and are exactly the conditions 1 and 2 in the lemma when $[a_2, \Delta_2]=1$.

If $[a_2, \Delta_2]=m$, and $a_2=ma_3$, $\Delta_2=m\Delta_3$, x_2 must be divisible by m, say $x_2=mx_3$, $y_2=y_3$, $z_2=z_3$. Then (18) becomes

$$mx_3^2-\Delta_3 y_3^2-a_3 z_3^2=0.$$

By the classical theorem, this is solvable if and only if the conditions 1 and 3 in the lemma are fulfilled. Hence the lemma is established.

I take the opportunity of expressing my thanks to Prof. Mordell for helping me in the preparation of my manuscript.

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* Dickson, *Introduction to the theory of numbers* (1929), ch. 8.

A MEAN VALUE THEOREM FOR ANALYTIC FUNCTIONS

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[Received 5 June, 1936.—Read 18 June, 1936.]

1. In a recent paper* I proved that the inequality

$$(1.1) \qquad |\zeta(\sigma+it)| > e^{-40\{(1/\delta)\log(1/\delta)\}^2} \quad (\tfrac{1}{2}+\delta \leqslant \sigma \leqslant 2)$$

is satisfied for arbitrarily large values of t. The proof consisted essentially in showing that, if a function $f(s)$ possesses the degree of smoothness on *vertical* lines indicated by the existence of the limit

$$(1.2) \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T |f(\sigma+it)|^2\,dt$$

for $a < \sigma < b$, then its modulus must exceed a fixed bound on certain *horizontal* lines, an inequality

$$(1.3) \qquad |f(\sigma+it)| \geqslant L(\delta) \quad (a+\delta \leqslant \sigma \leqslant b-\delta)$$

being satisfied for arbitrarily large values of t.

The content of (1.1) would be much increased if it could be shown that the values of t for which it is true form a set of density approximating to unity. An attempt to prove this was unsuccessful (and indeed there seems to be no reason to suspect that it is true), but it yielded the result that a similar but weaker inequality is true at any rate for a set of positive lower density. As usual, we define the lower and upper densities of a measurable set E, defined in $(0, \infty)$, to be

$$\varliminf_{T\to\infty} \frac{1}{T}\, m\{E\times(0, T)\}, \qquad \varlimsup_{T\to\infty} \frac{1}{T}\, m\{E\times(0, T)\},$$

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* "An inequality for the Riemann zeta-function", *Proc. London Math. Soc.* (2), 41 (1936), 544–552.

where

$$m\{E \times (0, T)\} = \text{measure of part of } E \text{ in } (0, T).$$

The method proves the following general result:

THEOREM 1. *Let $f(s)$ be regular in the strip $S(a < \sigma < b, t > 0)$, and let*

$$|f(\sigma+it)| < K_\delta t^{A_\delta} \quad (a+\delta \leqslant \sigma \leqslant b-\delta, \quad t \geqslant 1); \tag{1.4}$$

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \int_1^T |f(\sigma+it)|^\beta \, dt < \infty \quad (a < \sigma < b); \tag{1.5}$$

$$\underline{\lim_{T\to\infty}} \frac{1}{T} \int_1^T |f(\sigma+it)|^\alpha \, dt > 0 \quad (a < \sigma < b), \tag{1.6}$$

where α, β are positive integers and $\alpha < \beta$. Then, corresponding to a given positive number δ, an inequality

$$L_1(\delta) \leqslant |f(\sigma+it)| \leqslant L_2(\delta) \quad (a+\delta \leqslant \sigma \leqslant b-\delta) \tag{1.7}$$

is satisfied for values of t forming a set whose lower density is not less than a positive number $\mu(\delta)$.

In virtue of a theorem of Carlson*, (1.4) and (1.5) together imply that

$$\frac{1}{T} \int_1^T |f(\sigma+it)|^\beta \, dt < C_\delta \quad (a+\delta \leqslant \sigma \leqslant b-\delta, \ T \geqslant 1), \tag{1.8}$$

and we can, if we like, assume this in place of them. $L_1(\delta)$, $L_2(\delta)$, $\mu(\delta)$ are determinate functions of δ, a, b, α, β, and the limits occurring in (1.5), (1.6). Their values can be extracted from the argument, but they are too complicated to be worth recording. If, as in the case of the zeta-function, it is important to know them, the best plan is to approximate at each stage of the proof.

It will be noticed that mean value results for two different powers of $|f|$ are now required, and that these are assumed to be *integral* powers. Whether or not this last hypothesis is necessary I cannot say. Carlson's theorems apply to any power not less than unity, and in the work of Hardy, Ingham, and Pólya the theorems of Phragmén-Lindelöf type hold for mean values of any positive power; but in another theorem on the existence of mean values on isolated lines they found it expedient to restrict the discussion to the case of an even integral power. The point does not arise in the case of the zeta-function, since results for two integral powers are available*,

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* "Sur quelques valeurs moyennes d'une fonction analytique", *Comptes rendus*, 181 (1925), 397–8. The important memoir of Hardy, Ingham, and Pólya, "Theorems concerning mean values of analytic functions", *Proc. Royal Soc.* (A), 113 (1927), 542–569, should also be consulted. Theorem 1 extends, of course, to the bilateral means employed by the latter writers.

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2 dt \to \zeta(2\sigma) \quad (\sigma > \tfrac{1}{2}), \tag{1.9}$$

and

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^4 dt \to \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} \quad \text{uniformly in} \quad \sigma \geqslant \sigma_0 > \tfrac{1}{2}. \tag{1.10}$$

The result for the zeta-function is

THEOREM 2. *If δ is sufficiently small, the inequality*

$$\delta^{30\delta^{-3}} < |\zeta(\sigma+it)| < \delta^{-3} \quad (\tfrac{1}{2}+\delta \leqslant \sigma \leqslant 2) \tag{1.11}$$

is satisfied for values of t forming a set whose lower density is not less than $\delta^3/1600$.

This will be proved first and the modifications needed to establish Theorem 1 will be indicated at the end.

2. As before, the proof depends mainly on the following two principles.

LEMMA 1. *If*

(*a*) *$f(s)$ is regular in $|s| < R$;*

(*b*) $|f(s)| \leqslant M \quad (|s| < R)$;

(*c*) $|f(a_k)| \leqslant L \quad (k = 0, 1, \ldots, N)$,

where $a_0, a_1, \ldots, a_N$ are any distinct points in $|s| \leqslant rR$ $(r < 1)$, then

$$|f(s)| \leqslant M\left(\frac{2r}{1+r^2}\right)^{N+1} + \frac{6}{1-r^4} L\Delta(2rR)^N \quad (|s| \leqslant rR), \tag{2.1}$$

where

$$\Delta = \sum_{k=0}^{N} \prod_i{}' \frac{1}{|a_i - a_k|}. \tag{2.2}$$

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* Titchmarsh, *The zeta-function of Riemann* (Cambridge Tract, 1930), 30, 38.

LEMMA 2. *Let* $f(s)$ *be regular in* $(a<\sigma<b,\ t\geqslant t_0)$, *let* k *be a positive integer, and let*

$$I(T,\sigma)=\int_{t_0}^{T}|f(\sigma+it)|^k\,dt,$$

$M(t)=$ *maximum of* $|f(s)|$ *in the rectangle with corners* $a'+\delta+i(t+\delta)$, $b'-\delta+i(t+\delta)$, $a'+\delta+i(t+d-\delta)$, $b'-\delta+i(t+d-\delta)$, *where* $a<a'<b'<b$. *Then*

$$(2.3)\quad \int_{T_1}^{T_2}\{M(t)\}^k\,dt\leqslant\frac{1}{2\pi\delta}\int_{a'}^{b'}\{I(T_2,u)-I(T_1,u)+I(T_2+d,u)$$

$$-I(T_1+d,u)\}\,du+\frac{d}{2\pi\delta}\{I(T_2+\theta_1 d,a')$$

$$-I(T_1+\theta_2 d,a')+I(T_2+\theta_1 d,b')-I(T_1+\theta_2 d,b')\},$$

where $0<\theta_1,\theta_2<1$.

The first lemma was quoted in the former paper* and the second is the formula (20) of it.

The following inequality will also be used:

LEMMA 3. *If* $A>0$ *and* $0\leqslant\psi(t)\leqslant 1$, *then*

$$(2.4)\qquad \frac{1}{T}\int_0^T A^{\psi(t)}\,dt\leqslant 1-(1-A)\frac{1}{T}\int_0^T\psi(t)\,dt.$$

To prove this, consider the function

$$u(x)=A^x-1+(1-A)x.$$

Evidently $u(0)=u(1)=0$, and $u''(x)=A^x(\log A)^2\geqslant 0$, so that $u'(x)$ increases as x increases. Hence $u(x)\leqslant 0$ in $0\leqslant x\leqslant 1$, and so

$$A^{\psi(t)}\leqslant 1-(1-A)\psi(t).$$

(2.4) follows by integration.

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* For a proof see J. M. Whittaker, *Interpolatory function theory* (Cambridge Tract, 1935), 57.

Finally, we need a simple result on density:

LEMMA 4. *If the lower density of E is at least α and the lower density of F is at least β, then the lower density of EF is at least $\alpha+\beta-1$.*

The upper densities of E_1, F_1, the complements of E, F, do not exceed $1-\alpha$, $1-\beta$; whence it is evident that the upper density of E_1+F_1 does not exceed $(1-\alpha)+(1-\beta)$. The lower density of EF, the complement of E_1+F_1, is therefore at least $1-(1-\alpha)-(1-\beta)$.

It is convenient to collect here a number of symbols which will be used in the subsequent work. Most of them have a geometrical significance which will become apparent later.

$$r_1=\tfrac{3}{4}-\delta, \quad r_2=\tfrac{3}{4}, \quad r=\frac{r_1}{r_2}. \tag{2.5}$$

$$\theta=\{1-(\tfrac{3}{2}-4\delta)^2(\tfrac{3}{2}-2\delta)^{-2}\}^{\frac{1}{2}}, \quad h=r_1\theta. \tag{2.6}$$

$$N=\text{integral part of } \delta^{-3}, \quad \lambda=\frac{2h}{N}. \tag{2.7}$$

$$L=\delta^{\frac{1}{3}N}. \tag{2.8}$$

It is readily proved that, for sufficiently small δ,

$$\frac{2r}{1+r^2}<1-\tfrac{1}{2}\delta^2, \quad 1-r^4>3\delta, \quad \theta>\sqrt{\delta}. \tag{2.9}$$

3. Denote by Q the set of positive values of t such that

$$|\zeta(\sigma+it)|\leqslant L \tag{3.1}$$

is satisfied for at least one value of σ in $\frac{1}{2}+4\delta\leqslant\sigma\leqslant 2$. It follows from the continuity of $\zeta(s)$ that Q is a closed set. Denote its characteristic function (equal to 1 when t is in Q and to 0 otherwise) by $\phi(t)$.

Let t_1 be the first point of Q which exceeds unity by at least λ; let t_2 be the first point of Q which exceeds t_1 by at least λ; and so on. Now take any interval (X, Y) on the t-axis and denote by t_p the largest t_k which does not exceed X and by t_q the smallest t_k which is not less than Y. It follows from the definition of the t_k's that

$$(t_{p+1}-X-\lambda)+(t_{p+2}-t_{p+1}-\lambda)+\ldots+(Y-t_{q-1}-\lambda)\leqslant m\{CQ\times(X, Y)\},$$

and so that

$$(q-p)\lambda\geqslant Y-X-m\{CQ\times(X, Y)\}=m\{Q\times(X, Y)\}. \tag{3.2}$$

Next, denote by $N(t)$ the number of points t_k in the open interval $(t-h, t+h)$. Then, by (3.2),

$$
\begin{aligned}
(3.3)\quad \lambda\int_1^T \{N(t)+1\}\,dt
&\geqslant \int_1^T m\{E\times(t-h,\, t+h)\}\,dt \\
&= \int_1^T \left\{\int_{t-h}^{t+h} \phi(u)\,du\right\} dt \\
&= \int_1^T \{\Phi(t+h)-\Phi(t-h)\}\,dt, \quad \text{where} \quad \Phi(t)=\int_1^t \phi(u)\,du, \\
&= \int_{1+h}^{T+h} \Phi(t)\,dt - \int_{1-h}^{T-h} \Phi(t)\,dt \\
&= \int_{T-h}^{T+h} \Phi(t)\,dt - \int_{1-h}^{1+h} \Phi(t)\,dt \\
&= 2h\Phi(T+\theta' h) - 2h\Phi(1+\theta'' h) \quad (-1<\theta',\theta''<1) \\
&= 2h\int_{1+\theta'' h}^{T+\theta' h} \phi(t)\,dt = 2hm\{Q\times(1+\theta'' h,\, T+\theta' h)\}.
\end{aligned}
$$

If we now write

$$(3.4)\qquad N(t)=N\psi(t),$$

the last inequality becomes, on dividing by λNT,

$$(3.5)\qquad \frac{1}{T}\int_1^T \psi(t)\,dt \geqslant \frac{1}{T}\, m\{Q\times(1+\theta'' h,\, T+\theta' h)\} - \frac{1}{N},$$

and so

$$(3.6)\qquad \varlimsup_{T\to\infty} \frac{1}{T}\int_1^T \psi(t)\,dt \geqslant \text{upper density of } Q - \frac{1}{N}.$$

Moreover it follows from the fact that the interval $(t-h, t+h)$ can be divided into N parts of length λ, the minimum distance between two successive points t_k, that

$$(3.7)\qquad 0 \leqslant \psi(t) \leqslant 1.$$

4. Now write

(4.1) $U(t)$ = maximum of $|\zeta(s)|$ in the rectangle with corners

$$\tfrac12+2\delta+i(t+\delta), \quad 2+2\delta+i(t+\delta),$$
$$\tfrac12+2\delta+i(t+2-\delta), \quad 2+2\delta+i(t+2-\delta),$$

$$(4.2)\qquad K(T,\sigma)=\int_1^T |\zeta(\sigma+it)|^4\,dt.$$

It follows from (2.3) that

$$\int_1^T \{U(t)\}^4 \, dt \leqslant \frac{1}{2\pi\delta} \int_{\frac{1}{2}+\delta}^{2+3\delta} \{K(T, u)+K(T+2, u)+O(1)\} \, du$$

$$+\frac{1}{\pi\delta} \{K(T+\phi_1, \tfrac{1}{2}+\delta)+K(T+\phi_1, 2+3\delta)+O(1)\} \quad (0<\phi_1<2)$$

$$=\frac{1}{2\pi\delta} \int_{\frac{1}{2}+\delta}^{2+3\delta} \left\{2T \frac{\zeta^4(2u)}{\zeta(4u)}+o(T)\right\} du$$

$$+\frac{1}{\pi\delta} \left\{T \frac{\zeta^4(1+2\delta)}{\zeta(2+4\delta)}+T \frac{\zeta^4(4+6\delta)}{\zeta(8+12\delta)}+o(T)\right\}.$$

The important term is

$$\frac{\zeta^4(1+2\delta)}{\zeta(2+4\delta)}<\frac{1}{\zeta(2)} \left\{\frac{1}{2\delta}+O(1)\right\} \sim \frac{1}{16\zeta(2)\delta^4},$$

so that

$$\int_1^T \{U(t)\}^4 \, dt<\frac{T}{16\pi\delta^5} \quad (0<\delta<\delta_0). \tag{4.3}$$

5. Draw horizontal lines $l_1, l_2, \ldots$ across the strip $S_0(\frac{1}{2}+4\delta \leqslant \sigma \leqslant 2)$ with ordinates $t_1, t_2, \ldots$, and consider the circles C_1, C_2 of centre

$$\tfrac{5}{4}+2\delta+i(t+\tfrac{3}{4}+\delta),$$

radii r_1, r_2. C_2 is contained in the rectangle in which the maximum modulus of $\zeta(s)$ is $U(t)$. Moreover, C_1 has in common with S_0 a rectangle Z of horizontal side $\frac{3}{2}-4\delta$ and vertical side $2h$. The number of lines l_k which lie inside Z is

$$N(t+\tfrac{3}{4}+\delta)=n+1 \quad \text{(say)}.$$

It has been seen that this number cannot exceed N, so that $n<N$. For the moment we may suppose, on the other hand, that $n \geqslant 0$, *i.e.* that there is at least one line l_k in Z.

Now apply (2.1) to $\zeta(s)$ in C_2. Each of the $n+1$ lines crossing Z contains a point at which

$$|\zeta(s)| \leqslant L.$$

Let $a_0, a_1, \ldots, a_n$ be such points, each on a different line, arranged in order of increasing ordinates. Now the lines are parallel and distant at least λ apart. Hence

$$|a_i-a_k| \geqslant (k-i)\lambda \quad (i=0, 1, \ldots, k-1),$$

$$\geqslant (i-k)\lambda \quad (i=k+1, \ldots, n),$$

so that

$$\Delta = \sum_{k=0}^{n} \prod_i{}' \frac{1}{|a_i - a_k|} \leqslant \frac{1}{\lambda^n} \sum_{k=0}^{n} \frac{1}{k(k-1)\ldots 2.1.1.2\ldots(n-k)}$$

$$= \frac{2^n}{\lambda^n n!} < \left(\frac{2e}{\lambda n}\right)^n,$$

and, in virtue of (2.6) and (2.7),

$$\Delta(2r_1)^n < \left(\frac{4er_1}{\lambda n}\right)^n = \left(\frac{2eN}{\theta n}\right)^n.$$

Now it is easy to verify that $(ae/x)^x$ is an increasing function in $0 \leqslant x \leqslant a$, so that

$$\left(\frac{2eN}{\theta n}\right)^n \leqslant \left(\frac{2eN}{\theta N}\right)^N = \left(\frac{2e}{\theta}\right)^N.$$

It now follows from (2.1) that, in C_1,

$$|\zeta(s)|^2 \leqslant \{U(t)\}^2 \left(\frac{2r}{1+r^2}\right)^{n+1} + \frac{6}{1-r^4} L^2 \Delta (2r_1)^n$$

$$\leqslant \{U(t)\}^2 \left(\frac{2r}{1+r^2}\right)^{n+1} + \frac{6}{1-r^4} L^2 \left(\frac{2e}{\theta}\right)^N,$$

and this obviously holds good in the case $n+1=0$, which we excluded above. Hence, since the point $\frac{1}{2}+3\delta+i(t+\frac{3}{4}+\delta)$ is on C_1,

$$(5.1)\quad \int_1^T |\zeta\{\tfrac{1}{2}+3\delta+i(t+\tfrac{3}{4}+\delta)\}|^2\, dt$$

$$\leqslant \int_1^T \{U(t)\}^2 \left(\frac{2r}{1+r^2}\right)^{n+1} dt + \frac{6}{1-r^4} L^2 \left(\frac{2e}{\theta}\right)^N T.$$

We now apply Hölder's inequality to the first term on the right and make use of (4.3) and (2.4).

$$\int_1^T \{U(t)\}^2 \left(\frac{2r}{1+r^2}\right)^{n+1} dt$$

$$\leqslant \left\{\int_1^T \{U(t)\}^4\, dt\right\}^{\frac{1}{2}} \left\{\int_1^T \left(\frac{2r}{1+r^2}\right)^{2n+2} dt\right\}^{\frac{1}{2}}$$

$$\leqslant \left(\frac{T}{16\pi\delta^5}\right)^{\frac{1}{2}} \left\{\int_0^T \left(\frac{2r}{1+r^2}\right)^{2N\psi(t+\frac{3}{4}+\delta)} dt\right\}^{\frac{1}{2}}$$

$$\leqslant T(16\pi\delta^5)^{-\frac{1}{2}} \left[1 - \left\{1 - \left(\frac{2r}{1+r^2}\right)^{2N}\right\} \frac{1}{T} \int_0^T \psi(t+\tfrac{3}{4}+\delta)\, dt\right]^{\frac{1}{2}}.$$

Substitute this in (5.1), divide by T, and make T tend to infinity. Remembering (1.9), we find that this process gives

$$\zeta(1+6\delta) \leqslant (16\pi\delta^5)^{-\frac{1}{2}} \left[1-\left\{1-\left(\frac{2r}{1+r^2}\right)^{2N}\right\} \varlimsup_{T\to\infty} \frac{1}{T}\int_0^T \psi(t+\tfrac{3}{4}+\delta)\,dt\right]^{\frac{1}{2}} \\ +\frac{6}{1-r^4}\,L^2\left(\frac{2e}{\theta}\right)^N.$$

Hence, making use of (3.6), we see that the upper density of Q

$$\leqslant \varlimsup_{T\to\infty} \frac{1}{T}\int_0^T \psi(t+\tfrac{3}{4}+\delta)\,dt+\frac{1}{N}$$

$$\leqslant \left\{1-\left(\frac{2r}{1+r^2}\right)^{2N}\right\}^{-1}\left[1-16\pi\delta^5\left\{\zeta(1+6\delta)-\frac{6}{1-r^4}\,L^2\left(\frac{2e}{\theta}\right)^N\right\}^2\right]+\frac{1}{N}.$$

Now, recalling (2.9), we have

$$\left(\frac{2r}{1+r^2}\right)^{2N} < (1-\tfrac{1}{2}\delta^2)^{2N} = e^{2N\log(1-\frac{1}{2}\delta^2)} < e^{-2\delta^{-3}\frac{1}{2}\delta^2} = e^{-1/\delta},$$

$$\frac{6}{1-r^4}\,L^2\left(\frac{2e}{\theta}\right)^N < \frac{6}{3\delta}\,\delta^{\frac{2}{3}N}\left(\frac{2e}{\sqrt{\delta}}\right)^N < 1,$$

and so
$$\zeta(1+6\delta)-\frac{6}{1-r^4}\,L^2\left(\frac{2e}{\theta}\right)^N = \frac{1}{6\delta}+O(1) > \frac{1}{7\delta} \quad (0<\delta<\delta_1).$$

Hence

$$\text{upper density of } Q \leqslant \{1+O(e^{-1/\delta})\}\left\{1-\frac{16\pi\delta^5}{49\delta^2}\right\}+\delta^3$$

$$< 1-\frac{51}{49}\,\delta^3+\delta^3 = 1-\frac{2\delta^3}{49},$$

and so
$$\text{lower density of } CQ > \frac{2\delta^3}{49}.$$

6. Consider next the set R of values of t such that

$$U(t) < \frac{1}{64\delta^3}.$$

It follows, from results given in § 4 of the previous paper, that

$$\int_1^T \{U(t)\}^2\,dt < \frac{T}{\delta^2},$$

and so the values of t in $(1, T)$ for which

$$U(t) \geqslant \frac{1}{64\delta^3}$$

form a set of measure less than $4096\delta^4 T$. Hence the lower density of R is at least $1-4096\delta^4$, and so, by Lemma 4, the lower density of $R \times CQ$ is at least

$$\frac{2\delta^3}{49}-4096\delta^4 > \frac{1}{25}\delta^3 \quad (0<\delta<\delta_2).$$

If t belongs to $R \times CQ$, we have

$$L < |\zeta(\sigma+it)| < \frac{1}{64\delta^3} \quad (\tfrac{1}{2}+4\delta \leqslant \sigma \leqslant 2),$$

and, on writing δ in place of 4δ, we obtain Theorem 2.

7. The proof of Theorem 1 follows the same course. We apply the fundamental inequality (2.1) to $\{f(s)\}^\alpha$, and integrate, obtaining a result of the form

$$\int_1^T |f\{a+3\delta+i(t+y)\}|^\alpha dt \leqslant \int_1^T \{U(t)\}^\alpha \left(\frac{2r}{1+r^2}\right)^{n+1} dt + \frac{6}{1-r^4} L^\alpha \left(\frac{2e}{\theta}\right)^N T,$$

where y, θ depend on a, b, δ, and N, L are to be chosen. Hölder's inequality gives

$$\int_1^T \{U(t)\}^\alpha \left(\frac{2r}{1+r^2}\right)^{n+1} dt \leqslant \left\{\int_1^T \{U(t)\}^\beta dt\right\}^{\alpha/\beta} \left\{\int_1^T \left(\frac{2r}{1+r^2}\right)^{\beta(n+1)/(\beta-\alpha)} dt\right\}^{1-\alpha/\beta}.$$

The first factor is dealt with by combining (1.8) with (2.3) and the second factor is treated as before. We arrive at an inequality of the form

upper density of Q

$$\leqslant \left\{1-\left(\frac{2r}{1+r^2}\right)^{\beta N/(\beta-\alpha)}\right\}^{-1} \left[1-\left\{A(\delta)-B(\delta)\, L^\alpha \left(\frac{2e}{\theta}\right)^N\right\}^{\beta/(\beta-\alpha)}\right]+\frac{1}{N},$$

where $A(\delta)$, $B(\delta)$ depend on the limits (1.5), (1.6). It is now a case of choosing N so large and L so small that the right-hand side becomes less than unity, and the rest of the argument runs as before.

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FUNCTIONS WITH DOMINANT SINGULARITIES OF THE GENERALIZED ALGEBRAIC-LOGARITHMIC TYPE

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[Received 24 April, 1935.—Read 25 April, 1935.—Received in revised form 18 June, 1936.]

1. *Introduction.* Consider the function $f(z)$, defined in the neighbourhood of the origin by the series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots, \tag{1}$$

of unit radius of convergence. Hadamard[†] has defined the order of $f(z)$ on $|z| = 1$ by means of the H-operator (or the R-operator) and the notion of finite span. He has also defined the order of $f(z)$ on an arc and at a point and proved that all three are unique. The order is in each case a real number and the order ω of $f(z)$ on $|z| = 1$ is determined by the relation

$$\omega = 1 + \overline{\lim_{n\to\infty}} \frac{\log|c_n|}{\log n}. \tag{2}$$

Further, the order on an arc (or on the circle) is the upper bound of the orders at the points contained by the arc (or circle).

The problem of finding the orders of the singular points on the circle of convergence is the second stage in the problem of the location and classification of those singularities. It is curious that the theory of order has given rise to considerable discussion, while the preliminary theory, dealing with the separation of singularities, is hardly known at all, if we except the special case of polar singularities. There is a reason for this, perhaps, in the fact that the theory of order is complete and without restriction, while the problem of the location of singularities on the circle of convergence has been solved only for a limited class of singularities.

[†] 2, 3, 4, 5, 7, 8, 9.

Nevertheless, it embodies a theory of some power, and the effectiveness of the theory of order, from a practical point of view, is dependent mainly on the state of development of the complementary theory.

Hadamard[†] considered a function $f(z)$ whose singularities on $|z|=1$ are of finite order. He obtained results which are conveniently stated in the two theorems below.

THEOREM A_1. *The order of* $\{1-(z/z')\}f(z)$ *is the same as that of* $f(z)$ *at every point of* $|z|=1$ *save possibly at the point* $z=z'$ *when this point is singular for* $f(z)$.

In fact, Hadamard remarked[‡] that, if the order of $\{1-(z/z')\}f(z)$ is less than that of $f(z)$ at $z=z'$, then this point must be either an isolated singular point or, if the singularities on the arc are dense in the neighbourhood of $z=z'$, it must be a point of order discretely[§] higher than that of the neighbouring singularities.

THEOREM B_1. *If there exists a polynomial* $P(z)$, *such that the order of* $P(z)f(z)$ *on* $|z|=1$ *is less than the order of* $f(z)$, *then there is a unique polynomial of lowest degree (with constant term unity) for which this is true.*

Evidently, from the remarks following Theorem A_1, $f(z)$ must have at the zeros of this polynomial singularities of order discretely higher than the order at any other point on $|z|=1$.

Hadamard[||] has given necessary and sufficient conditions, in terms of the coefficients of (1), for the existence of such a polynomial and has found a means of determining it when these conditions are satisfied. Unfortunately the *methods* employed are extremely involved and have not so far lent themselves to direct application. He made use of the *results*, however, for the purpose of determining the singular points of $f(z)$ on $|z|=1$ when these singularities are either polar, algebraic, logarithmic, or any combination of these. This was, in effect, made possible by an implicit use of the asymptotic values of the coefficients in the expansion (1), when $f(z)$ represents a function possessing on $|z|=1$ singularities of the appropriate type.

[†] **4**, 138–186.

[‡] **4**, 183, § 49.

[§] A point z' is said to be of order "discretely higher" than that of the neighbouring singularities if there is a positive number δ such that the order of $f(z)$ at z' exceeds its order at any point in a certain neighbourhood of z' by at least δ.

[||] **4**, 140, 147.

We shall extend the results so far obtained to include the generalized algebraic-logarithmic singularities considered by Jungen and others[†]. This type of singularity includes, as special cases, the singularities considered by Hadamard and we shall refer to the whole class as *singularities of algebraic-logarithmic type*.

Jungen[‡] discussed the asymptotic properties of the coefficients of the expansion (1) and the distribution of the coefficients of very small modulus, when the singularities of $f(z)$ on $|z| = 1$ are all of algebraic-logarithmic type. In §§ 2–4 we extend these results in considering the distribution of the coefficients of greatest modulus (which are of particular importance to the Hadamard theory) in the more general case in which only the *dominant singularities* (see § 4) of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type. These singularities are of finite order, and the following results, first obtained by Hadamard[§] in the restricted case, are proved for the more general case in § 4. They are set out in the form of three theorems. Theorems A_2 and B_2 complete, in the cases considered, the results of Theorems A_1 and B_1, while Theorem C_2 is a natural corollary to Theorem B_2. Actually, we obtain, in Theorems A and B of § 4, improved forms of Theorems A_2 and B_2.

THEOREM A_2. *At a singular point of algebraic-logarithmic type at* $z = z'$ *the order of* $\{1-(z/z')\}f(z)$ *is less by one than the order of* $f(z)$, *provided that the singularity is not a simple pole*[||].

THEOREM B_2. *If the singularities of highest order* ω *of* $f(z)$ *on* $|z| = 1$ *are all of algebraic-logarithmic type, then there exists a unique polynomial* $P(z)$, *of lowest degree* p, *where*

$$P(z) \equiv \left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\dots\left(1-\frac{z}{z_p}\right),$$

such that the order of $P(z)f(z)$ *on* $|z| = 1$ *is less than the order* ω *of* $f(z)$ *and the singularities of* $f(z)$ *of order* ω *are located at the points* $z = z_\nu (\nu = 1, 2, \dots p)$.

It should be noted that nothing is stipulated with regard to the remaining singularities of $f(z)$ on $|z| = 1$ except that they are not of the highest order. They may, for example, be everywhere dense on $|z| = 1$. On the other hand, if the remaining singularities on $|z| = 1$ are also of algebraic-logarithmic type, they can be located as in Theorem B_2. For, from Theorems A_2 and B_2, it follows that the singularities of $f(z)$ on $|z| = 1$ of next highest order ω' ($< \omega$) will be the singularities of highest order on $|z| = 1$ for the function $\{P(z)\}^m f(z)$, where m is the greatest integer in $\omega - \omega' + 1$. A second application of Theorem B_2, this time to $\{P(z)\}^m f(z)$, will enable us to locate all the singular points of $f(z)$ on $|z| = 1$ of order ω'. By a finite number of repetitions of the process, all the singular points of $f(z)$ on $|z| = 1$ can be located, in succession, in accordance with their orders, for, if all the singular points of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type, their number is finite†.

[†] **6**, 266.

[‡] **6**, 266–291.

[§] **4**, 185.

[||] In this case $\{1-(z/z')\}f(z)$ is regular at $z = z'$ and the order at a regular point is $-\infty$.

From this we deduce the existence of a unique polynomial $Q(z)$, the product of $P(z)$ and its successors, which selects every singular point of $f(z)$ on $|z| = 1$ once and once only. It is convenient, for further reference, to quote this result in the following form.

THEOREM C_2. *If the singularities of $f(z)$ on $|z| = 1$ are all of algebraic-logarithmic type, there exists a unique polynomial $Q(z)$, of lowest degree (with constant term unity), such that the order of $\{Q(z)\}^k f(z)$ on $|z| = 1$ is $\omega - k$ at most. If there is at least one non-polar singularity of $f(z)$ on $|z| = 1$ of order ω, then the order of $\{Q(z)\}^k f(z)$ on $|z| = 1$ is exactly $\omega - k$.*

It is probable that these results hold for a more extensive class of singularities than that considered above. Even with the restriction that the dominant singularities (see § 4) on the circle of convergence are of algebraic-logarithmic type, it is interesting to find that a number of classical theorems‡, due to Pólya, Szegö, Mandelbrojt, Vivanti, Dienes, and Fabry, proved by a diversity of methods, are united in the Hadamard theory of order and its extensions. Fortunately, the restriction on the nature of the dominant singularities is accompanied by a sharpening of the results. This is natural, for a knowledge of the order of $f(z)$ at every point of $|z| = 1$ is sufficient to determine not only the positions of the singularities (which is the object of most of the theorems in their original forms), but also the comparative strengths of those singularities. The extensions, therefore, involve the idea of *dominant* singularities, in the sense explained in § 4, that is, roughly speaking, the strongest among those singularities of $f(z)$ on $|z| = 1$ which are of highest order§.

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† Such singularities are either poles or isolated critical points of $f(z)$.

‡ References are given below.

§ Equality of order is measured only in the scale of powers of n and finer differences are not registered on this scale.

We shall consider the above-mentioned theorems in relation to the dominant singularities on the circle of convergence. Proofs of the theorems are given for the case in which the dominant singularities are of algebraic-logarithmic type. Not only are these apparently diverse theorems brought together to form part of a coherent theory, but their connections with that theory exhibit in a remarkable manner their relationships with one another. The theorems have been grouped so as to display these connections as far as possible, and the treatment is prefaced by certain extensions of Hadamard's results on polar singularities which serve as an introduction to the subsequent theory (§ 5). The proofs which follow (§§ 5-9) are characterized by that simplicity inherent in the ideas of the Hadamard order.

2. *The generalized algebraic-logarithmic singularity.* Jungen[†] has discussed, in considerable detail, the nature of the expansion, in the neighbourhood of a singularity, of a function which is an integral of a homogeneous linear differential equation of Fuchsian type. The expression of such a function, in a sufficiently small neighbourhood of a singular point $z=z'$, consists of a finite number of elements of the form

$$(z-z')^{-s}\{\log(z-z')\}^k\phi(z),$$

where s is complex, k a non-negative integer, and $\phi(z)$ is regular and non-zero at $z=z'$. Jungen[‡] has proved that, if the expansion (1) represents the element

$$(z-z')^{-s}\{\log(z-z')\}^k, \tag{3}$$

then

$$|c_n| = Cn^{\sigma-1}(\log n)^k\{1+O(1/\log n)\} \quad (s\neq 0,\ -1,\ -2,\ \ldots), \tag{4}$$

where $\mathfrak{R}(s)=\sigma$, and, if $k>0$,

$$|c_n| = C'n^{s-1}(\log n)^{k-1}\{1+O(1/\log n)\} \quad (s=0,\ -1,\ -2,\ \ldots),$$

where C, C' are positive constants.

The element (3) is said to be of *type* (s, k), and of *weight* $[\sigma, k]$ in the first case and $[s, k-1]$ in the second. If there are two elements of weights $[\alpha, \beta]$, $[\alpha', \beta']$, respectively, and if either $\alpha>\alpha'$ or $\alpha=\alpha'$ and $\beta>\beta'$, then the former is said to be the stronger.

[†] **6**, 266–291.

[‡] **6**, 273.

The weight of a singularity is defined to be the weight of the strongest element that it contains. Thus two types of singular point arise, that in which there is a strongest element and that in which there are two or more elements of greatest weight.

Taking first the case in which there is only one element of greatest weight at $z=z_1$, if the weight $[\sigma, k]$ at this point surpasses the weights of the singularities at $z_2, z_3, \ldots z_q$, the other singular points of $f(z)$ on $|z|=1$, Jungen[†] has proved that (4) still holds.

Taking next the case in which l of the singularities are of greatest weight, Jungen[‡] has proved that, if at least one of these singularities possesses *only one element of greatest weight*, then positive constants A and B exist such that

$$(5) \qquad An^{\sigma-1}(\log n)^k \leqslant |c_{n-1}|+|c_{n-2}|+\ldots+|c_{n-l}| \leqslant Bn^{\sigma-1}(\log n)^k,$$

for every $n > n_0$.

Both in this case and the preceding, it follows that a positive constant C_1 exists, such that, for every n,

$$(6) \qquad |c_n| \leqslant C_1 n^{\sigma-1}(\log n)^k,$$

and that a positive constant C_2 $(\leqslant C_1)$ exists, such that, for an infinite sequence of n,

$$(7) \qquad |c_n| \geqslant C_2 n^{\sigma-1}(\log n)^k.$$

Suppose now that at the point $z=1$ there are r elements of greatest weight $[\sigma, k]$. Jungen[§] has proved that, if (1) represents the expansion of a function possessing at $z=1$ these elements and others of less weight, together with other singularities on $|z|=1$ of less weight, then

$$(8) \qquad c_n = n^{\sigma-1}(\log n)^k\{A_1 e^{i\alpha_1 \log n}+\ldots+A_r e^{i\alpha_r \log n}+O(1/\log n)\},$$

where the A_ν are non-zero constants and the α_ν are real and different.

Any sequence of coefficients satisfying (7) for a given *positive* C_2 will be called a sequence of *dominant* coefficients. Result (4) shows that in the first case every coefficient is a dominant one, while (5) shows that in the second case at least one in every l successive coefficients is a dominant one, so that in both cases the problem of the distribution of dominant coefficients is simple. This is far from being so in the general case, or even in the third case, typified by (8).

[†] 6, 275, 277–8.

[‡] 6, 287–8.

[§] 6, 281.

In § 3 we obtain results concerning the size and distribution of the dominant coefficients in the general case in which there are several singularities of greatest weight, each containing more than one element of greatest weight. Such results depend directly on corresponding results in the simpler case in which there is only one singularity of greatest weight. Certain of these latter results are established by means of a lemma, due to Jungen[†].

Lemma 1. *The upper density*[‡] *of the indices n for which*

$$\left|\sum_{\nu=1}^{r} A_\nu e^{i\alpha_\nu \log n}\right| < \lambda$$

is less than $10Nd(\lambda)$, *where* N *is a constant independent of* λ, *and* $d(\lambda)$ *is a function of* λ *satisfying the condition* $\lim\limits_{\lambda\to 0} d(\lambda) = 0$.

It is evidently possible to select a positive λ sufficiently small to ensure that the upper density in question is less than 1. If we take C_2 to be a value of λ for which this is true, it follows that the lower density[‡] of the sequence of n satisfying the relation

$$\left|\sum_{\nu=1}^{r} A_\nu e^{i\alpha_\nu \log n}\right| \geqslant C_2$$

is positive. Further, when we take for C_1 the value $\sum\limits_{\nu=1}^{r} |A_\nu|$, it follows that

$$\left|\sum_{\nu=1}^{r} A_\nu e^{i\alpha_\nu \log n}\right| \leqslant C_1$$

for every n.

When we apply these results to (8), it is evident that a positive constant C, lying between C_1 and C_2, exists such that

$$\overline{\lim_{n\to\infty}}\, |c_n|/n^{\sigma-1}(\log n)^k = C, \tag{9}$$

and this is true in all three cases so far considered.

This result can be further improved, for we can take for C_1 any number greater than C, and for C_2 any positive number less than C. In fact, the choice of C_1 determines the integer n_0 for which (6) is satisfied with $n > n_0$, while the choice of C_2 determines the upper and lower densities of the sequence of n for which (7) holds. In particular, Lemma 2 results from the preceding remarks.

[†] **6**, 283.

[‡] **11**, 556–562.

LEMMA 2. *Given a positive number $h < 1$, it is possible to select a positive number C_2 such that* (7) *is true for a sequence of lower density h.*

On the other hand, for every positive ϵ, however small, it is possible to find an infinite sequence of n such that (7) is satisfied for $C_2 = C - \epsilon$. We elaborate this remark in the following lemma, which shows that, in the case under discussion, the dominant terms occur consecutively in sets of increasing extent.

LEMMA 3. *For every positive η, however small, there exists an infinite sequence of n such that*

$$(C+\eta)\, n^{\sigma-1}(\log n)^k \geqslant |c_{n-r}| \geqslant (C-\eta)\, n^{\sigma-1}(\log n)^k \tag{10}$$

holds for $r = 0, \pm 1, \pm 2, \ldots, \pm q$, where q is $o(n/\log n)$.

To prove this it is necessary to consider in detail the asymptotic form of c_n.

The contribution† to c_n from the singularity of greatest weight $[\sigma, k]$ of $f(z)$ on $|z| = 1$ takes the form

$$\begin{aligned} (11) \quad c_n' = {} & n^{\sigma-1}(\log n)^k \sum_{\nu=1}^{r} \left\{ \phi_{0,\nu}(n) + \frac{\phi_{1,\nu}(n)}{\log n} + \ldots + \frac{\phi_{k,\nu}(n)}{(\log n)^k} \right\} e^{i\alpha_\nu \log n} \\ & + n^{\sigma'-1}(\log n)^{k'} \sum_{\nu=1}^{r'} \left\{ \psi_{0,\nu}(n) + \frac{\psi_{1,\nu}(n)}{\log n} + \ldots + \frac{\psi_{k',\nu}(n)}{\log (n)^{k'}} \right\} e^{i\alpha_\nu' \log n} + \ldots, \end{aligned}$$

where the $\phi_{\mu,\nu}(n)$, $\psi_{\mu,\nu}(n)$, ..., are all of the form

$$\gamma_0 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + \ldots,$$

where the γ_ν are constants; the total number of elements is finite, and the weights $[\sigma, k]$, $[\sigma', k']$, ... are in descending order.

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† Jungen's expression of **6**, 281, is rewritten in greater detail, using Theorem A of p. 269.

The number of the remaining singularities is finite and it therefore follows from (6) that, if K_1 is a suitable constant,

$$|c_n''| < K_1 n^{\sigma-1}(\log n)^{k-1}, \tag{12}$$

where $c_n' + c_n'' = c_n$.

From (9), corresponding to every positive ϵ, there exists an infinite sequence of n such that

$$(C+\epsilon)\, n^{\sigma-1}(\log n)^k \geqslant |c_n| \geqslant (C-\epsilon)\, n^{\sigma-1}(\log n)^k. \tag{13}$$

If n belongs to this subsequence, it follows from (11) and (12) that

$$\begin{aligned} |c_n| \sim |c_{n\pm q}| \leqslant |c_n - c_{n\pm q}| &< |c_n' - c_{n\pm q}'| + 2K_1 n^{\sigma-1}(\log n)^{k-1} \\ &< q n^{\sigma-2}(\log n)^k \{K_2 + O(1/\log n)\} + 2K_1 n^{\sigma-1}(\log n)^{k-1} \\ &< K n^{\sigma-1}(\log n)^{k-1}, \end{aligned}$$

provided that q is $o(n/\log n)$, K_2 and K being positive constants.

Hence

$$|c_{n\pm q}| > |c_n| - K n^{\sigma-1}(\log n)^{k-1} > \left\{(C-\epsilon) - \frac{K}{\log n}\right\} n^{\sigma-1}(\log n)^k,$$

and corresponding to every $\eta > \epsilon$ an n_0 can be found such that

$$|c_{n\pm q}| > (C-\eta)\, n^{\sigma-1}(\log n)^k$$

holds for an infinite sequence of $n > n_0$, so long as q is $o(n/\log n)$.

On combining this with the left-hand side of (13), it follows that to every positive η, sufficiently small, the relations

$$(C+\eta)\, n^{\sigma-1}(\log n)^k \geqslant |c_{n-r}| \geqslant (C-\eta)\, n^{\sigma-1}(\log n)^k$$

$$(r = 0, \pm 1, \pm 2, \ldots, \pm q),$$

in which q is $o(n/\log n)$, hold for an infinite sequence of values of n.

Lemmas 2 and 3 concern the distribution of the largest values of c_n. Jungen†, using Lemma 1, obtained the following result concerning small values of c_n, in the case under consideration.

LEMMA 4. *If $\epsilon(n)$ is a function tending to zero with $1/n$, then*

$$|c_n| \geqslant \epsilon(n)\, n^{\sigma-1}(\log n)^k$$

for every n save possibly a sequence of density zero.

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† **6**, 276–284.

This shows that, when the expansion (1) represents a function in the third case, the density of a set of coefficients of sufficiently small modulus is zero. Certain results on the distribution of these coefficients arise from Lemma 4.

LEMMA 5. *If $f(z)$ has on $|z|=1$ only one singularity of greatest weight, containing r elements of greatest weight $[\sigma, k]$ together with others of less weight, and if the remaining singularities are of weight less than $[\sigma-r+1, k]$, then at least one in every r successive coefficients satisfies the relation*

$$|c_n| \geqslant A n^{\sigma-r} (\log n)^k,$$

where A is a positive constant and $n > n_0$.

If the theorem is not true, then a function $\eta(n)$ exists, tending to zero with $1/n$, such that the sequence of coefficients for which $\epsilon(n)$ in Lemma 4 has the value $\eta(n)/n^{r-1}$ contains a subsequence consisting of sets of r successive terms all of which satisfy the inequality of Lemma 4.

The expression for c_n' in (11) may be written in the form

$$(11') \quad c_n' = n^{\sigma-1} (\log n)^k \sum_{\nu=1}^{r} \left\{ A_\nu + \frac{B_\nu}{\log n} + \frac{C_\nu}{(\log n)^2} + \dots \right\} e^{i\alpha_\nu \log n}$$
$$+ n^{\sigma'-1} (\log n)^{k'} \sum_{\nu=1}^{r'} \left\{ A_\nu' + \frac{B_\nu'}{\log n} + \frac{C_\nu'}{(\log n)^2} + \dots \right\} e^{i\alpha_\nu' \log n} + \dots,$$

where the A_ν, B_ν, C_ν, ... are constants, of which the A_ν are non-zero†.

Relation (12) is now replaced by

$$|c_n''| < K_1 n^{\sigma-r} (\log n)^{k-1},$$

where K_1 is a suitable constant, and therefore, *if n is a member of the above-mentioned sequence*, it follows that

$$(14) \qquad c_n' = c_n - c_n'' = O\{n^{\sigma-r} (\log n)^k \theta(n)\},$$

where $\theta(n)$ tends to zero with $1/n$ more slowly than either $\eta(n)$ or $1/\log n$.

If we take successive differences of a set of r consecutive terms (11′), belonging to the subsequence and satisfying (14), we obtain the relations

$$\sum_{\nu=1}^{r} \{\alpha_\nu^\lambda A_\nu + O(1/\log n)\} e^{i\alpha_\nu \log n} = O\{\theta(n)/n^{r-\lambda-1}\}$$
$$(\lambda = 0, 1, 2, \dots, r-1).$$

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† 6, 281.

This gives a system of linear equations in the A_ν, namely

$$(15) \qquad \sum_{\nu=1}^{r} a_\nu^\lambda A_\nu e^{ia_\nu \log n} = \theta_\lambda(n) \quad (\lambda = 0, 1, 2, \ldots, r-1),$$

where $|\theta_\lambda(n)| < K\theta(n)$, and K is a suitable constant.

The modulus of the discriminant of the system is

$$\prod_{\nu=1}^{r} |e^{ia_\nu \log n}| \prod_{\lambda, \mu=1}^{r}{}' (a_\lambda - a_\mu)^2 = \prod_{\lambda, \mu=1}^{r}{}' (a_\lambda - a_\mu)^2,$$

and is positive, since the a_ν are all real and different[†].

If we solve (15) for the A_ν, it follows that, since the discriminant is a non-zero constant, every A_ν tends to zero with $\theta(n)$. The A_ν are, however, preassigned non-zero constants. This contradiction proves the lemma, which may be formulated as follows, for future reference : *there exist positive constants A and B such that, for every $n > n_0$*,

$$(16) \qquad An^{\sigma-r}(\log n)^k \leqslant |c_{n-1}| + |c_{n-2}| + \ldots + |c_{n-r}| \leqslant Bn^{\sigma-1}(\log n)^k,$$

the inequality on the right being a consequence of (9).

This result is a special case of a theorem due to Pólya[‡].

3. *The general case.* We now consider the case of a function $f(z)$ having l singularities of greatest weight, each containing more than one element of greatest weight, with a total of q ($\geqslant l$) algebraic-logarithmic singularities on $|z| = 1$. In the following, generalizations of the result (9) and of Lemmas 2–5 are obtained for such functions. The method used is developed in the proof of Theorem D$_2$, which is a generalisation of (9).

THEOREM D$_2$. *If the singularities of $f(z)$ on $|z| = 1$ are all of algebraic-logarithmic type, and if the strongest singularities are of weight $[\sigma, k]$, then there exists a positive constant C such that*

$$(9) \qquad \varlimsup_{n\to\infty} |c_n| / n^{\sigma-1}(\log n)^k = C.$$

Suppose that the singularities of greatest weight are at $z_1, z_2, \ldots, z_l$. If we write

$$(17) \quad f^*(z) = \Sigma c_n^* z^n = \prod_{\nu=2}^{l} \left(1 - \frac{z}{z_\nu}\right) f(z)$$
$$\equiv (1 + b_1 z + b_2 z^2 + \ldots + b_{l-1} z^{l-1}) f(z),$$

[†] 6, 281.

[‡] See 6, 287, foot-note 3.

it follows that $f^*(z)$ is the sum of two functions, one of which has a unique multiple singularity of weight $[\sigma, k]$ at $z = z_1$, and the other has singularities of weight $[\sigma-1, k]$ at $z_2, z_3, \ldots, z_l$, together with $q-l$ other singularities of weight at most $[\sigma', k']$, less than $[\sigma, k]$. Since the number of such singularities is finite, it follows from (6) that the coefficients of the latter function are $O\{n^{\sigma-2}(\log n)^k\}$ or $O\{n^{\sigma'-1}(\log n)^{k'}\}$, according as the weight $[\sigma-1, k]$ is greater or less than $[\sigma', k']$. The first function is of the type already discussed (third case), and it possesses an infinite sequence of coefficients satisfying (7). It follows, therefore, that there is an infinite sequence of c_n^* satisfying (7). From (17), we have

$$|c_n^*| = \left|\sum_{\nu=0}^{l-1} b_\nu c_{n-\nu}\right| \leqslant M \sum_{\nu=0}^{l-1} |c_{n-\nu}|,$$

where M is $\max(1, |b_1|, |b_2|, \ldots, |b_{l-1}|)$, and hence, from the above, there exists an infinite sequence of n for which

$$\sum_{\nu=0}^{l-1} |c_{n-\nu}| \geqslant M^{-1}|c_n^*| \geqslant C_2' n^{\sigma-1}(\log n)^k, \tag{18}$$

where C_2' is a positive constant. From this it follows that a positive constant C_2 exists, such that, for an infinite sequence of n,

$$|c_n| \geqslant C_2 n^{\sigma-1}(\log n)^k, \tag{7}$$

for otherwise (18) would not hold.

Since the coefficient c_n may be obtained as the sum of a finite number of expressions of the form (8), it follows that c_n is $O\{n^{\sigma-1}(\log n)^k\}$, and that therefore a positive constant C_1 exists such that, for every $n > n_0$,

$$C_1 n^{\sigma-1}(\log n)^k \geqslant |c_n|. \tag{6}$$

These last two results establish the theorem.

As already remarked, $f^*(z)$ is a function of the type which is the subject of Lemmas 2–5. Applying to $f^*(z)$ the results of Lemma 2, we see that, corresponding to any positive $h < 1$, a positive constant C_2^* exists such that

$$|c_n^*| \geqslant C_2^* n^{\sigma-1}(\log n)^k$$

holds for a sequence of n of lower density h. Hence, from (18), a positive constant C_2'' exists such that

$$\sum_{\nu=0}^{l-1} |c_{n-\nu}| \geqslant C_2'' n^{\sigma-1}(\log n)^k$$

holds for a sequence of n of lower density h. This gives rise to the following theorem:

THEOREM E_2. *If the singularities of $f(z)$ on $|z| = 1$ are all of algebraic-logarithmic type, and if there are l singularities of greatest weight $[\sigma, k]$, then for every positive $h < 1$ there exists a positive constant C_2 such that*

$$|c_n| \geqslant C_2 n^{\sigma-1}(\log n)^k$$

holds for a sequence of n of lower density at least h/l.

Since Lemma 3 also applies to $f^*(z)$, the following theorem results from combining (18) with (10).

THEOREM F_2. *If the singularities of $f(z)$ on $|z| = 1$ are all of algebraic-logarithmic type, and if there are l singularities of greatest weight $[\sigma, k]$, then for every $\eta > 0$ there exists an infinite sequence of n such that*

$$\sum_{\nu=0}^{l-1} |c_{n-\nu-r}| \geqslant \frac{C^*-\eta}{M} n^{\sigma-1}(\log n)^k$$

holds for $r = 0, \pm 1, \pm 2, \ldots, \pm q$, where q is $o(n/\log n)$.

This theorem may be presented in another form, of considerable use in the sequel. We notice that

$$l \sum_{\nu=0}^{N-1} |c_{n\pm\nu}| > \sum_{\mu=0}^{N-l} \sum_{\nu=0}^{l-1} |c_{n\pm\nu\pm\mu}| > (N-l) \frac{C^*-\eta}{M} n^{\sigma-1}(\log n)^k,$$

and, when we write $C' = (C^*-\eta)/Ml$, the result may be stated as follows:

THEOREM F_2'. *If the singularities of $f(z)$ on $|z| = 1$ are all of algebraic-logarithmic type, and if there are l singularities of greatest weight $[\sigma, k]$, then for every positive $C' < C^*/Ml$ there exists an infinite sequence of n such that*

$$\sum_{\nu=0}^{N-1} |c_{n\pm\nu}| > (N-l)\, C' n^{\sigma-1}(\log n)^k$$

for every N $(> l)$ which is $o(n/\log n)$.

Lemma 3 shows that, when $f(z)$ has a unique singularity of greatest weight on $|z| = 1$, each member of any dominant sequence derived from (9) is contained in an interval of $o(n/\log n)$ consecutive terms every one of which is a dominant term. The essential condition on the sequence of dominant terms derived from (13) is that $C-\epsilon$ should be positive. The dominant terms associated with any set of indices of extent $o(n/\log n)$ then satisfy (10).

We now proceed to establish a corresponding result in the more general case here considered. Take first a sequence of dominant terms derived from (9) of Theorem D_2 by means of (13). The dominant portion of any c_n belonging to such a sequence consists of the sum of l terms of type (8). Consequently for each index n of the sequence the coefficient c_n^* of at least one of the l singularities must satisfy a relation

$$|c_n^*|/n^{\sigma-1}(\log n)^k \geqslant (C-\epsilon)/l.$$

On replacing $(C^*-\eta)$ by $(C-\epsilon)/l$, we see that Theorems F_2 and F_2' continue to hold. Thus every member c_n of a sequence of dominant terms satisfying (13) is contained in an interval of extent $o(n/\log n)$ of which the members satisfy Theorems F_2 and F_2'. In fact, if (n') is the set of indices determining a sequence of dominant terms by means of (13), the corresponding intervals of Theorems F_2 and F_2' are given by $(n'-\theta_{n'}, n'+\theta_{n'})$, where θ_n is $o(n/\log n)$.

When we apply Lemma 4 to $f^*(z)$, it follows from (18) that

$$M \sum_{\nu=0}^{l-1} |c_{n-\nu}| \geqslant |c_n^*| \geqslant \epsilon(n)\, n^{\sigma-1} (\log n)^k$$

for every n save possibly a sequence of density zero. Hence we have

THEOREM G_2. *If the singularities of $f(z)$ on $|z|=1$ are all of algebraic-logarithmic type, and if there are only l singularities of greatest weight $[\sigma, k]$, then, if $\eta(n)$ is a function tending to zero with $1/n$,*

$$\sum_{\nu=0}^{l-1} |c_{n-\nu}| \geqslant \eta(n)\, n^{\sigma-1}(\log n)^k$$

for every n save possibly a sequence of density zero.

To generalize Lemma 5 we first select that multiple singularity of greatest weight $[\sigma, k]$ which has the smallest number r of elements of that weight. Secondly, it is necessary to ensure that the remaining singularities at $z_2, z_3, \ldots, z_q$ are reduced so as to be of weight less than $[\sigma-r+1, k]$.

This is attained by considering the product

$$(19)\quad f^{**}(z) = \Sigma c_n^{**} z^n$$

$$= \prod_{z=2}^{q} \left(1-\frac{z}{z_\nu}\right)^{r_\nu} f(z) \equiv (1+b_1' z+b_2' z^2+\ldots+b_{q'-1}' z^{q'-1}) f(z),$$

where $r_\nu \leqslant r$ is the smallest integer necessary to ensure that the weight of the singularity at $z=z_\nu$ is less than $[\sigma-r+1, k]$.

$f^{**}(z)$ now satisfies the requirements of Lemma 5 and hence, from (16), positive constants A^{**} and B^{**} exist such that, for every $n > n_0$,

$$(20) \quad A^{**}n^{\sigma-\nu}(\log n)^k \leqslant |c^{**}_{n-1}|+|c^{**}_{n-2}|+\ldots+|c^{**}_{n-\nu}| \leqslant B^{**}n^{\sigma-1}(\log n)^k.$$

From (19) it follows that

$$|c^{**}_n| = \left|\sum_{\nu=0}^{q'-1} b_\nu c_{n-\nu}\right| \leqslant M' \sum_{\nu=0}^{q'-1} |c_{n-\nu}|,$$

where $M' = \max(1, |b_1'|, \ldots, |b'_{q'-1}|)$, and hence that

$$\sum_{\nu=0}^{q'-1} |c_{n-\nu}| \geqslant M'^{-1}|c^{**}_n|.$$

Combining this with (20), we see that

$$\sum_{\nu=1}^{q'+r-1} |c_{n-\nu}| \geqslant An^{\sigma-r}(\log n)^k,$$

where $q'A > A^{**}$, giving an important result due to Pólya[†].

Theorem H_2. *If the singularities of $f(z)$ on $|z| = 1$ are all of algebraic-logarithmic type, of greatest weight $[\sigma, k]$, there exist positive numbers A and B such that, for $n > n_0$,*

$$An^{\sigma-r}(\log n)^k \leqslant |c_{n-1}|+|c_{n-2}|+\ldots+|c_{n-m}| \leqslant Bn^{\sigma-1}(\log n)^k,$$

where $m \leqslant qr$.

The right-hand side of the inequality follows from Theorem D_2. We notice that

$$m = r+q'-1 = r+\sum_{\nu=2}^{q} r_\nu, \quad \text{with} \quad r_\nu \leqslant r.$$

4. *Dominant singularities of algebraic-logarithmic type.* From the definition of weight in § 2, we obtain the following improved form of Theorem A_2.

Theorem A. *If the weight of a singular point of algebraic-logarithmic type of $f(z)$ at $z = z'$ is $[\sigma, k]$, then the weight of the corresponding singularity of $\{1-(z/z')\}f(z)$ at $z = z'$ is $[\sigma-1, k]$, unless the singularity is a simple pole.*

[†] **6**, 267.

The idea of the weight of a function on the circle of convergence $|z| = 1$ can be extended to include any function of finite order, for we can determine the weight $[\omega, \omega_1 - 1]$ of such a function, given by the expansion (1), by means of the formulae

$$\omega = 1 + \overline{\lim_{n\to\infty}} \frac{\log|c_n|}{\log n} \tag{2}$$

and

$$\omega_1 = 1 + \overline{\lim_{n\to\infty}} \frac{\log|c_n| - (\omega - 1)\log n}{\log^2 n}. \tag{21}$$

In the case of a function with singularities of algebraic-logarithmic type on $|z| = 1$ of greatest weight $[\sigma, k]$, it follows from Theorem D_2 that $\omega = \sigma$, $\omega_1 = 1 + k$.

Consider a function $f(z)$ consisting of the sum of two functions, $f_1(z)$ possessing only singularities of algebraic-logarithmic type on $|z| = 1$, of greatest weight $[\sigma, k]$, and $f_2(z)$ of weight $[\sigma', k']$, less than $[\sigma, k]$, on $|z| = 1$. On writing

$$f_1(z) \equiv \Sigma\, c_{n1} z^n, \quad f_2(z) \equiv \Sigma\, c_{n2} z^n,$$

and applying (2) and (21) to $f(z) \equiv \Sigma\,(c_{n1} + c_{n2}) z^n$, we see that the weight of $f(z)$ is also $[\sigma, k]$. In this case the singularities of weight $[\sigma, k]$ of $f(z)$ on $|z| = 1$ will be called the *dominant* singularities of $f(z)$ on $|z| = 1$. They are evidently of algebraic-logarithmic type. The results of §§ 2 and 3 may now be extended to functions with dominant singularities of algebraic-logarithmic type on the circle of convergence.

Suppose that there are l singularities of greatest weight $[\sigma, k]$ of $f_1(z)$ on $|z| = 1$ at $z_1, z_2, \ldots, z_l$. Then the weight of $L(z) f_1(z)$ on $|z| = 1$, where

$$L(z) \equiv \left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{z_2}\right)\ldots\left(1 - \frac{z}{z_l}\right),$$

is, from Theorem A, either $[\sigma - 1, k]$ or $[\sigma'', k'']$, the weight of the next strongest singularity of $f_1(z)$ on $|z| = 1$, whichever is the greater.

If we write $L(z) f_2(z) \equiv \Sigma\, c'_{n2} z^n$, it follows that, with a suitable constant M,

$$|c'_{n2}| < M \sum_0^l |c_{n-\nu, 2}|,$$

and the application of (2) and (21) then shows that the weight of $L(z) f_2(z)$ cannot exceed $[\sigma', k']$. Since

$$L(z) f(z) = L(z) f_1(z) + L(z) f_2(z),$$

it follows, as above, that the weight of $L(z) f(z)$ is the greater of the weights of

$L(z)f_1(z)$ and $L(z)f_2(z)$ and is therefore less than $[\sigma, k]$, the weight of $f(z)$. Moreover, $L(z)$ is the polynomial of lowest degree for which this is true, for any polynomial which does not contain all the factors $1-(z/z_\nu)$ $(\nu = 1, 2, \ldots, l)$ does not reduce the weight of $f_1(z)$ and consequently of $f(z)$. Hence we have

THEOREM B. *If the dominant singularities of* $f(z)$ *on* $|z| = 1$ *are all of algebraic-logarithmic type and of weight* $[\sigma, k]$, *then there exists a unique polynomial* $L(z)$, *of lowest degree, such that the weight of* $L(z)f(z)$ *on* $|z| = 1$ *is less than* $[\sigma, k]$ *and such that the singularities of* $f(z)$ *of weight* $[\sigma, k]$ *are located at the points* $z = z_\nu$ $(\nu = 1, 2, \ldots, l)$.

When the singularities of $f(z)$ of highest order ω on $|z| = 1$ are all of algebraic-logarithmic type the above proof can be repeated, in full detail, provided that the term "weight $[\sigma, k]$" is replaced by "order ω" and other "weights" by corresponding "orders", the term "dominant" being replaced by "of highest order". This leads to the proof of Theorem B_2, established by Hadamard† in the case of the simpler type of algebraic-logarithmic singularity. If $P(z)$ is the polynomial which selects the singularities of highest order of $f(z)$ on $|z| = 1$, and if these are of algebraic-logarithmic type, they can be further separated in accordance with their weights. For we have incidentally proved above that the weight of $f(z)$ on $|z| = 1$ is that of its dominant singularities and that the singularity at $z = z_r$ of $P(z)f(z)/\{1-(z/z_r)\}$ is the dominant singularity of this function on $|z| = 1$ and has, by Theorem A, the same weight as the corresponding singularity of $f(z)$. Since the singularity at $z = z_r$ is of algebraic-logarithmic type in the case of either function, its weight is determined by applying formulae (2) and (21) to the function $P(z)f(z)/\{1-(z/z_r)\}$.

If all the singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type, then, combining the above process with the methods following Theorem B_2, we can order all these singularities in accordance with their weights.

Theorem D_2 applied to $f_1(z)$ shows that a positive constant C exists such that

$$\overline{\lim_{n\to\infty}} |c_{n1}|/n^{\sigma-1}(\log n)^k = C,$$

while (2) and (21) applied to $f_2(z)$ show that to every positive ϵ corresponds an n_0 such that, for $n > n_0$,

$$(22) \qquad |c_{n2}| < n^{\sigma'-1}(\log n)^{k'+\epsilon},$$

where either $\sigma' < \sigma$ or $\sigma' = \sigma$ and $k' < k$. This leads to the result below.

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† **4**, 185.

THEOREM D. *If the dominant singularities of $f(z)$ on $|z|=1$ are of algebraic-logarithmic type, and if $f(z)$ is of weight $[\sigma, k]$, then there exists a positive constant C such that*

$$\overline{\lim_{n\to\infty}} |c_n|/n^{\sigma-1}(\log n)^k = C. \tag{9}$$

In the same way, by applying Theorem E_2 to $f_1(z)$, we see that for every positive $h<1$ there exists a positive constant C_2 such that

$$|c_{n1}| \geqslant C_2 n^{\sigma-1}(\log n)^k$$

holds for a sequence of n of lower density at least h/l. In conjunction with (22) this leads to the following extension of Theorem E_2:

THEOREM E. *If the dominant singularities of $f(z)$ on $|z|=1$ are of algebraic-logarithmic type, and if there are l singularities of greatest weight $[\sigma, k]$, then to every positive $h<1$ there exists a positive constant C_2 such that*

$$|c_n| \geqslant C_2 n^{\sigma-1}(\log n)^k$$

holds for a sequence of n of lower density at least h/l.

Similar considerations enable us to extend Theorem F_2 as follows:

THEOREM F. *If the l dominant singularities of $f(z)$ on $|z|=1$ are of algebraic-logarithmic type and of weight $[\sigma, k]$, then for every $\eta>0$ there exists an infinite sequence of n such that*

$$\sum_{\nu=0}^{l-1} |c_{n-\nu-r}| \geqslant \frac{C^*-\eta}{M} n^{\sigma-1}(\log n)^k$$

holds for $r=0, \pm 1, \pm 2, \ldots, \pm q$, where q is $o(n/\log n)$ and C^* and M are positive constants.

For, applying Theorem F_2 to $f_1(z)$, we obtain the result

$$\sum_{\nu=0}^{l-1} |c_{n-\nu-r,1}| \geqslant \frac{C_1^*-\eta_1}{M} n^{\sigma-1}(\log n)^k,$$

while, from (22),

$$\sum_{\nu=0}^{l-1} |c_{n-\nu-r,2}| < n^{\sigma'-1}(\log n)^{k'+\epsilon'},$$

where, by taking n sufficiently large, ϵ' can be made arbitrarily small.

Finally,

$$\sum_{\nu=0}^{l-1} |c_{n-\nu-r}| \geqslant \sum_{\nu=0}^{l-1} |c_{n-\nu-r,1}| - \sum_{\nu=0}^{l-1} |c_{n-\nu-r,2}| \geqslant \frac{C^*-\eta}{M} n^{\sigma-1}(\log n)^k,$$

for $\eta > \eta_1$, since from (9) applied to $f(z) = f_1(z) + f_2(z)$ we get $C_1^* = C^*$.

The second form of this theorem may be obtained from Theorem F in the way that Theorem F_2' was obtained from Theorem F_2.

THEOREM F′. *If the l dominant singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type and of weight $[\sigma, k]$, then for every positive $C' < C^*/Ml$ there exists an infinite sequence of n such that*

$$\sum_{\nu=0}^{N-1} |c_{n\pm\nu}| > (N-l)\, C' n^{\sigma-1}(\log n)^k$$

for every N $(> l)$, which is $o(n/\log n)$.

Since from (22) the coefficients of $f_2(z)$ make no effective contribution to the terms of a dominant sequence of coefficients of $f(z)$, it follows that a sequence of dominant terms of $f_1(z)$ gives rise to a sequence of dominant terms of $f(z)$ and that to every term whch is dominant for $f(z)$ corresponds a coefficient which is dominant for $f_1(z)$. Hence the remarks following Theorem F_2' apply equally well when $f(z)$ has dominant algebraic-logarithmic singularities on $|z| = 1$, provided that we use Theorem D in place of Theorem D_2. Thus every member of a sequence of dominant coefficients of $f(z)$ satisfying (13) is contained in an interval of extent $o(n/\log n)$ of which the members satisfy Theorems F and F′.

Theorem G_2 is apparently limited in its extension. For the condition complementary to (22) relating to $f_2(z)$ shows that there exists an infinite sequence of n, associated with any positive ϵ, sufficiently small, for which

$$|c_{n2}| > n^{\sigma'-1}(\log n)^{k'-\epsilon}.$$

Thus nothing is to be gained by choosing the function $\eta(n)$ to be smaller than $\eta'(n)$, where

$$\eta'(n)\, n^{\sigma-1}(\log n)^k \geqslant n^{\sigma'-1}(\log n)^{k'+\epsilon}$$

for every $n > n_0$ determined by a sufficiently small positive ϵ. On the other hand, it is evident that, if the inequality (23) below is true for a function $\eta'(n)$, it remains true when $\eta'(n)$ is replaced by a smaller function. This leads to

THEOREM G. *If the l dominant singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type and of weight $[\sigma, k]$, and if $\eta(n)$ tends to zero* *with* $1/n$, *then*

$$\sum_{\nu=0}^{l-1} |c_{n-\nu}| \geqslant \eta(n)\, n^{\sigma-1}(\log n)^k \tag{23}$$

for every n save possibly a sequence of density zero.

Theorem H_2 admits of only a limited extension.

We first effect a separation of $f(z)$ into two functions $\phi_1(z)$ and $\phi_2(z)$, the latter being of weight $[\sigma', k']$ and containing all the singularities of $f(z)$ on $|z|=1$ which are not of algebraic-logarithmic type together with such singularities of that type as are of weight not exceeding $[\sigma', k']$, the former containing all the algebraic-logarithmic singularities of weight exceeding $[\sigma', k']$, in number q. We call $\phi_1(z)$ the *dominant* function and $\phi_2(z)$ the *residual* function. Theorem H_2 depends for its proof on Lemma 5, which contains the essential condition that the residual singularities are of weight less than $[\sigma-r+1, k]$. In Theorem H_2 these singularities are of algebraic-logarithmic type and it is possible to satisfy the condition by multiplying $f(z)$ by a suitable polynomial, as in (19). Such a procedure is not possible in the case of $\phi_2(z)$, unless all the residual singularities are of special type. With the appropriate modification of the hypothesis, the proofs of Lemma 5 and Theorem H_2 lead to the following generalization of Pólya's result:

THEOREM H. *If the dominant function of* $f(z)$ *on* $|z|=1$, *of weight* $[\sigma, k]$, *contains* q *singularities, and the smallest number of dominant elements in a dominant singularity is* r, *and if the residual function is of weight less than* $[\sigma-r+1, k]$, *then there exist positive constants* A *and* B *such that, for* $n > n_0$,

$$An^{\sigma-r}(\log n)^k \leqslant |c_{n-1}|+|c_{n-2}|+\ldots+|c_{n-m}| \leqslant Bn^{\sigma-1}(\log n)^k,$$

where $m \leqslant qr$.

The value of m is determined as in Theorem H_2. It should be noted, however, that q has a different significance in the two theorems.

5. *Polar singularities on the circle of convergence.* Hadamard has found necessary and sufficient conditions in terms of the coefficients of (1) that $f(z)$ should have only polar singularities on $|z|=1$ and has shown how to locate those poles and determine their multiplicities. These results are well known†. When the singularities of $f(z)$ on $|z|=1$ are of algebraic-logarithmic type, the results of Theorems B_2 and C_2 enable us to locate the poles on $|z|=1$, if any, and to determine their orders.

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† **4**, 118–134. See also **2**, **5**, **7**, **8**, in which accounts (of various lengths) of this work are given.

Let $Q_1(z)$ be the polynomial of Theorem C_2 which selects all the singularities of $f(z)$ on $|z| = 1$ once and once only. Let $Q_2(z)$ be the corresponding polynomial for the function $Q_1(z)f(z)$; let $Q_3(z)$ be the corresponding polynomial for $Q_2(z)f(z)$; ...; let $Q_\nu(z)$ be the corresponding polynomial for $Q_{\nu-1}(z)f(z)$; .... If $f(z)$ has simple poles on $|z| = 1$, $Q_1(z)f(z)$ is free of those particular singularities and the polynomial $Q_1(z)/Q_2(z)$ then determines the location of the simple poles of $f(z)$. $Q_1(z)f(z)$ has, by Theorem C_2, its singularities on $|z| = 1$ at the same points as $f(z)$, except for the simple poles of $f(z)$, with orders reduced by one. In the same way the simple poles of $Q_1(z)f(z)$, if any, are selected by the polynomial $Q_2(z)/Q_3(z)$, and, at the points so determined, $f(z)$ has poles of the second order. Continuing in this way it is evident that each member of the sequence $Q_1(z), Q_2(z), \ldots, Q_\nu(z), \ldots$ divides its predecessor and that the affixes of the simple, second order, ..., ν-th order, ... poles of $f(z)$ on $|z| = 1$ are determined by the polynomials $Q_1(z)/Q_2(z)$, $Q_2(z)/Q_3(z)$, ..., $Q_\nu(z)/Q_{\nu+1}(z)$, ... Hence we have

Theorem 1. *A necessary and sufficient condition that $f(z)$ should have poles on $|z| = 1$ is that the polynomials $Q_1(z), Q_2(z), \ldots, Q_\nu(z), \ldots$ decrease in degree; the number of poles is determined by the ultimate defect in degree and their positions and orders are determined by the sequence of polynomials $Q_1(z)/Q_2(z)$, $Q_2(z)/Q_3(z)$, ..., $Q_\nu(z)/Q_{\nu+1}(z)$, ...*

If k is the highest order among the poles on $|z| = 1$, then the polynomial which selects the polar singularities in accordance with their multiplicities may be written in the form

$$Q_1(z)\, Q_2(z) \ldots Q_k(z)/\{Q_{k+1}(z)\}^k.$$

Evidently $Q_{k+1}(z)$ determines the affixes of all non-polar singularities. Their weights are determined as in § 4, following Theorem B.

If the polynomial $P(z)$ of Theorem B_2 is identical with the polynomial $Q_k(z)/Q_{k+1}(z)$, then the polar singularities of order k are dominant and there are no other dominant singularities on $|z| = 1$.

If $Q_k(z)/Q_{k+1}(z)$ is of smaller degree than $P(z)$, but divides $P(z)$, then the polar singularities of order k, although singularities of the highest order, are not dominant on $|z| = 1$, since the non-polar singularities of highest order will have a greater weight; for the weight of a polar singularity of highest order ω is $[\omega, 0]$, while that of a non-polar singularity of algebraic-logarithmic type is $[\omega, k]$, where k is positive.

If $P(z)$ and $Q_k(z)/Q_{k+1}(z)$ are prime to one another, then there are no poles among the singularities of highest order on $|z| = 1$.

THEOREM 2. *A necessary and sufficient condition that the dominant singularities of* $f(z)$ *on* $|z| = 1$ *should be all poles is that* $P(z)$ *should be identical with* $Q_k(z)/Q_{k+1}(z)$.

These results cannot be extended to the case in which the dominant singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type unless the order of the residual function is less than unity. In this case the methods of Theorem 1 enable us to select all the polar singularities of $f(z)$ on $|z| = 1$ and to determine their orders provided that we take, in place of the polynomial $Q(z)$, that polynomial which selects all the singularities of greater order than the residual function once and once only.

In any case, however, the methods of Theorem B enable each singularity of the dominant function to be located and its weight to be determined. Since the weight of a pole is necessarily of the form $[\omega, 0]$, where ω is a positive integer, the poles of the dominant function are readily obtained, and the polynomial selecting the set of poles of a given order arises as in the discussion following Theorem B_2.

6. *Theorems of Pólya and Szegö.* Kronecker† obtained necessary and sufficient conditions in terms of the coefficients of (1) in order that $f(z)$ should represent a rational function. Theorems 3 and 5, due respectively to Pólya and Szegö, give criteria for certain classes of rational functions. Pólya proved the following theorem‡:

THEOREM 3. *If all the coefficients of* $f(z) = \Sigma c_n z^n$ *are integers and if* $f(z)$ *is regular in a domain whose inner conform radius with respect to* $z = 0$ *exceeds* 1, *then* $f(z)$ *is rational.*

By hypothesis, the circle of convergence is not a natural boundary and the theorem is eventually concerned with a finite number of singularities on the circle of convergence. We prove a modified form of the theorem for the case in which the singularities on $|z| = 1$ are all of algebraic-logarithmic type, making use of Theorem C_2.

THEOREM 4. *If the singularities of* $f(z)$ *on* $|z| = 1$ *are all of algebraic-logarithmic type, and if, for every* $n > n_0$, *all the* c_n *are integers, then* $f(z)$ *is rational.*

Since, for $n > n_0$, every c_n is either a real or complex integer or zero, the order ω of $f(z)$ on $|z| = 1$ is greater than or equal to 1, from (2), and is finite, by hypothesis.

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† 2, 321–2.

‡ 2, 323–4.

Let $Q(z)$ be the polynomial which selects all the singularities of $f(z)$ on $|z|=1$. By Theorem C_2, an integer k exists such that the order of $\{Q(z)\}^k f(z)$ on $|z|=1$ is $\varpi<1$. Let

$$\{Q(z)\}^k \equiv 1+a_1' z+a_2' z^2+\ldots+a_q' z^q.$$

Then, from (2), for every $\epsilon>0$, there exists an n_0 such that, for $n>n_0$,

$$|c_n+a_1' c_{n-1}+\ldots+a_q' c_{n-q}| < \frac{1}{n^{1-\varpi-\epsilon}}.$$

In fact, for $n>n_0$,

$$c_{n+r}+a_1' c_{n+r-1}+\ldots+a_q' c_{n+r-q}=\epsilon_{n+r} \quad (r=0,\ 1,\ 2,\ \ldots), \tag{24}$$

where $|\epsilon_{n+r}|<1/(n+r)^{1-\varpi-\epsilon}$.

Consider the determinant

$$\begin{vmatrix} c_{n+r} & \ldots\ c_n & c_{n-1} & \ldots\ c_{n-q} \\ c_{n+r+1} & \ldots\ c_{n+1} & c_n & \ldots\ c_{n-q+1} \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \\ c_{n+q+2r} & \ldots\ c_{n+q+r} & c_{n+q+r-1} & \ldots\ c_{n+r} \end{vmatrix} = \begin{vmatrix} \epsilon_{n+r} & \ldots\ \epsilon_n & c_{n-1} & \ldots\ c_{n-q} \\ \epsilon_{n+r+1} & \ldots\ \epsilon_{n+1} & c_n & \ldots\ c_{n-q+1} \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \\ \epsilon_{n+q+2r} & \ldots\ \epsilon_{n+q+r} & c_{n+q+r-1} & \ldots\ c_{n+r} \end{vmatrix},$$

an equality which holds in virtue of equations (24). The modulus of the determinant on the left-hand side is evidently an integer or zero, since its elements are all integral or zero. On the other hand, the modulus of the determinant on the right is less than

$$(q+r+1)!\ \{(n+q+r-1)^{\omega-1+\epsilon}\}^q/(n^{1-\varpi-\epsilon})^{r+1}.$$

Consequently, if we choose $r>r_0$ to satisfy the inequality

$$(1-\varpi-\epsilon)(r+1)>(\omega-1+\epsilon)q,$$

it follows that this determinant is of modulus less than 1 for a sufficiently large n.

Hence, for $r>r_0$, $n>n_0$, every determinant of the above type vanishes, and, by Kronecker's theorem†, $f(z)$ is rational.

If the coefficients of Theorems 3, 4 are real, then the poles in question will be at pairs of conjugate complex points on $|z|=1$, including possibly the points $z=1$ or $z=-1$. If the coefficients are positive, it follows from

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† *Loc. cit.*

Theorem 14 that one of the *dominant* poles is at $z=1$ and that there may be others at pairs of conjugate points on $|z|=1$, including possibly $z=-1$.

Szegö has proved the following theorem†:

THEOREM 5. *If among the coefficients* c_n *of* $f(z)=\Sigma c_n z^n$ *there are only a finite number of different numbers, then either* $f(z)=Q(z)/(1-z^m)$, *where* $Q(z)$ *is a polynomial and* m *is a positive integer, or else* $f(z)$ *cannot be continued beyond* $|z|=1$.

Notice that the function Σz^{ν_n}, where $\lim\limits_{n\to\infty} \nu_n/n=\infty$, satisfies the hypothesis and has the circle of convergence as a natural boundary‡.

We prove a modified form of this theorem, based on Theorem B.

THEOREM 6. *If among the coefficients of* $f(z)=\Sigma c_n z^n$ *there are only a finite number of different numbers and if the dominant singularities on* $|z|=1$ *are all of algebraic-logarithmic type, then* $f(z)=Q(z)/(1-z^m)$, *where* $Q(z)$ *is a polynomial and* m *is a positive integer.*

Since the coefficients of $f(z)$ are bounded, it is evident, from (2) and (21), that the weight of $f(z)$ on $|z|=1$ is $[1, 0]$. Let the polynomial $L(z)$ of lowest degree l in Theorem B, which selects the dominant singularities of $f(z)$ on $|z|=1$, be

$$L(z)\equiv 1+a_1 z+a_2 z^2+\ldots+a_l z^l. \tag{25}$$

Since the number of c_n and a_r which are different is finite, it is clear that the function $L(z)f(z)$ is also one whose Taylor series has only a finite number of different coefficients. The weight of $L(z)f(z)$ on $|z|=1$ is therefore also $[1, 0]$, unless all the coefficients, from a certain point onwards, vanish, *i.e.* unless $L(z)f(z)=M(z)$, where $M(z)$ is a polynomial. Since, from Theorem B, $L(z)f(z)$ must be of smaller weight than $f(z)$ on $|z|=1$, it follows that $f(z)$ is rational. Applying the same procedure as in the last part of Szegö's proof, we deduce that, after a certain suffix, all the coefficients are repeated regularly and therefore that $f(z)$ is of the form $Q(z)/(1-z^m)$.

The function $(1+z^2)^{-1}$ satisfies the conditions of Theorems 5 and 6 and has its only singularities at $z=\pm i$. We cannot, however, deduce from Theorem 6 that $m=l$ and that the points $z=e^{2k\pi i/m}$ are all singular for $f(z)$. The reason for this is that $Q(z)$ and $1-z^m$ are not necessarily prime to one another. It is, however, true that $m \geqslant l$ and that all the poles of $f(z)$ are to be found at some of the points $e^{2k\pi i/m}$. In particular, if, from a certain index, the coefficients are all real, the poles are at pairs of conjugate points on $|z| = 1$, including possibly $z = 1$ or $z = -1$.

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† 2, 324–7.

‡ 2, 373.

It is evident from Vivanti's theorem (Theorem 13) that, if the coefficients are all positive for $n > n_0$, then there is a pole at $z = 1$. This can also be shown quite simply, as follows.

Assume the contrary to be the case. Then $f(z)$ can be written in the form

$$\frac{Q(z)}{1-z^m} = \frac{Q_1(z)}{1+z+\ldots+z^{m-1}} \equiv c_0+c_1 z+c_2 z^2+\ldots+c_n z^n+\ldots,$$

where, for $n > n_0$, $c_n \geqslant 0$ and $Q_1(z)$ is a polynomial. The identity

$$Q_1(z) \equiv (1+z+\ldots+z^{m-1})(c_0+c_1 z+c_2 z^2+\ldots+c_n z^n+\ldots)$$

can then only be satisfied if $c_n = 0$ for every $n > n_0'$. Since this is not possible, it follows that $Q(z)$ and $1-z$ must be prime to one another and $z = 1$ is therefore a singular point of $f(z)$.

7. *Theorems of Mandelbrojt.* The following theorem, due to Mandelbrojt[†], has been the starting point of a number of interesting investigations[‡].

Theorem 7. *If $f(z) \equiv \Sigma c_{\lambda_m} z^{\lambda_m}$ is meromorphic on its circle of convergence and has thereon k poles of the highest order, then*

$$\varlimsup_{m\to\infty} (\lambda_{m+1}-\lambda_m) < k+1.$$

In other words, the number of consecutive zero coefficients is less than k for $m > m_0$.

There have been a number of generalizations of this theorem to include algebraic-logarithmic singularities[‡]. Although result (5) and Theorems H_2 and H are probably the most complete *formal* extensions of this theorem so far attained, yet Theorems E–G bear intimately on the problem, which is really concerned with the distribution of zero coefficients. In view of the fact that the restriction, that the dominant singularities on the circle of convergence are of algebraic-logarithmic type, is not particularly heavy, the information available from Theorems E–H on the distribution of the coefficients in accordance with their size is surprisingly complete.

[†] 2, 381; 7, 262–3.

[‡] The literature on this and related problems is extensive. See 7, 273, foot-note.

Under the different hypotheses, result (5) shows that the number of consecutive zero coefficients is less than l, the number of singularities of greatest weight, of which one at least contains only a single element of greatest weight; while Theorem H_2 states that the number of consecutive zero coefficients is less than qr, where q is the total number of singularities on $|z| = 1$ and r the smallest number of strongest elements in a dominant singularity. Each of these two theorems is concerned with functions having only singularities of algebraic-logarithmic type on the circle of convergence. Theorem H shows that in the contrary case, if the residual function is of weight less than $[\sigma - r + 1, k]$, where $[\sigma, k]$ is the weight of the dominant function, then the number of consecutive zero coefficients is less than qr, where r has the same significance as before, but q is now the total number of singularities in the dominant function. These are all of algebraic-logarithmic type and are located on the circle of convergence.

Theorems E–G give detailed information on the distribution of zero coefficients. Theorem E can be restated as follows:

THEOREM 8. *If the dominant singularities of $f(z)$ on $|z| = 1$, l in number, are of algebraic-logarithmic type, then*

$$\varliminf_{m\to\infty} (\lambda_{m+1} - \lambda_m) < l+1.$$

For if this were not true every non-zero coefficient would be followed by at least l consecutive zero coefficients (from a certain point onwards), and the lower density of the sequence of n in Theorem E would be at most $1/(l+1)$. Since we can choose $h > l/(l+1)$, the lower density, which is at least h/l, exceeds $1/(l+1)$ in this case.

Theorems G and F together give the following result:

THEOREM 9. *If the dominant singularities of $f(z)$ on $|z| = 1$, l in number, are of algebraic-logarithmic type, then, for every m,*

$$\lambda_{m+1} - \lambda_m < l+1, \tag{26}$$

save possibly for a sequence of density zero. In particular, condition (26) *is satisfied in every neighbourhood of a dominant c_n of extent at least $o(n/\log n)$.*

Mandelbrojt extended Theorem 7 in the following manner:

THEOREM 10. *If*

$$\varlimsup_{m\to\infty} (\lambda_{m+1} - \lambda_m) = \infty, \tag{27}$$

then $f(z)$ has on its circle of convergence at least one singularity which is not a pole.

Theorem H_2 leads to the following extension of this result:

THEOREM 11. *If (27) holds, then $f(z)$ has on its circle of convergence at least one singularity which is not of algebraic-logarithmic type.*

Theorem H, which deals with this case, produces the further extension:

THEOREM 12. *If (27) holds and $f(z)$ has on its circle of convergence at least one singularity which is not of algebraic-logarithmic type, then the residual function is of weight at least $[\sigma-r+1, k]$, where σ, r, k are defined as in Theorem H.*

If such singularities are not dominant, Theorem 9 shows that the totality of zero coefficients for which $\lambda_{m+1}-\lambda_m > l$ is of density zero, so that large gaps must be very infrequent if they exist at all.

8. *The theorems of Vivanti and Dienes.* Vivanti's theorem[†] starts from rather more general hypotheses than the theorems of § 6 and consequently the information to be derived is less precise. It is, however, of fundamental importance as the basis of the theorems of Dienes and Fabry.

THEOREM 13. *If the radius of convergence of $\Sigma c_n z^n$ is 1, and if, for $n > n_0$, every $c_n \geqslant 0$, then $z=1$ is a singular point of $\Sigma c_n z^n$.*

We extend this result in the following theorem, the proof of which depends on Theorems B and F. Like Theorem 13, it is the first of a sequence of important results.

THEOREM 14. *If the radius of convergence of $\Sigma c_n z^n$ is 1, and if, for $n > n_0$, every $c_n \geqslant 0$, and if the dominant singularities on $|z|=1$ are of algebraic-logarithmic type, then one of them is at $z=1$, and the others (if any) are at pairs of conjugate points on $|z|=1$, including possibly $z=-1$.*

Let $L(z)$, as in (25), be the polynomial of Theorem B which selects all the singular points of greatest weight $[\sigma, k]$ on $|z|=1$. If $[\sigma', k']$, less than $[\sigma, k]$, is the weight of $L(z)f(z)$ on $|z|=1$, then, applying (2) and (21) to

$$L(z)f(z) = \Sigma(c_n+a_1 c_{n-1}+\ldots+a_l c_{n-l}) z^n,$$

[†] 2, 226.

we see that for every $\epsilon > 0$ there exists an n_0 such that, for $n > n_0$,

$$(28) \qquad |c_n + a_1 c_{n-1} + \ldots + a_l c_{n-l}| < n^{\sigma'-1} (\log n)^{k'+\epsilon}.$$

This result can be written

$$c_n + a_1 c_{n-1} + \ldots + a_l c_{n-l} = \alpha_n n^{\sigma'-1} (\log n)^{k'+\epsilon} \quad (n > n_0),$$

where $|\alpha_n| < 1$. Taking $(\kappa+1)l+1$ successive relations and adding, we obtain

$$\begin{aligned}(29) \quad & a_l c_{n-(\kappa+1)l} + (a_{l-1}+a_l) c_{n-(\kappa+1)l+1} + \ldots + (a_1+a_2+\ldots+a_l) c_{n-\kappa l+1} \\ & + (1+a_1+\ldots+a_l)\left(\sum_{\nu=0}^{\kappa l} c_{n-\nu}\right) + (1+a_1+\ldots+a_{l-1}) c_{n+1} + \ldots \\ & + (1+a_1) c_{n+l-1} + c_{n+l} = \sum_{\nu=0}^{(\kappa+1)l} \alpha_{n-\nu+l} (n-\nu+l)^{\sigma'-1} \{\log (n-\nu+l)\}^{k'+\epsilon}.\end{aligned}$$

If we write $\kappa l + 1 = N$ in Theorem F′, it follows that, for an infinite sequence of n determined by the positive constant C',

$$\begin{aligned}(30) \qquad |1+a_1+\ldots+a_l| \left(\sum_{\nu=0}^{\kappa l} c_{n-\nu}\right) &= |L(1)| \left(\sum_{\nu=0}^{\kappa l} c_{n-\nu}\right) \\ &\geqslant |L(1)| (\kappa-1) l C' n^{\sigma-1} (\log n)^k.\end{aligned}$$

Now the sum of the coefficients of the remaining c_n on the left-hand side of (29) cannot exceed $l(1+|a_1|+\ldots+|a_l|) \leqslant l \,.\, 2^l$, and the size of the largest c_n is governed by (9). In fact, corresponding to n_0 there exists an $\epsilon' > 0$ such that, for $n > n_0$,

$$(31) \qquad \max(|c_{n-(\kappa+1)l}|, \ldots, |c_{n+l}|) < (C+\epsilon') n^{\sigma-1} (\log n)^k \{1+\eta'(n)\},$$

where $\lim\limits_{n\to\infty} \eta'(n) = 0$ and C is a suitable positive constant.

Combining (30) and (31), we see that the modulus of the left-hand side of (29) is, for a sufficiently large κ, at least

$$\{|L(1)| (\kappa-1) l C' - l \,.\, 2^l (C+\epsilon'')\} n^{\sigma-1} (\log n)^k,$$

where $\epsilon'' > \epsilon'$, for an infinite sequence of $n > n_0$.

These results hold for any κ which is $o(n/\log n)$. Consequently, unless $L(1) = 0$, it follows that κ can be determined as the smallest integer satisfying the inequality

$$(\kappa-1) L(1) \,.\, C' > 2^l (C+\epsilon''),$$

and relation (29) now takes the form

$$K_n n^{\sigma-1}(\log n)^k = K_n' n^{\sigma'-1}(\log n)^{k'+\epsilon},$$

where K_n, K_n' are bounded and $|K_n| > K$, a positive constant.

Since such relations hold for an infinite sequence of n, we arrive at a contradiction, for either $\sigma > \sigma'$ or $\sigma = \sigma'$ and $k > k'$. It is, therefore, necessary that $L(1) = 0$ and that $f(z)$ has, in consequence, a dominant singularity at $z = 1$.

Further, the a_ν must be real. For, since all except a finite number of the coefficients c_ν of $f(z)$ are real, the expansions of $f(z)$ and $\bar{f}(z)$, the conjugate function, differ only in a finite number of terms. Hence these functions have their singularities on $|z| = 1$ at the same points, which must, therefore, be pairs of conjugate points or the points $z = \pm 1$. This remark proves the second part of the theorem.

Dienes' theorem† is a generalization of that due to Vivanti.

THEOREM 15. *If the radius of convergence of* $f(z) = \Sigma c_n z^n$ *is* 1, *and if, for* $n > n_0$, *the argument of* c_n *is between* $-\alpha$ *and* $+\alpha$, *where* $0 < \alpha < \frac{1}{2}\pi$, *then* $z = 1$ *is a singular point of* $f(z)$.

Dienes' theorem can be extended in the same way as Vivanti's theorem in the case where the dominant singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type.

Write
$$f(z) = f_1(z) + if_2(z),$$
where
$$f_1(z) = \Sigma c_n' z^n, \quad f_2(z) = \Sigma c_n'' z^n,$$
so that
$$c_n = c_n' + ic_n''$$
with c_n', c_n'' real and c_n' positive. It follows that

$$|c_n| < c_n' \sec \alpha, \quad |c_n''| < c_n \tan \alpha, \quad |c_n| > c_n', \ |c_n''|. \tag{32}$$

If the weight of $f(z)$ on $|z| = 1$ is $[\sigma, k]$, then, from Theorem D, a positive constant C exists such that

$$\overline{\lim_{n\to\infty}} |c_n|/n^{\sigma-1}(\log n)^k = C.$$

From (32) it follows that there exists a constant C', where $C \cos \alpha \leqslant C' \leqslant C$, such that

$$\overline{\lim_{n\to\infty}} c_n'/n^{\sigma-1}(\log n)^k = C', \quad \overline{\lim_{n\to\infty}} |c_n''|/n^{\sigma-1}(\log n)^k < C,$$

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† 2, 227.

and therefore that the weights on $|z|=1$ of $f(z)$, $f_1(z)$ are the same and are at least as great as that of $f_2(z)$. On writing

$$f(z)=f_1(z)+if_2(z), \quad 2f_1(z)=f(z)+\bar{f}(z),$$
$$\bar{f}(z)=f_1(z)-if_2(z), \quad 2if_2(z)=f(z)-\bar{f}(z),$$

we see that $f_1(z)$ and $f_2(z)$ have dominant singularities of algebraic-logarithmic type at the same points as $f(z)$ and $\bar{f}(z)$ on $|z|=1$, except in so far as these singularities may cancel. In general, some or all of these may cancel, but in the case considered $f_1(z)$ possesses at least one dominant singularity of algebraic-logarithmic type on $|z|=1$, for otherwise its weight would be less than that of $f(z)$ on $|z|=1$.

Let $L(z)$ be the polynomial of Theorem B which selects all the singular points of greatest weight $[\sigma, k]$ of $f(z)$ on $|z|=1$. Then, certainly, $L(z)\bar{L}(z)$, where $\bar{L}(z)$ has coefficients conjugate to $L(z)$, reduces the weight of $f(z)$ on $|z|=1$. Writing

$$L(z)\bar{L}(z)=1+a_1'z+\ldots+a_{2l}'z^{2l},$$

suppose that the weight of $L(z)\bar{L}(z)f(z)$ on $|z|=1$ is $[\sigma', k']$. Since

$$L(z)\bar{L}(z)f(z)=L(z)\bar{L}(z)f_1(z)+iL(z)\bar{L}(z)f_2(z),$$

it follows that the coefficients in the expansion of $L(z)\bar{L}(z)f_1(z)$ are the real parts of the coefficients in the expansion of $L(z)\bar{L}(z)f(z)$, and, therefore, that the weight of the former function on $|z|=1$ is at most $[\sigma', k']$.

Now $f_1(z)$ has dominant algebraic-logarithmic singularities on $|z|=1$ of weight $[\sigma, k]$, and its weight is reduced by the polynomial $L(z)\bar{L}(z)$. Further, its coefficients, from a certain index onward, are positive. We can therefore repeat the argument of Theorem 14, replacing the $f(z)$ therein by $f_1(z)$, and the $L(z)$ by $L(z)\bar{L}(z)$. As in Theorem 14, we conclude that $L(1)\bar{L}(1)=0$. Hence $L(z)$ contains the factor $1-z$, and therefore $z=1$ is a dominant singularity of $f(z)$ on $|z|=1$.

THEOREM 16. *If the dominant singularities of* $f(z)=\Sigma c_n z^n$ *on* $|z|=1$ *are of algebraic-logarithmic type, and if, for* $n>n_0$, *the argument of* c_n *is between* $-a$ *and* $+a$, *where* $0<a<\frac{1}{2}\pi$, *then* $f(z)$ *has a dominant singularity at* $z=1$.

9. *Theorems of Fabry.* Fabry proved a very general theorem†, giving sufficient conditions in order that $z=1$ should be a singular point of

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† 2, 377.

$f(z)$ on its circle of convergence $|z|=1$. It contains the theorems of Vivanti and Dienes as particular cases.

THEOREM 17. *Suppose that the radius of convergence of* $\Sigma c_n z^n$ *is* 1; *let* σ_n *denote the number of changes of sign of* $\mathfrak{R}(c_{m_n+\nu}e^{-\gamma_n i})$, *where* ν *runs through integer values between* $-\theta m_n$ *and* θm_n, $0<\theta<1$. *If there exist real numbers* γ_n *such that*

$$(a) \quad \lim_{n\to\infty} [\mathfrak{R}(c_{m_n}e^{-\gamma_n i})]^{1/m_n}=1, \qquad (b) \quad \lim_{n\to\infty} \sigma_n/m_n=0,$$

then $z=1$ *is a singular point of* $\Sigma c_n z^n$.

We obtain an extension of this theorem in the case for which the dominant singularities of $f(z)=\Sigma c_n z^n$ on $|z|=1$ are of algebraic-logarithmic type. The theorem is based on a similar set of conditions, and although the restriction that the dominant singularities are of algebraic-logarithmic type is added, the other conditions are somewhat easier.

THEOREM 18. *Suppose that the dominant singularities of* $f(z)=\Sigma c_n z^n$, *of weight* $[\sigma, k]$ *on* $|z|=1$, *are of algebraic-logarithmic type; let* σ_n *denote the number of changes of sign of* $\mathfrak{R}(c_{m_n+\nu}e^{-\gamma_n i})$, *where* ν *runs through integral values between* $-\theta_{m_n}$ *and* θ_{m_n}, *where* θ_n *is* $o(n/\log n)$. *If there exist real numbers* γ_n *such that*

$$(a) \quad \varliminf_{n\to\infty} \{\mathfrak{R}(c_{m_n}e^{-\gamma_n i})/m_n^{\sigma-1}(\log m_n)^k\}>0, \qquad (b) \quad \lim_{n\to\infty} \sigma_n/\theta_{m_n}=0,$$

then $f(z)$ *has a dominant singularity at* $z=1$.

Condition (a) shows that c_{m_n} is a dominant coefficient, and thus the coefficients with indices in the interval $-\theta_{m_n}<\nu<\theta_{m_n}$ satisfy the relations of Theorems F and F′. Suppose first that the coefficients belonging to this interval are such that the arguments of every $\mathfrak{R}(c_{m_n+\nu}e^{-\gamma_n i})$ lie between $-\alpha$ and $+\alpha$, where $0<\alpha<\frac{1}{2}\pi$.

Let $L(z)$ be the polynomial of Theorem B which selects all the singular points of greatest weight $[\sigma, k]$ of $f(z)$ on $|z|=1$. Employing the notation and methods of Theorem 14, we obtain the result

$$c_n+a_1c_{n-1}+\ldots+a_l c_{n-l}=\alpha_n n^{\sigma'-1}(\log n)^{k'+\epsilon} \quad (n>n_0).$$

Using n' to indicate indices in the interval $-\theta_{m_n}<\nu<\theta_{m_n}$, we obtain

$$(33) \quad c_{n'}e^{-\gamma_n i}+a_1c_{n'-1}e^{-\gamma_n i}+\ldots+a_l c_{n'-l}e^{-\gamma_n i}=\alpha_{n'}e^{-\gamma_n i}n'^{\sigma'-1}(\log n')^{k'+\epsilon}.$$

From (32) and Theorem F′, it follows that

$$\sum_{\nu=0}^{N-1} \mathfrak{R}(c_{n'-\nu}e^{-\gamma_n i}) > \cos\alpha \sum_{\nu=0}^{N-1} |c_{n'-\nu}| > (N-l)\, C' \cos\alpha\, n'^{\sigma-1}(\log n')^k.$$

When we add together $\{(\kappa+1)\,l+1\}$ successive relations (33), it follows that, as in Theorem 14, the real part of the left-hand side exceeds $K_1 n'^{\sigma-1}(\log n')^k$, where K_1 is a positive constant, while the right-hand side is $O\{n'^{\sigma'-1}(\log n')^{k'+\epsilon}\}$, and this is true for an infinite sequence of intervals. As in the proof of Theorem 14, the contradiction requires that $L(1)=0$.

This proves the theorem when $\mathfrak{R}(c_{m_n+\nu}e^{-\gamma_n i})$ is positive for every ν in the interval $-\theta_{m_n} < \nu < \theta_{m_n}$.

If condition (*b*) holds, $\mathfrak{R}(c_{m_n+\nu}e^{-\gamma_n i})$ will be negative for certain values of ν in the interval. For these terms γ_n must be replaced by $\gamma_n' = \gamma_n + \pi$. The hypothesis that $\mathfrak{R}(c_{m_n+\nu}e^{-\gamma_n i})$ is positive is now true for each subinterval.

The essential condition for the proofs of Theorems 14, 16, and 18 is that the interval should be large enough to enable κ to satisfy the inequality of Theorem 14,

$$(\kappa-1)\, L(1)\, C' > 2^l(C+\epsilon''),$$

with $L(1) \neq 0$. As the value of $L(1)$ is not known, it is necessary that the interval should become unbounded with n and the condition that the interval is of extent $o(n/\log n)$ makes this possible.

Condition (*b*) ensures that the same is true *for at least one subinterval* in each interval $-\theta_{m_n} < \nu < \theta_{m_n}$. Consequently the proof still holds when any interval is replaced by the appropriate subinterval over which γ_n remains unaltered.

This proves the theorem as stated.

A particular case of Fabry's theorem is†

THEOREM 19. *If* $\lim\limits_{n\to\infty} c_n/c_{n+1} = e^{i\alpha}$ *exists, then* $e^{i\alpha}$ *is a singular point of* $\Sigma\, c_n z^n$.

If we take $\alpha = 0$, it follows that, for every n,

$$c_{n+1} = (1+\epsilon_n)\, c_n, \tag{34}$$

where $\lim\limits_{n\to\infty} \epsilon_n = 0$.

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† 2, 377–9.

Suppose that the dominant singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type and of weight $[\sigma, k]$. Then, from Theorem D,

$$(35) \qquad |c_n| > C' n^{\sigma-1}(\log n)^k,$$

where C' is positive, for an infinite sequence of n. Let $L(z)$ be the polynomial of Theorem B which selects the dominant singularities of $f(z)$ on $|z| = 1$. It follows, as in (28), that

$$(28) \qquad |c_n + a_1 c_{n-1} + \dots + a_l c_{n-l}| < n^{\sigma'-1}(\log n)^{k'+\epsilon}$$

holds for every $n > n_0$ determined by $\epsilon > 0$.

From (34), (35), and (28), we have

$$|(1+a_1+\dots+a_l) + O(\epsilon_n, \epsilon_{n+1}, \dots, \epsilon_{n-l+1})| < 1/n^{\sigma-\sigma'}(\log n)^{k-k'-\epsilon'},$$

for every $n > n_0$ determined by an arbitrarily small $\epsilon' > \epsilon$. Hence

$$L(1) = 1 + a_1 + a_2 + \dots + a_l = 0,$$

and therefore $1-z$ is a factor of $L(z)$.

This proves that $z = 1$ is a dominant singularity of $f(z)$ on $|z| = 1$.

Moreover, it is the only dominant singularity of $f(z)$ on $|z| = 1$. For suppose that there are others. In this case $(1-z)f(z)$ and $f(z)$ are both of weight $[\sigma, k]$ on $|z| = 1$. Hence positive constants C_1, C exist such that

$$(9) \qquad \overline{\lim_{n\to\infty}}\, |\epsilon_n c_n|/n^{\sigma-1}(\log n)^k = C_1, \quad \overline{\lim_{n\to\infty}}\, |c_n|/n^{\sigma-1}(\log n)^k = C.$$

The former result shows that there is an infinite sequence of n such that

$$|\epsilon_n c_n| > C_1' n^{\sigma-1}(\log n)^k,$$

where $C_1' < C_1$, is positive, and the latter shows that, for every $n > n_0$,

$$|c_n| < C' n^{\sigma-1}(\log n)^k,$$

where $C' > C$.

Since $\lim_{n\to\infty} \epsilon_n = 0$, these two results are contradictory. Hence we have

Theorem 20. *If the dominant singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type and if $\lim_{n\to\infty} c_n/c_{n+1} = 1$ exists, then the strongest singularity of $f(z)$ on $|z| = 1$ is at $z = 1$.*

In the case in which all the singularities of $f(z)$ on $|z| = 1$ are of algebraic-logarithmic type, Bernstein and Rey Pastor† have proved Theorem 20 under the modified hypothesis that *there exists a sequence of values of n of density* 1 *for which* $\lim (c_n/c_{n+1}) = 1$.

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† 1; 10.

This restriction does not affect the validity of the proof of Theorem 20, since the relation (34) is then true for every n save possibly a sequence of density zero, and by Theorem E_2 it is possible to choose C' so that (35) holds for a sequence of values of n of positive lower density. Consequently (36) remains true for an infinite sequence of values of n, and this is sufficient to establish relation (37). The remainder of the proof follows as before.

The converse theorem was proved by Jungen[†].

References.

1. V. Bernstein, "Alcune osservazione sopra un teorema di Fabry", *Atti. Accad. naz. Lincei, Rend.* (6), 21 (1935), 780–785.
2. P. Dienes, *The Taylor series* (1931).
3. P. Dienes, *Leçons sur les singularités des fonctions analytiques* (1913).
4. J. Hadamard, "Essai sur l'étude des fonctions données par leur développement de Taylor", *Journal de Math.* (4), 8 (1892), 101–186.
5. J. Hadamard et M. Mandelbrojt, *La série de Taylor et son prolongement analytique* (1926).
6. R. Jungen, "Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence", *Commentarii Math. Helvetici*, 3 (1931), 266–306.
7. M. Mandelbrojt, *Modern researches on the singularities of functions defined by Taylor's series* (1927).
8. M. Mandelbrojt, *Les singularités des fonctions analytiques représentées par une série de Taylor* (1932).
9. N. E. Nörlund, *Leçons sur les séries d'interpolation* (1926).
10. J. Rey Pastor, "Sobre las singularidades algebrico-logaritmica de las funciones analiticas", *Bol. del Sem. Matemático instituto mat. Hispano-Americano*, 4 (1935), 41–46.
11. G. Pólya, "Untersuchungen über Lücken und Singularitäten von Potenzreihen", *Math. Zeitschrift*, 29 (1929), 549–640.

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[†] 6, 276–285.

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

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[Received 28 May, 1936.—Read 12 November, 1936.]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable. In § 8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†. These results have valuable applications. In particular, it is shown (§ 11) that the Hilbertian Entscheidungsproblem can have no solution.

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† Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte Math. Phys.*, 38 (1931), 173–198.

In a recent paper Alonzo Church† has introduced an idea of "effective calculability", which is equivalent to my "computability", but is very differently defined. Church also reaches similar conclusions about the Entscheidungsproblem‡. The proof of equivalence between "computability" and "effective calculability" is outlined in an appendix to the present paper.

1. *Computing machines.*

We have said that the computable numbers are those whose decimals are calculable by finite means. This requires rather more explicit definition. No real attempt will be made to justify the definitions given until we reach § 9. For the present I shall only say that the justification lies in the fact that the human memory is necessarily limited.

We may compare a man in the process of computing a real number to a machine which is only capable of a finite number of conditions $q_1, q_2, \ldots, q_R$ which will be called "m-configurations". The machine is supplied with a "tape" (the analogue of paper) running through it, and divided into sections (called "squares") each capable of bearing a "symbol". At any moment there is just one square, say the r-th, bearing the symbol $\mathfrak{S}(r)$ which is "in the machine". We may call this square the "scanned square". The symbol on the scanned square may be called the "scanned symbol". The "scanned symbol" is the only one of which the machine is, so to speak, "directly aware". However, by altering its m-configuration the machine can effectively remember some of the symbols which it has "seen" (scanned) previously. The possible behaviour of the machine at any moment is determined by the m-configuration q_n and the scanned symbol $\mathfrak{S}(r)$. This pair $q_n, \mathfrak{S}(r)$ will be called the "configuration": thus the configuration determines the possible behaviour of the machine. In some of the configurations in which the scanned square is blank (*i.e.* bears no symbol) the machine writes down a new symbol on the scanned square: in other configurations it erases the scanned symbol. The machine may also change the square which is being scanned, but only by shifting it one place to right or left. In addition to any of these operations the m-configuration may be changed. Some of the symbols written down will form the sequence of figures which is the decimal of the real number which is being computed. The others are just rough notes to "assist the memory". It will only be these rough notes which will be liable to erasure.

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† Alonzo Church, "An unsolvable problem of elementary number theory", *American J. of Math.*, 58 (1936), 345–363.

‡ Alonzo Church, "A note on the Entscheidungsproblem", *J. of Symbolic Logic*, 1 (1936), 40–41.

It is my contention that these operations include all those which are used in the computation of a number. The defence of this contention will be easier when the theory of the machines is familiar to the reader. In the next section I therefore proceed with the development of the theory and assume that it is understood what is meant by "machine", "tape", "scanned", etc.

2. *Definitions.*

Automatic machines.

If at each stage the motion of a machine (in the sense of § 1) is *completely* determined by the configuration, we shall call the machine an "automatic machine" (or a-machine).

For some purposes we might use machines (choice machines or c-machines) whose motion is only partially determined by the configuration (hence the use of the word "possible" in § 1). When such a machine reaches one of these ambiguous configurations, it cannot go on until some arbitrary choice has been made by an external operator. This would be the case if we were using machines to deal with axiomatic systems. In this paper I deal only with automatic machines, and will therefore often omit the prefix a-.

Computing machines.

If an a-machine prints two kinds of symbols, of which the first kind (called figures) consists entirely of 0 and 1 (the others being called symbols of the second kind), then the machine will be called a computing machine. If the machine is supplied with a blank tape and set in motion, starting from the correct initial m-configuration, the subsequence of the symbols printed by it which are of the first kind will be called the *sequence computed by the machine*. The real number whose expression as a binary decimal is obtained by prefacing this sequence by a decimal point is called the *number computed by the machine*.

At any stage of the motion of the machine, the number of the scanned square, the complete sequence of all symbols on the tape, and the m-configuration will be said to describe the *complete configuration* at that stage. The changes of the machine and tape between successive complete configurations will be called the *moves* of the machine.

Circular and circle-free machines.

If a computing machine never writes down more than a finite number of symbols of the first kind, it will be called *circular*. Otherwise it is said to be *circle-free.*

A machine will be circular if it reaches a configuration from which there is no possible move, or if it goes on moving, and possibly printing symbols of the second kind, but cannot print any more symbols of the first kind. The significance of the term "circular" will be explained in § 8.

Computable sequences and numbers.

A sequence is said to be computable if it can be computed by a circle-free machine. A number is computable if it differs by an integer from the number computed by a circle-free machine.

We shall avoid confusion by speaking more often of computable sequences than of computable numbers.

3. *Examples of computing machines.*

I. A machine can be constructed to compute the sequence 010101 .... The machine is to have the four *m*-configurations "𝔟", "𝔠", "𝔨", "𝔢" and is capable of printing "0" and "1". The behaviour of the machine is described in the following table in which "*R*" means "the machine moves so that it scans the square immediately on the right of the one it was scanning previously". Similarly for "*L*". "*E*" means "the scanned symbol is erased" and "*P*" stands for "prints". This table (and all succeeding tables of the same kind) is to be understood to mean that for a configuration described in the first two columns the operations in the third column are carried out successively, and the machine then goes over into the *m*-configuration described in the last column. When the second column is left blank, it is understood that the behaviour of the third and fourth columns applies for any symbol and for no symbol. The machine starts in the *m*-configuration 𝔟 with a blank tape.

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
𝔟	None	*P*0, *R*	𝔠
𝔠	None	*R*	𝔢
𝔢	None	*P*1, *R*	𝔨
𝔨	None	*R*	𝔟

If (contrary to the description in § 1) we allow the letters L, R to appear more than once in the operations column we can simplify the table considerably.

m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$	None	$P0$	$\mathfrak{b}$
	0	$R, R, P1$	$\mathfrak{b}$
	1	$R, R, P0$	$\mathfrak{b}$

II. As a slightly more difficult example we can construct a machine to compute the sequence 001011011101111011111.... The machine is to be capable of five m-configurations, viz. "$\mathfrak{o}$", "$\mathfrak{q}$", "$\mathfrak{p}$", "$\mathfrak{f}$", "$\mathfrak{b}$" and of printing "ə", "x", "0", "1". The first three symbols on the tape will be "əə0"; the other figures follow on alternate squares. On the intermediate squares we never print anything but "x". These letters serve to "keep the place" for us and are erased when we have finished with them. We also arrange that in the sequence of figures on alternate squares there shall be no blanks.

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$		$Pə, R, Pə, R, P0, R, R, P0, L, L$	$\mathfrak{o}$
$\mathfrak{o}$	1	R, Px, L, L, L	$\mathfrak{o}$
	0		$\mathfrak{q}$
$\mathfrak{q}$	Any (0 or 1)	R, R	$\mathfrak{q}$
	None	$P1, L$	$\mathfrak{p}$
$\mathfrak{p}$	x	E, R	$\mathfrak{q}$
	ə	R	$\mathfrak{f}$
	None	L, L	$\mathfrak{p}$
$\mathfrak{f}$	Any	R, R	$\mathfrak{f}$
	None	$P0, L, L$	$\mathfrak{o}$

To illustrate the working of this machine a table is given below of the first few complete configurations. These complete configurations are described by writing down the sequence of symbols which are on the tape, with the m-configuration written below the scanned symbol. The successive complete configurations are separated by colons.

```
      : ə ə 0   0 : ə ə 0   0 : ə ə 0   0 : ə ə 0   0     : ə ə 0   0   1 :
   𝔟        𝔬           𝔮               𝔮               𝔮             𝔭
   ə ə 0    0   1 : ə ə 0   0   1 : ə ə 0   0   1 : ə ə 0   0   1 :
          𝔭           𝔭                 𝔣                   𝔣
   ə ə 0    0   1 : ə ə 0   0   1     : ə ə 0   0   1   0 :
                𝔣                   𝔣               𝔬
   ə ə 0    0   1 x 0 : ....
            𝔬
```

This table could also be written in the form

𝔟 : ə ə 𝔬 0   0 : ə ə 𝔮 0   0 : ..., (C)

in which a space has been made on the left of the scanned symbol and the m-configuration written in this space. This form is less easy to follow, but we shall make use of it later for theoretical purposes.

The convention of writing the figures only on alternate squares is very useful: I shall always make use of it. I shall call the one sequence of alternate squares F-squares and the other sequence E-squares. The symbols on E-squares will be liable to erasure. The symbols on F-squares form a continuous sequence. There are no blanks until the end is reached. There is no need to have more than one E-square between each pair of F-squares: an apparent need of more E-squares can be satisfied by having a sufficiently rich variety of symbols capable of being printed on E-squares. If a symbol β is on an F-square S and a symbol α is on the E-square next on the right of S, then S and β will be said to be *marked* with α. The process of printing this α will be called marking β (or S) with α.

4. *Abbreviated tables.*

There are certain types of process used by nearly all machines, and these, in some machines, are used in many connections. These processes include copying down sequences of symbols, comparing sequences, erasing all symbols of a given form, etc. Where such processes are concerned we can abbreviate the tables for the m-configurations considerably by the use of "skeleton tables". In skeleton tables there appear capital German letters and small Greek letters. These are of the nature of "variables". By replacing each capital German letter throughout by an m-configuration and each small Greek letter by a symbol, we obtain the table for an *m*-configuration.

The skeleton tables are to be regarded as nothing but abbreviations: they are not essential. So long as the reader understands how to obtain the complete tables from the skeleton tables, there is no need to give any exact definitions in this connection.

Let us consider an example:

m-config.	*Symbol*	*Behaviour*	*Final m-config.*
$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$	ə	L	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	not ə	L	$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{B}$

From the *m*-configuration $\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$ the machine finds the symbol of form α which is farthest to the left (the "first α") and the *m*-configuration then becomes $\mathfrak{C}$. If there is no α then the *m*-configuration becomes $\mathfrak{B}$.

If we were to replace $\mathfrak{C}$ throughout by $\mathfrak{q}$ (say), $\mathfrak{B}$ by $\mathfrak{r}$, and α by x, we should have a complete table for the *m*-configuration $\mathfrak{f}(\mathfrak{q}, \mathfrak{r}, x)$. $\mathfrak{f}$ is called an "*m*-configuration function" or "*m*-function".

The only expressions which are admissible for substitution in an *m*-function are the *m*-configurations and symbols of the machine. These have to be enumerated more or less explicitly: they may include expressions such as $\mathfrak{p}(\mathfrak{e}, x)$; indeed they must if there are any *m*-functions used at all. If we did not insist on this explicit enumeration, but simply stated that the machine had certain *m*-configurations (enumerated) and all *m*-configurations obtainable by substitution of *m*-configurations in certain *m*-functions, we should usually get an infinity of *m*-configurations; *e.g.*, we might say that the machine was to have the *m*-configuration $\mathfrak{q}$ and all *m*-configurations obtainable by substituting an *m*-configuration for $\mathfrak{C}$ in $\mathfrak{p}(\mathfrak{C})$. Then it would have $\mathfrak{q}$, $\mathfrak{p}(\mathfrak{q})$, $\mathfrak{p}\big(\mathfrak{p}(\mathfrak{q})\big)$, $\mathfrak{p}\Big(\mathfrak{p}\big(\mathfrak{p}(\mathfrak{q})\big)\Big)$, ... as *m*-configurations.

Our interpretation rule then is this. We are given the names of the *m*-configurations of the machine, mostly expressed in terms of *m*-functions. We are also given skeleton tables. All we want is the complete table for the *m*-configurations of the machine. This is obtained by repeated substitution in the skeleton tables.

Further examples.

(In the explanations the symbol "→" is used to signify "the machine goes into the m-configuration. . . .")

$\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}\big(\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha\big)$	From $\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$ the first α is erased and $\rightarrow \mathfrak{C}$. If there is no $\alpha \rightarrow \mathfrak{B}$.
$\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$	E	$\mathfrak{C}$	
$\mathfrak{e}(\mathfrak{B}, \alpha)$		$\mathfrak{e}\big(\mathfrak{e}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha\big)$	From $\mathfrak{e}(\mathfrak{B}, \alpha)$ all letters α are erased and $\rightarrow \mathfrak{B}$.

The last example seems somewhat more difficult to interpret than most. Let us suppose that in the list of m-configurations of some machine there appears $\mathfrak{e}(\mathfrak{b}, x)$ $(= \mathfrak{q}$, say). The table is

	$\mathfrak{e}(\mathfrak{b}, x)$	$\mathfrak{e}\big(\mathfrak{e}(\mathfrak{b}, x), \mathfrak{b}, x\big)$
or	$\mathfrak{q}$	$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$.

Or, in greater detail:

$\mathfrak{q}$		$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$
$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$		$\mathfrak{f}\big(\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x), \mathfrak{b}, x\big)$
$\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x)$	E	$\mathfrak{q}$.

In this we could replace $\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x)$ by $\mathfrak{q}'$ and then give the table for $\mathfrak{f}$ (with the right substitutions) and eventually reach a table in which no m-functions appeared.

$\mathfrak{pe}(\mathfrak{C}, \beta)$			$\mathfrak{f}\big(\mathfrak{pe}_1(\mathfrak{C}, \beta), \mathfrak{C}, \text{ə}\big)$	From $\mathfrak{pe}$ $(\mathfrak{C}, \beta)$ the machine prints β at the end of the sequence of symbols and $\rightarrow \mathfrak{C}$.
$\mathfrak{pe}_1(\mathfrak{C}, \beta)$	Any	R, R	$\mathfrak{pe}_1(\mathfrak{C}, \beta)$	
	None	$P\beta$	$\mathfrak{C}$	
$\mathfrak{l}(\mathfrak{C})$		L	$\mathfrak{C}$	From $\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$ it does the same as for $\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$ but moves to the left before $\rightarrow \mathfrak{C}$.
$\mathfrak{r}(\mathfrak{C})$		R	$\mathfrak{C}$	
$\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$			$\mathfrak{f}\big(\mathfrak{l}(\mathfrak{C}), \mathfrak{B}, \alpha\big)$	
$\mathfrak{f}''(\mathfrak{C}, \mathfrak{B}, \alpha)$			$\mathfrak{f}\big(\mathfrak{r}(\mathfrak{C}), \mathfrak{B}, \alpha\big)$	
$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$			$\mathfrak{f}'\big(\mathfrak{c}_1(\mathfrak{C}), \mathfrak{B}, \alpha\big)$	$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$. The machine writes at the end the first symbol marked α and $\rightarrow \mathfrak{C}$.
$\mathfrak{c}_1(\mathfrak{C})$	β		$\mathfrak{pe}(\mathfrak{C}, \beta)$	

The last line stands for the totality of lines obtainable from it by replacing β by any symbol which may occur on the tape of the machine concerned.

$\mathfrak{ce}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{c}\left(\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha\right)$	$\mathfrak{ce}(\mathfrak{B}, \alpha)$. The machine copies down in order at the end all symbols marked α and erases the letters α; $\rightarrow \mathfrak{B}$.
$\mathfrak{ce}(\mathfrak{B}, \alpha)$		$\mathfrak{ce}\left(\mathfrak{ce}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha\right)$	
$\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$		$\mathfrak{f}\left(\mathfrak{re}_1(\mathfrak{C}, \mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha\right)$	$\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$. The machine replaces the first α by β and $\rightarrow \mathfrak{C} \rightarrow \mathfrak{B}$ if there is no α.
$\mathfrak{re}_1(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$	$E, P\beta$	$\mathfrak{C}$	
$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$		$\mathfrak{re}\left(\mathfrak{re}(\mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha, \beta\right)$	$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$. The machine replaces all letters α by β; $\rightarrow \mathfrak{B}$.
$\mathfrak{cr}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{c}\left(\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, a), \mathfrak{B}, \alpha\right)$	$\mathfrak{cr}(\mathfrak{B}, \alpha)$ differs from $\mathfrak{ce}(\mathfrak{B}, \alpha)$ only in that the letters α are not erased. The m-configuration $\mathfrak{cr}(\mathfrak{B}, \alpha)$ is taken up when no letters "a" are on the tape.
$\mathfrak{cr}(\mathfrak{B}, \alpha)$		$\mathfrak{cr}\left(\mathfrak{cr}(\mathfrak{B}, \alpha), \mathfrak{re}(\mathfrak{B}, a, \alpha), \alpha\right)$	

$\mathfrak{cp}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$		$\mathfrak{f}'\left(\mathfrak{cp}_1(\mathfrak{C}_1 \mathfrak{A}, \beta), \mathfrak{f}(\mathfrak{A}, \mathfrak{E}, \beta), \alpha\right)$
$\mathfrak{cp}_1(\mathfrak{C}, \mathfrak{A}, \beta)$	γ	$\mathfrak{f}'\left(\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma), \mathfrak{A}, \beta\right)$
$\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma)$	γ	$\mathfrak{C}$
	not γ	$\mathfrak{A}$.

The first symbol marked α and the first marked β are compared. If there is neither α nor β, $\rightarrow \mathfrak{E}$. If there are both and the symbols are alike, $\rightarrow \mathfrak{C}$. Otherwise $\rightarrow \mathfrak{A}$.

$$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta) \qquad \mathfrak{cp}\left(\mathfrak{e}\left(\mathfrak{e}(\mathfrak{C}, \mathfrak{C}, \beta), \mathfrak{C}, \alpha\right), \mathfrak{A}, \mathfrak{E}, \alpha, \beta\right)$$

$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$ differs from $\mathfrak{cp}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$ in that in the case when there is similarity the first α and β are erased.

$$\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta) \qquad \mathfrak{cpe}\left(\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta), \mathfrak{A}, \mathfrak{E}, \alpha, \beta\right).$$

$\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta)$. The sequence of symbols marked α is compared with the sequence marked β. $\rightarrow \mathfrak{E}$ if they are similar. Otherwise $\rightarrow \mathfrak{A}$. Some of the symbols α and β are erased.

$\mathfrak{q}(\mathfrak{C})$	Any	R	$\mathfrak{q}(\mathfrak{C})$	$\mathfrak{q}(\mathfrak{C}, \alpha)$. The machine finds the last symbol of form α. $\rightarrow \mathfrak{C}$.
	None	R	$\mathfrak{q}_1(\mathfrak{C})$	
$\mathfrak{q}_1(\mathfrak{C})$	Any	R	$\mathfrak{q}(\mathfrak{C})$	
	None		$\mathfrak{C}$	
$\mathfrak{q}(\mathfrak{C}, \alpha)$			$\mathfrak{q}\big(\mathfrak{q}_1(\mathfrak{C}, \alpha)\big)$	
$\mathfrak{q}_1(\mathfrak{C}, \alpha)$	α		$\mathfrak{C}$	
	not α	L	$\mathfrak{q}_1(\mathfrak{C}, \alpha)$	
$\mathfrak{pe}_2(\mathfrak{C}, \alpha, \beta)$			$\mathfrak{pe}\big(\mathfrak{pe}(\mathfrak{C}, \beta), \alpha\big)$	$\mathfrak{pe}_2(\mathfrak{C}, \alpha, \beta)$. The machine prints $\alpha\ \beta$ at the end.
$\mathfrak{ce}_2(\mathfrak{B}, \alpha, \beta)$			$\mathfrak{ce}\big(\mathfrak{ce}(\mathfrak{B}, \beta), \alpha\big)$	$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$. The machine copies down at the end first the symbols marked α, then those marked β, and finally those marked γ; it erases the symbols α, β, γ.
$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$			$\mathfrak{ce}\big(\mathfrak{ce}_2(\mathfrak{B}, \beta, \gamma), \alpha\big)$	
$\mathfrak{e}(\mathfrak{C})$	ə	R	$\mathfrak{e}_1(\mathfrak{C})$	From $\mathfrak{e}(\mathfrak{C})$ the marks are erased from all marked symbols. $\rightarrow \mathfrak{C}$.
	Not ə	L	$\mathfrak{e}(\mathfrak{C})$	
$\mathfrak{e}_1(\mathfrak{C})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{C})$	
	None		$\mathfrak{C}$	

5. *Enumeration of computable sequences.*

A computable sequence γ is determined by a description of a machine which computes γ. Thus the sequence 001011011101111... is determined by the table on p. 234, and, in fact, any computable sequence is capable of being described in terms of such a table.

It will be useful to put these tables into a kind of standard form. In the first place let us suppose that the table is given in the same form as the first table, for example, I on p. 233. That is to say, that the entry in the operations column is always of one of the forms $E : E$, $R : E$, $L : Pa : Pa$, $R : Pa$, $L : R : L :$ or no entry at all. The table can always be put into this form by introducing more m-configurations. Now let us give numbers to the m-configurations, calling them q_1, ..., q_R, as in § 1. The initial m-configuration is always to be called q_1. We also give numbers to the symbols S_1, ..., S_m

and, in particular, blank $= S_0$, $0 = S_1$, $1 = S_2$. The lines of the table are now of form

m-config.	*Symbol*	*Operations*	*Final m-config.*	
q_i	S_j	PS_k, L	q_m	(N_1)
q_i	S_j	PS_k, R	q_m	(N_2)
q_i	S_j	PS_k	q_m	(N_3)

Lines such as

$$q_i \qquad S_j \qquad E, R \qquad q_m$$

are to be written as

$$q_i \qquad S_j \qquad PS_0, R \qquad q_m$$

and lines such as

$$q_i \qquad S_j \qquad R \qquad q_m$$

to be written as

$$q_i \qquad S_j \qquad PS_j, R \qquad q_m$$

In this way we reduce each line of the table to a line of one of the forms (N_1), (N_2), (N_3).

From each line of form (N_1) let us form an expression $q_i S_j S_k L q_m$; from each line of form (N_2) we form an expression $q_i S_j S_k R q_m$; and from each line of form (N_3) we form an expression $q_i S_j S_k N q_m$.

Let us write down all expressions so formed from the table for the machine and separate them by semi-colons. In this way we obtain a complete description of the machine. In this description we shall replace q_i by the letter "D" followed by the letter "A" repeated i times, and S_j by "D" followed by "C" repeated j times. This new description of the machine may be called the *standard description* (S.D). It is made up entirely from the letters "A", "C", "D", "L", "R", "N", and from ";".

If finally we replace "A" by "1", "C" by "2", "D" by "3", "L" by "4", "R" by "5", "N" by "6", and ";" by "7" we shall have a description of the machine in the form of an arabic numeral. The integer represented by this numeral may be called a *description number* (D.N) of the machine. The D.N determine the S.D and the structure of the machine uniquely. The machine whose D.N is n may be described as $\mathcal{M}(n)$.

To each computable sequence there corresponds at least one description number, while to no description number does there correspond more than one computable sequence. The computable sequences and numbers are therefore enumerable.

Let us find a description number for the machine I of § 3. When we rename the m-configurations its table becomes:

q_1	S_0	PS_1, R	q_2
q_2	S_0	PS_0, R	q_3
q_3	S_0	PS_2, R	q_4
q_4	S_0	PS_0, R	q_1

Other tables could be obtained by adding irrelevant lines such as

q_1	S_1	PS_1, R	q_2

Our first standard form would be

$$q_1 S_0 S_1 R q_2;\ q_2 S_0 S_0 R q_3;\ q_3 S_0 S_2 R q_4;\ q_4 S_0 S_0 R q_1;.$$

The standard description is

$DADDCRDAA;DAADDRDAAA;$

$DAAADDCCRDAAAA;DAAAADDRDA;$

A description number is

31332531173113353111731113322531111731111335317

and so is

3133253117311335311173111332253111173111133531731323253117

A number which is a description number of a circle-free machine will be called a *satisfactory* number. In § 8 it is shown that there can be no general process for determining whether a given number is satisfactory or not.

6. *The universal computing machine.*

It is possible to invent a single machine which can be used to compute any computable sequence. If this machine $\mathcal{U}$ is supplied with a tape on the beginning of which is written the S.D of some computing machine $\mathcal{M}$, then $\mathcal{U}$ will compute the same sequence as $\mathcal{M}$. In this section I explain in outline the behaviour of the machine. The next section is devoted to giving the complete table for $\mathcal{U}$.

Let us first suppose that we have a machine $\mathcal{M}'$ which will write down on the F-squares the successive complete configurations of $\mathcal{M}$. These might be expressed in the same form as on p. 235, using the second description, (C), with all symbols on one line. Or, better, we could transform this description (as in § 5) by replacing each m-configuration by "D" followed by "A" repeated the appropriate number of times, and by replacing each symbol by "D" followed by "C" repeated the appropriate number of times. The numbers of letters "A" and "C" are to agree with the numbers chosen in § 5, so that, in particular, "0" is replaced by "DC", "1" by "DCC", and the blanks by "D". These substitutions are to be made after the complete configurations have been put together, as in (C). Difficulties arise if we do the substitution first. In each complete configuration the blanks would all have to be replaced by "D", so that the complete configuration would not be expressed as a finite sequence of symbols.

If in the description of the machine II of § 3 we replace "$\mathfrak{o}$" by "DAA", "ə" by "$DCCC$", "$\mathfrak{q}$" by "$DAAA$", then the sequence (C) becomes:

$$DA:DCCCDCCCDAADCDDC:DCCCDCCCDAAADCDDC:\ldots \quad (C_1)$$

(This is the sequence of symbols on F-squares.)

It is not difficult to see that if $\mathcal{M}$ can be constructed, then so can $\mathcal{M}'$. The manner of operation of $\mathcal{M}'$ could be made to depend on having the rules of operation (*i.e.*, the S.D) of $\mathcal{M}$ written somewhere within itself (*i.e.* within $\mathcal{M}'$); each step could be carried out by referring to these rules. We have only to regard the rules as being capable of being taken out and exchanged for others and we have something very akin to the universal machine.

One thing is lacking: at present the machine $\mathcal{M}'$ prints no figures. We may correct this by printing between each successive pair of complete configurations the figures which appear in the new configuration but not in the old. Then (C_1) becomes

$$DDA:0:0:DCCCDCCCDAADCDDC:DCCC.\ldots \qquad (C_2)$$

It is not altogether obvious that the E-squares leave enough room for the necessary "rough work", but this is, in fact, the case.

The sequences of letters between the colons in expressions such as (C_1) may be used as standard descriptions of the complete configurations. When the letters are replaced by figures, as in § 5, we shall have a numerical description of the complete configuration, which may be called its description number.

7. *Detailed description of the universal machine.*

A table is given below of the behaviour of this universal machine. The m-configurations of which the machine is capable are all those occurring in the first and last columns of the table, together with all those which occur when we write out the unabbreviated tables of those which appear in the table in the form of m-functions. *E.g.*, $\mathfrak{e}(\mathfrak{anf})$ appears in the table and is an m-function. Its unabbreviated table is (see p. 239)

$\mathfrak{e}(\mathfrak{anf})$	ə	R	$\mathfrak{e}_1(\mathfrak{anf})$
	not ə	L	$\mathfrak{e}(\mathfrak{anf})$
$\mathfrak{e}_1(\mathfrak{anf})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{anf})$
	None		$\mathfrak{anf}$

Consequently $\mathfrak{e}_1(\mathfrak{anf})$ is an m-configuration of $\mathcal{U}$.

When $\mathcal{U}$ is ready to start work the tape running through it bears on it the symbol ə on an F-square and again ə on the next E-square; after this, on F-squares only, comes the S.D of the machine followed by a double colon "::" (a single symbol, on an F-square). The S.D consists of a number of instructions, separated by semi-colons.

Each instruction consists of five consecutive parts

(i) "D" followed by a sequence of letters "A". This describes the relevant m-configuration.

(ii) "D" followed by a sequence of letters "C". This describes the scanned symbol.

(iii) "D" followed by another sequence of letters "C". This describes the symbol into which the scanned symbol is to be changed.

(iv) "L", "R", or "N", describing whether the machine is to move to left, right, or not at all.

(v) "D" followed by a sequence of letters "A". This describes the final m-configuration.

The machine $\mathcal{U}$ is to be capable of printing "A", "C", "D", "0", "1", "u", "v", "w", "x", "y", "z". The S.D is formed from ";", "A", "C", "D", "L", "R", "N".

Subsidiary skeleton table.

$\mathfrak{con}(\mathfrak{C}, \alpha)$	Not A	R, R	$\mathfrak{con}(\mathfrak{C}, \alpha)$
	A	L, Pa, R	$\mathfrak{con}_1(\mathfrak{C}, \alpha)$
$\mathfrak{con}_1(\mathfrak{C}, \alpha)$	A	R, Pa, R	$\mathfrak{con}_1(\mathfrak{C}, \alpha)$
	D	R, Pa, R	$\mathfrak{con}_2(\mathfrak{C}, \alpha)$
$\mathfrak{con}_2(\mathfrak{C}, \alpha)$	C	R, Pa, R	$\mathfrak{con}_2(\mathfrak{C}, \alpha)$
	Not C	R, R	$\mathfrak{C}$

$\mathfrak{con}(\mathfrak{C}, \alpha)$. Starting from an F-square, S say, the sequence C of symbols describing a configuration closest on the right of S is marked out with letters α. $\rightarrow \mathfrak{C}$.

$\mathfrak{con}(\mathfrak{C}, \ )$. In the final configuration the machine is scanning the square which is four squares to the right of the last square of C. C is left unmarked.

The table for $\mathfrak{U}$.

$\mathfrak{b}$		$\mathfrak{f}(\mathfrak{b}_1, \mathfrak{b}_1, ::)$
$\mathfrak{b}_1$	$R, R, P:, R, R, PD, R, R, PA$	$\mathfrak{anf}$

$\mathfrak{b}$. The machine prints $:DA$ on the F-squares after $::$ $\rightarrow \mathfrak{anf}$.

$\mathfrak{anf}$	?	$\mathfrak{g}(\mathfrak{anf}_1, :)$
$\mathfrak{anf}_1$		$\mathfrak{con}(\mathfrak{kom}, y)$

$\mathfrak{anf}$. The machine marks the configuration in the last complete configuration with y. $\rightarrow \mathfrak{kom}$.

$\mathfrak{kom}$	;	R, Pz, L	$\mathfrak{con}(\mathfrak{kmp}, x)$
	z	L, L	$\mathfrak{kom}$
	not z nor ;	L	$\mathfrak{kom}$

$\mathfrak{kom}$. The machine finds the last semi-colon not marked with z. It marks this semi-colon with z and the configuration following it with x.

$\mathfrak{kmp}$	$\mathfrak{cpe}\Big(\mathfrak{e}(\mathfrak{kom}, x, y), \mathfrak{sim}, x, y\Big)$

$\mathfrak{kmp}$. The machine compares the sequences marked x and y. It erases all letters x and y. $\rightarrow \mathfrak{sim}$ if they are alike. Otherwise $\rightarrow \mathfrak{kom}$.

$\mathfrak{anf}$. Taking the long view, the last instruction relevant to the last configuration is found. It can be recognised afterwards as the instruction following the last semi-colon marked z. $\rightarrow \mathfrak{sim}$.

$\mathfrak{sim}$			$\mathfrak{f}'(\mathfrak{sim}_1, \mathfrak{sim}_1, z)$
$\mathfrak{sim}_1$			$\mathfrak{con}(\mathfrak{sim}_2, \ )$
$\mathfrak{sim}_2$	A		$\mathfrak{sim}_3$
	not A	R, Pu, R, R, R	$\mathfrak{sim}_2$
$\mathfrak{sim}_3$	not A	L, Py	$\mathfrak{e}(\mathfrak{mk}, z)$
	A	L, Py, R, R, R	$\mathfrak{sim}_3$

$\mathfrak{sim}$. The machine marks out the instructions. That part of the instructions which refers to operations to be carried out is marked with u, and the final m-configuration with y. The letters z are erased.

$\mathfrak{mk}$			$\mathfrak{g}(\mathfrak{mk}, :)$
$\mathfrak{mk}_1$	not A	R, R	$\mathfrak{mk}_1$
	A	L, L, L, L	$\mathfrak{mk}_2$
$\mathfrak{mk}_2$	C	R, Px, L, L, L	$\mathfrak{mk}_2$
	:		$\mathfrak{mk}_4$
	D	R, Px, L, L, L	$\mathfrak{mk}_3$
$\mathfrak{mk}_3$	not :	R, Pv, L, L, L	$\mathfrak{mk}_3$
	:		$\mathfrak{mk}_4$
$\mathfrak{mk}_4$			$\mathfrak{con}\Big(\mathfrak{l}\big(\mathfrak{l}(\mathfrak{mk}_5)\big), \ \Big)$
$\mathfrak{mk}_5$	Any	R, Pw, R	$\mathfrak{mk}_5$
	None	$P:$	$\mathfrak{sh}$

$\mathfrak{mk}$. The last complete configuration is marked out into four sections. The configuration is left unmarked. The symbol directly preceding it is marked with x. The remainder of the complete configuration is divided into two parts, of which the first is marked with v and the last with w. A colon is printed after the whole. → $\mathfrak{sh}$.

$\mathfrak{sh}$			$\mathfrak{f}(\mathfrak{sh}_1, \mathfrak{inst}, u)$
$\mathfrak{sh}_1$		L, L, L	$\mathfrak{sh}_2$
$\mathfrak{sh}_2$	D	R, R, R, R	$\mathfrak{sh}_2$
	not D		$\mathfrak{inst}$
$\mathfrak{sh}_3$	C	R, R	$\mathfrak{sh}_4$
	not C		$\mathfrak{inst}$
$\mathfrak{sh}_4$	C	R, R	$\mathfrak{sh}_5$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 0, :)$
$\mathfrak{sh}_5$	C		$\mathfrak{inst}$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 1, :)$

$\mathfrak{sh}$. The instructions (marked u) are examined. If it is found that they involve "Print 0" or "Print 1", then 0: or 1: is printed at the end.

$\mathfrak{inst}$			$\mathfrak{g}\left(\mathfrak{l}(\mathfrak{inst}_1),\ u\right)$	$\mathfrak{inst}$. The next complete configuration is written down, carrying out the marked instructions. The letters u, v, w, x, y are erased. $\rightarrow \mathfrak{anf}$.
$\mathfrak{inst}_1$	α	R, E	$\mathfrak{inst}_1(\alpha)$	
$\mathfrak{inst}_1(L)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, y, x, u, w)$	
$\mathfrak{inst}_1(R)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, x, u, y, w)$	
$\mathfrak{inst}_1(N)$			$\mathfrak{ec}_5(\mathfrak{ov}, v, x, y, u, w)$	
$\mathfrak{ov}$			$\mathfrak{e}(\mathfrak{anf})$	

8. *Application of the diagonal process.*

It may be thought that arguments which prove that the real numbers are not enumerable would also prove that the computable numbers and sequences cannot be enumerable*. It might, for instance, be thought that the limit of a sequence of computable numbers must be computable. This is clearly only true if the sequence of computable numbers is defined by some rule.

Or we might apply the diagonal process. "If the computable sequences are enumerable, let α_n be the n-th computable sequence, and let $\phi_n(m)$ be the m-th figure in α_n. Let β be the sequence with $1-\phi_n(n)$ as its n-th figure. Since β is computable, there exists a number K such that $1-\phi_n(n) = \phi_K(n)$ all n. Putting $n = K$, we have $1 = 2\phi_K(K)$, *i.e.* 1 is even. This is impossible. The computable sequences are therefore not enumerable".

The fallacy in this argument lies in the assumption that β is computable. It would be true if we could enumerate the computable sequences by finite means, but the problem of enumerating computable sequences is equivalent to the problem of finding out whether a given number is the D.N of a circle-free machine, and we have no general process for doing this in a finite number of steps. In fact, by applying the diagonal process argument correctly, we can show that there cannot be any such general process.

The simplest and most direct proof of this is by showing that, if this general process exists, then there is a machine which computes β. This proof, although perfectly sound, has the disadvantage that it may leave the reader with a feeling that "there must be something wrong". The proof which I shall give has not this disadvantage, and gives a certain insight into the significance of the idea "circle-free". It depends not on constructing β, but on constructing β', whose n-th figure is $\phi_n(n)$.

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* *Cf.* Hobson, *Theory of functions of a real variable* (2nd ed., 1921), 87, 88.

Let us suppose that there is such a process; that is to say, that we can invent a machine $\mathfrak{D}$ which, when supplied with the S.D of any computing machine $\mathfrak{M}$ will test this S.D and if $\mathfrak{M}$ is circular will mark the S.D with the symbol "u" and if it is circle-free will mark it with "s". By combining the machines $\mathfrak{D}$ and $\mathfrak{U}$ we could construct a machine $\mathfrak{H}$ to compute the sequence β'. The machine $\mathfrak{D}$ may require a tape. We may suppose that it uses the E-squares beyond all symbols on F-squares, and that when it has reached its verdict all the rough work done by $\mathfrak{D}$ is erased.

The machine $\mathfrak{H}$ has its motion divided into sections. In the first $N-1$ sections, among other things, the integers 1, 2, ..., $N-1$ have been written down and tested by the machine $\mathfrak{D}$. A certain number, say $R(N-1)$, of them have been found to be the D.N's of circle-free machines. In the N-th section the machine $\mathfrak{D}$ tests the number N. If N is satisfactory, *i.e.*, if it is the D.N of a circle-free machine, then $R(N) = 1+R(N-1)$ and the first $R(N)$ figures of the sequence of which a D.N is N are calculated. The $R(N)$-th figure of this sequence is written down as one of the figures of the sequence β' computed by $\mathfrak{H}$. If N is not satisfactory, then $R(N) = R(N-1)$ and the machine goes on to the $(N+1)$-th section of its motion.

From the construction of $\mathfrak{H}$ we can see that $\mathfrak{H}$ is circle-free. Each section of the motion of $\mathfrak{H}$ comes to an end after a finite number of steps. For, by our assumption about $\mathfrak{D}$, the decision as to whether N is satisfactory is reached in a finite number of steps. If N is not satisfactory, then the N-th section is finished. If N is satisfactory, this means that the machine $\mathfrak{M}(N)$ whose D.N is N is circle-free, and therefore its $R(N)$-th figure can be calculated in a finite number of steps. When this figure has been calculated and written down as the $R(N)$-th figure of β', the N-th section is finished. Hence $\mathfrak{H}$ is circle-free.

Now let K be the D.N of $\mathfrak{H}$. What does $\mathfrak{H}$ do in the K-th section of its motion? It must test whether K is satisfactory, giving a verdict "s" or "u". Since K is the D.N of $\mathfrak{H}$ and since $\mathfrak{H}$ is circle-free, the verdict cannot be "u". On the other hand the verdict cannot be "s". For if it were, then in the K-th section of its motion $\mathfrak{H}$ would be bound to compute the first $R(K-1)+1 = R(K)$ figures of the sequence computed by the machine with K as its D.N and to write down the $R(K)$-th as a figure of the sequence computed by $\mathfrak{H}$. The computation of the first $R(K)-1$ figures would be carried out all right, but the instructions for calculating the $R(K)$-th would amount to "calculate the first $R(K)$ figures computed by H and write down the $R(K)$-th". This $R(K)$-th figure would never be found. *I.e.*, $\mathfrak{H}$ is circular, contrary both to what we have found in the last paragraph and to the verdict "s". Thus both verdicts are impossible and we conclude that there can be no machine $\mathfrak{D}$.

We can show further that *there can be no machine* $\mathcal{E}$ *which, when supplied with the S.D of an arbitrary machine* $\mathcal{M}$, *will determine whether* $\mathcal{M}$ *ever prints a given symbol* (0 *say*).

We will first show that, if there is a machine $\mathcal{E}$, then there is a general process for determining whether a given machine $\mathcal{M}$ prints 0 infinitely often. Let $\mathcal{M}_1$ be a machine which prints the same sequence as $\mathcal{M}$, except that in the position where the first 0 printed by $\mathcal{M}$ stands, $\mathcal{M}_1$ prints $\bar{0}$. $\mathcal{M}_2$ is to have the first two symbols 0 replaced by $\bar{0}$, and so on. Thus, if $\mathcal{M}$ were to print

$$A\,B\,A\,0\,1\,A\,A\,B\,0\,0\,1\,0\,A\,B\ldots,$$

then $\mathcal{M}_1$ would print

$$A\,B\,A\,\bar{0}\,1\,A\,A\,B\,0\,0\,1\,0\,A\,B\ldots$$

and $\mathcal{M}_2$ would print

$$A\,B\,A\,\bar{0}\,1\,A\,A\,B\,\bar{0}\,0\,1\,0\,A\,B\ldots.$$

Now let $\mathcal{F}$ be a machine which, when supplied with the S.D of $\mathcal{M}$, will write down successively the S.D of $\mathcal{M}$, of $\mathcal{M}_1$, of $\mathcal{M}_2$, ... (there is such a machine). We combine $\mathcal{F}$ with $\mathcal{E}$ and obtain a new machine, $\mathcal{G}$. In the motion of $\mathcal{G}$ first $\mathcal{F}$ is used to write down the S.D of $\mathcal{M}$, and then $\mathcal{E}$ tests it, : 0 : is written if it is found that $\mathcal{M}$ never prints 0; then $\mathcal{F}$ writes the S.D of $\mathcal{M}_1$, and this is tested, : 0 : being printed if and only if $\mathcal{M}_1$ never prints 0, and so on. Now let us test $\mathcal{G}$ with $\mathcal{E}$. If it is found that $\mathcal{G}$ never prints 0, then $\mathcal{M}$ prints 0 infinitely often; if $\mathcal{G}$ prints 0 sometimes, then $\mathcal{M}$ does not print 0 infinitely often.

Similarly there is a general process for determining whether $\mathcal{M}$ prints 1 infinitely often. By a combination of these processes we have a process for determining whether $\mathcal{M}$ prints an infinity of figures, *i.e.* we have a process for determining whether $\mathcal{M}$ is circle-free. There can therefore be no machine $\mathcal{E}$.

The expression "there is a general process for determining ..." has been used throughout this section as equivalent to "there is a machine which will determine ...". This usage can be justified if and only if we can justify our definition of "computable". For each of these "general process" problems can be expressed as a problem concerning a general process for determining whether a given integer n has a property $G(n)$ [*e.g.* $G(n)$ might mean "n is satisfactory" or "n is the Gödel representation of a provable formula"], and this is equivalent to computing a number whose n-th figure is 1 if $G(n)$ is true and 0 if it is false.

9. *The extent of the computable numbers.*

No attempt has yet been made to show that the "computable" numbers include all numbers which would naturally be regarded as computable. All arguments which can be given are bound to be, fundamentally, appeals to intuition, and for this reason rather unsatisfactory mathematically. The real question at issue is "What are the possible processes which can be carried out in computing a number?"

The arguments which I shall use are of three kinds.

(*a*) A direct appeal to intuition.

(*b*) A proof of the equivalence of two definitions (in case the new definition has a greater intuitive appeal).

(*c*) Giving examples of large classes of numbers which are computable.

Once it is granted that computable numbers are all "computable", several other propositions of the same character follow. In particular, it follows that, if there is a general process for determining whether a formula of the Hilbert function calculus is provable, then the determination can be carried out by a machine.

I. [Type (*a*)]. This argument is only an elaboration of the ideas of § 1.

Computing is normally done by writing certain symbols on paper. We may suppose this paper is divided into squares like a child's arithmetic book. In elementary arithmetic the two-dimensional character of the paper is sometimes used. But such a use is always avoidable, and I think that it will be agreed that the two-dimensional character of paper is no essential of computation. I assume then that the computation is carried out on one-dimensional paper, *i.e.* on a tape divided into squares. I shall also suppose that the number of symbols which may be printed is finite. If we were to allow an infinity of symbols, then there would be symbols differing to an arbitrarily small extent†. The effect of this restriction of the number of symbols is not very serious. It is always possible to use sequences of symbols in the place of single symbols. Thus an Arabic numeral such as

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† If we regard a symbol as literally printed on a square we may suppose that the square is $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. The symbol is defined as a set of points in this square, viz. the set occupied by printer's ink. If these sets are restricted to be measurable, we can define the "distance" between two symbols as the cost of transforming one symbol into the other if the cost of moving unit area of printer's ink unit distance is unity, and there is an infinite supply of ink at $x = 2$, $y = 0$. With this topology the symbols form a conditionally compact space.

17 or 999999999999999 is normally treated as a single symbol. Similarly in any European language words are treated as single symbols (Chinese, however, attempts to have an enumerable infinity of symbols). The differences from our point of view between the single and compound symbols is that the compound symbols, if they are too lengthy, cannot be observed at one glance. This is in accordance with experience. We cannot tell at a glance whether 9999999999999999 and 999999999999999 are the same.

The behaviour of the computer at any moment is determined by the symbols which he is observing, and his " state of mind " at that moment. We may suppose that there is a bound B to the number of symbols or squares which the computer can observe at one moment. If he wishes to observe more, he must use successive observations. We will also suppose that the number of states of mind which need be taken into account is finite. The reasons for this are of the same character as those which restrict the number of symbols. If we admitted an infinity of states of mind, some of them will be " arbitrarily close " and will be confused. Again, the restriction is not one which seriously affects computation, since the use of more complicated states of mind can be avoided by writing more symbols on the tape.

Let us imagine the operations performed by the computer to be split up into "simple operations" which are so elementary that it is not easy to imagine them further divided. Every such operation consists of some change of the physical system consisting of the computer and his tape. We know the state of the system if we know the sequence of symbols on the tape, which of these are observed by the computer (possibly with a special order), and the state of mind of the computer. We may suppose that in a simple operation not more than one symbol is altered. Any other changes can be split up into simple changes of this kind. The situation in regard to the squares whose symbols may be altered in this way is the same as in regard to the observed squares. We may, therefore, without loss of generality, assume that the squares whose symbols are changed are always "observed" squares.

Besides these changes of symbols, the simple operations must include changes of distribution of observed squares. The new observed squares must be immediately recognisable by the computer. I think it is reasonable to suppose that they can only be squares whose distance from the closest of the immediately previously observed squares does not exceed a certain fixed amount. Let us say that each of the new observed squares is within L squares of an immediately previously observed square.

In connection with "immediate recognisability", it may be thought that there are other kinds of square which are immediately recognisable. In particular, squares marked by special symbols might be taken as immediately recognisable. Now if these squares are marked only by single symbols there can be only a finite number of them, and we should not upset our theory by adjoining these marked squares to the observed squares. If, on the other hand, they are marked by a sequence of symbols, we cannot regard the process of recognition as a simple process. This is a fundamental point and should be illustrated. In most mathematical papers the equations and theorems are numbered. Normally the numbers do not go beyond (say) 1000. It is, therefore, possible to recognise a theorem at a glance by its number. But if the paper was very long, we might reach Theorem 157767733443477; then, further on in the paper, we might find "... hence (applying Theorem 157767733443477) we have ...". In order to make sure which was the relevant theorem we should have to compare the two numbers figure by figure, possibly ticking the figures off in pencil to make sure of their not being counted twice. If in spite of this it is still thought that there are other "immediately recognisable" squares, it does not upset my contention so long as these squares can be found by some process of which my type of machine is capable. This idea is developed in III below.

The simple operations must therefore include:

(*a*) Changes of the symbol on one of the observed squares.

(*b*) Changes of one of the squares observed to another square within L squares of one of the previously observed squares.

It may be that some of these changes necessarily involve a change of state of mind. The most general single operation must therefore be taken to be one of the following:

(*A*) A possible change (*a*) of symbol together with a possible change of state of mind.

(*B*) A possible change (*b*) of observed squares, together with a possible change of state of mind.

The operation actually performed is determined, as has been suggested on p. 250, by the state of mind of the computer and the observed symbols. In particular, they determine the state of mind of the computer after the operation is carried out.

We may now construct a machine to do the work of this computer. To each state of mind of the computer corresponds an "*m*-configuration" of the machine. The machine scans B squares corresponding to the B squares observed by the computer. In any move the machine can change a symbol on a scanned square or can change any one of the scanned squares to another square distant not more than L squares from one of the other scanned squares. The move which is done, and the succeeding configuration, are determined by the scanned symbol and the m-configuration. The machines just described do not differ very essentially from computing machines as defined in § 2, and corresponding to any machine of this type a computing machine can be constructed to compute the same sequence, that is to say the sequence computed by the computer.

II. [Type (*b*)].

If the notation of the Hilbert functional calculus† is modified so as to be systematic, and so as to involve only a finite number of symbols, it becomes possible to construct an automatic‡ machine $\mathcal{K}$, which will find all the provable formulae of the calculus§.

Now let α be a sequence, and let us denote by $G_\alpha(x)$ the proposition "The x-th figure of α is 1", so that|| $-G_\alpha(x)$ means "The x-th figure of α is 0". Suppose further that we can find a set of properties which define the sequence α and which can be expressed in terms of $G_\alpha(x)$ and of the propositional functions $N(x)$ meaning "x is a non-negative integer" and $F(x, y)$ meaning "$y = x+1$". When we join all these formulae together conjunctively, we shall have a formula, $\mathfrak{A}$ say, which defines α. The terms of $\mathfrak{A}$ must include the necessary parts of the Peano axioms, viz.,

$$(\exists u) N(u) \,\&\, (x) \Big(N(x) \to (\exists y) F(x, y) \Big) \,\&\, \Big(F(x, y) \to N(y) \Big),$$

which we will abbreviate to P.

When we say "$\mathfrak{A}$ defines α", we mean that $-\mathfrak{A}$ is not a provable formula, and also that, for each n, one of the following formulae (A_n) or (B_n) is provable.

$$\mathfrak{A} \,\&\, F^{(n)} \to G_\alpha(u^{(n)}), \qquad (\mathrm{A}_n)¶$$

$$\mathfrak{A} \,\&\, F^{(n)} \to \Big(-G_\alpha(u^{(n)}) \Big), \qquad (\mathrm{B}_n),$$

where $F^{(n)}$ stands for $F(u, u') \,\&\, F(u', u'') \,\&\, \ldots F(u^{(n-1)}, u^{(n)})$.

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† The expression "the functional calculus" is used throughout to mean the *restricted* Hilbert functional calculus.

‡ It is most natural to construct first a choice machine (§ 2) to do this. But it is then easy to construct the required automatic machine. We can suppose that the choices are always choices between two possibilities 0 and 1. Each proof will then be determined by a sequence of choices $i_1, i_2, \ldots, i_n$ ($i_1 = 0$ or 1, $i_2 = 0$ or $1, \ldots, i_n = 0$ or 1), and hence the number $2^n + i_1 2^{n-1} + i_2 2^{n-2} + \ldots + i_n$ completely determines the proof. The automatic machine carries out successively proof 1, proof 2, proof 3, ...

§ The author has found a description of such a machine.

|| The negation sign is written before an expression and not over it.

¶ A sequence of r primes is denoted by $^{(r)}$.

I say that α is then a computable sequence: a machine $\mathcal{K}_\alpha$ to compute α can be obtained by a fairly simple modification of $\mathcal{K}$.

We divide the motion of $\mathcal{K}_\alpha$ into sections. The n-th section is devoted to finding the n-th figure of α. After the $(n-1)$-th section is finished a double colon : : is printed after all the symbols, and the succeeding work is done wholly on the squares to the right of this double colon. The first step is to write the letter "*A*" followed by the formula (A_n) and then "*B*" followed by (B_n). The machine $\mathcal{K}_\alpha$ then starts to do the work of $\mathcal{K}$, but whenever a provable formula is found, this formula is compared with (A_n) and with (B_n). If it is the same formula as (A_n), then the figure "1" is printed, and the n-th section is finished. If it is (B_n), then "0" is printed and the section is finished. If it is different from both, then the work of $\mathcal{K}$ is continued from the point at which it had been abandoned. Sooner or later one of the formulae (A_n) or (B_n) is reached; this follows from our hypotheses about α and $\mathfrak{A}$, and the known nature of $\mathcal{K}$. Hence the n-th section will eventually be finished. $\mathcal{K}_\alpha$ is circle-free; α is computable.

It can also be shown that the numbers α definable in this way by the use of axioms include all the computable numbers. This is done by describing computing machines in terms of the function calculus.

It must be remembered that we have attached rather a special meaning to the phrase "$\mathfrak{A}$ defines α". The computable numbers do not include all (in the ordinary sense) definable numbers. Let δ be a sequence whose n-th figure is 1 or 0 according as n is or is not satisfactory. It is an immediate consequence of the theorem of § 8 that δ is not computable. It is (so far as we know at present) possible that any assigned number of figures of δ can be calculated, but not by a uniform process. When sufficiently many figures of δ have been calculated, an essentially new method is necessary in order to obtain more figures.

III. This may be regarded as a modification of I or as a corollary of II.

We suppose, as in I, that the computation is carried out on a tape; but we avoid introducing the "state of mind" by considering a more physical and definite counterpart of it. It is always possible for the computer to break off from his work, to go away and forget all about it, and later to come back and go on with it. If he does this he must leave a note of instructions (written in some standard form) explaining how the work is to be continued. This note is the counterpart of the "state of mind". We will suppose that the computer works in such a desultory manner that he never does more than one step at a sitting. The note of instructions must enable him to carry out one step and write the next note. Thus the state of progress of the computation at any stage is completely determined by the note of instructions and the symbols on the tape. That is, the state of the system may be described by a single expression (sequence of symbols), consisting of the symbols on the tape followed by Δ (which we suppose not to appear elsewhere) and then by the note of instructions. This expression may be called the "state formula". We know that the state formula at any given stage is determined by the state formula before the last step was made, and we assume that the relation of these two formulae is expressible in the functional calculus. In other words, we assume that there is an axiom $\mathfrak{A}$ which expresses the rules governing the behaviour of the computer, in terms of the relation of the state formula at any stage to the state formula at the preceding stage. If this is so, we can construct a machine to write down the successive state formulae, and hence to compute the required number.

10. *Examples of large classes of numbers which are computable.*

It will be useful to begin with definitions of a computable function of an integral variable and of a computable variable, etc. There are many equivalent ways of defining a computable function of an integral variable. The simplest is, possibly, as follows. If γ is a computable sequence in which 0 appears infinitely† often, and n is an integer, then let us define $\xi(\gamma, n)$ to be the number of figures 1 between the n-th and the $(n+1)$-th figure 0 in γ. Then $\phi(n)$ is computable if, for all n and some γ, $\phi(n) = \xi(\gamma, n)$. An equivalent definition is this. Let $H(x, y)$ mean $\phi(x) = y$. Then, if we can find a contradiction-free axiom $\mathfrak{A}_\phi$, such that $\mathfrak{A}_\phi \rightarrow P$, and if for each integer n there exists an integer N, such that

$$\mathfrak{A}_\phi \mathbin{\&} F^{(N)} \rightarrow H(u^{(n)}, u^{(\phi(n))}),$$

and such that, if $m \neq \phi(n)$, then, for some N',

$$\mathfrak{A}_\phi \mathbin{\&} F^{(N')} \rightarrow \big(-H(u^{(n)}, u^{(m)}\big),$$

then ϕ may be said to be a computable function.

We cannot define general computable functions of a real variable, since there is no general method of describing a real number, but we can define a computable function of a computable variable. If n is satisfactory, let γ_n be the number computed by $\mathcal{M}(n)$, and let

$$a_n = \tan\big(\pi(\gamma_n - \tfrac{1}{2})\big),$$

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† If $\mathcal{M}$ computes γ, then the problem whether $\mathcal{M}$ prints 0 infinitely often is of the same character as the problem whether $\mathcal{M}$ is circle-free.

unless $\gamma_n = 0$ or $\gamma_n = 1$, in either of which cases $\alpha_n = 0$. Then, as n runs through the satisfactory numbers, α_n runs through the computable numbers†. Now let $\phi(n)$ be a computable function which can be shown to be such that for any satisfactory argument its value is satisfactory‡. Then the function f, defined by $f(\alpha_n) = \alpha_{\phi(n)}$, is a computable function and all computable functions of a computable variable are expressible in this form.

Similar definitions may be given of computable functions of several variables, computable-valued functions of an integral variable, etc.

I shall enunciate a number of theorems about computability, but I shall prove only (ii) and a theorem similar to (iii).

(i) A computable function of a computable function of an integral or computable variable is computable.

(ii) Any function of an integral variable defined recursively in terms of computable functions is computable. *I.e.* if $\phi(m, n)$ is computable, and r is some integer, then $\eta(n)$ is computable, where

$$\eta(0) = r,$$

$$\eta(n) = \phi\big(n, \eta(n-1)\big).$$

(iii) If $\phi\ (m, n)$ is a computable function of two integral variables, then $\phi(n, n)$ is a computable function of n.

(iv) If $\phi(n)$ is a computable function whose value is always 0 or 1, then the sequence whose n-th figure is $\phi(n)$ is computable.

Dedekind's theorem does not hold in the ordinary form if we replace "real" throughout by "computable". But it holds in the following form:

(v) If $G(\alpha)$ is a propositional function of the computable numbers and

$$(a) \quad (\exists\alpha)(\exists\beta)\Big\{G(\alpha)\ \&\ \big(-G(\beta)\big)\Big\},$$

$$(b) \quad G(\alpha)\ \&\ \big(-G(\beta)\big) \rightarrow (\alpha < \beta),$$

and there is a general process for determining the truth value of $G(\alpha)$, then

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[†] A function α_n may be defined in many other ways so as to run through the computable numbers.

[‡] Although it is not possible to find a general process for determining whether a given number is satisfactory, it is often possible to show that certain classes of numbers are satisfactory.

there is a computable number ξ such that

$$G(\alpha) \rightarrow \alpha \leqslant \xi,$$

$$-G(\alpha) \rightarrow \alpha \geqslant \xi.$$

In other words, the theorem holds for any section of the computables such that there is a general process for determining to which class a given number belongs.

Owing to this restriction of Dedekind's theorem, we cannot say that a computable bounded increasing sequence of computable numbers has a computable limit. This may possibly be understood by considering a sequence such as

$$-1, -\tfrac{1}{2}, -\tfrac{1}{4}, -\tfrac{1}{8}, -\tfrac{1}{16}, \tfrac{1}{2}, \ldots .$$

On the other hand, (v) enables us to prove

(vi) If α and β are computable and $\alpha < \beta$ and $\phi(\alpha) < 0 < \phi(\beta)$, where $\phi(\alpha)$ is a computable increasing continuous function, then there is a unique computable number γ, satisfying $\alpha < \gamma < \beta$ and $\phi(\gamma) = 0$.

Computable convergence.

We shall say that a sequence β_n of computable numbers *converges computably* if there is a computable integral valued function $N(\epsilon)$ of the computable variable ϵ, such that we can show that, if $\epsilon > 0$ and $n > N(\epsilon)$ and $m > N(\epsilon)$, then $|\beta_n - \beta_m| < \epsilon$.

We can then show that

(vii) A power series whose coefficients form a computable sequence of computable numbers is computably convergent at all computable points in the interior of its interval of convergence.

(viii) The limit of a computably convergent sequence is computable.

And with the obvious definition of "uniformly computably convergent":

(ix) The limit of a uniformly computably convergent computable sequence of computable functions is a computable function. Hence

(x) The sum of a power series whose coefficients form a computable sequence is a computable function in the interior of its interval of convergence.

From (viii) and $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \ldots)$ we deduce that π is computable.

From $e = 1 + 1 + \dfrac{1}{2\,!} + \dfrac{1}{3\,!} + \ldots$ we deduce that e is computable.

From (vi) we deduce that all real algebraic numbers are computable.

From (vi) and (x) we deduce that the real zeros of the Bessel functions are computable.

Proof of (ii).

Let $H(x, y)$ mean "$\eta(x) = y$", and let $K(x, y, z)$ mean "$\phi(x, y) = z$". $\mathfrak{A}_\phi$ is the axiom for $\phi(x, y)$. We take $\mathfrak{A}_\eta$ to be

$$\begin{aligned} &\mathfrak{A}_\phi \mathbin{\&} P \mathbin{\&} \big(F(x, y) \to G(x, y)\big) \mathbin{\&} \big(G(x, y) \mathbin{\&} G(y, z) \to G(x, z)\big) \\ &\qquad \mathbin{\&} \big(F^{(r)} \to H(u, u^{(r)})\big) \mathbin{\&} \big(F(v, w) \mathbin{\&} H(v, x) \mathbin{\&} K(w, x, z) \to H(w, z)\big) \\ &\qquad \mathbin{\&} \Big[H(w, z) \mathbin{\&} G(z, t) \vee G(t, z) \to \big(-H(w, t)\big)\Big]. \end{aligned}$$

I shall not give the proof of consistency of $\mathfrak{A}_\eta$. Such a proof may be constructed by the methods used in Hilbert and Bernays, *Grundlagen der Mathematik* (Berlin, 1934), p. 209 *et seq*. The consistency is also clear from the meaning.

Suppose that, for some n, N, we have shown

$$\mathfrak{A}_\eta \mathbin{\&} F^{(N)} \to H(u^{(n-1)}, u^{(\eta(n-1))}),$$

then, for some M,

$$\mathfrak{A}_\phi \mathbin{\&} F^{(M)} \to K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}),$$

$$\begin{aligned} \mathfrak{A}_\eta \mathbin{\&} F^{(M)} \to F(u^{(n-1)}, u^{(n)}) &\mathbin{\&} H(u^{(n-1)}, u^{(\eta(n-1))}) \\ &\mathbin{\&} K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}), \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{A}_\eta \mathbin{\&} F^{(M)} \to [F(u^{(n-1)}, u^{(n)}) \mathbin{\&} H(u^{(n-1)}, u^{(\eta(n-1))}) \\ &\qquad\qquad \mathbin{\&} K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}) \to H(u^{(n)}, u^{(\eta(n))})]. \end{aligned}$$

Hence $$\mathfrak{A}_\eta \mathbin{\&} F^{(M)} \to H(u^{(n)}, u^{(\eta(n))}).$$

Also $$\mathfrak{A}_\eta \mathbin{\&} F^{(r)} \to H(u, u^{(\eta(0))}).$$

Hence for each n some formula of the form

$$\mathfrak{A}_\eta \mathbin{\&} F^{(M)} \to H(u^{(n)}, u^{(\eta(n))})$$

is provable. Also, if $M' \geqslant M$ and $M' \geqslant m$ and $m \neq \eta(u)$, then

$$\mathfrak{A}_\eta \mathbin{\&} F^{(M')} \to G(u^{\eta((n))}, u^{(m)}) \nu G(u^{(m)}, u^{(\eta(n))})$$

and

$$\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow \Big[\{ G(u^{(\eta(n))}, u^{(m)}) \,v\, G(u^{(m)}, u^{(\eta(n)}) \\ \& \, H(u^{(n)}, u^{(\eta(n))} \} \rightarrow \big(-H(u^{(n)}, u^{(m)}) \big) \Big].$$

Hence $$\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow \big(-H(u^{(n)}, u^{(m)}) \big).$$

The conditions of our second definition of a computable function are therefore satisfied. Consequently η is a computable function.

Proof of a modified form of (iii).

Suppose that we are given a machine $\mathfrak{N}$, which, starting with a tape bearing on it əə followed by a sequence of any number of letters "F" on F-squares and in the m-configuration b, will compute a sequence γ_n depending on the number n of letters "F". If $\phi_n(m)$ is the m-th figure of γ_n, then the sequence β whose n-th figure is $\phi_n(n)$ is computable.

We suppose that the table for $\mathfrak{N}$ has been written out in such a way that in each line only one operation appears in the operations column. We also suppose that Ξ, Θ, $\bar{0}$, and $\bar{1}$ do not occur in the table, and we replace ə throughout by Θ, 0 by $\bar{0}$, and 1 by $\bar{1}$. Further substitutions are then made. Any line of form

$\mathfrak{A}$	a	$P\bar{0}$	$\mathfrak{B}$

we replace by

$\mathfrak{A}$	a	$P\bar{0}$	$\mathfrak{re}(\mathfrak{B}, \mathfrak{u}, h, k)$

and any line of the form

	$\mathfrak{A}$	a	$P\bar{1}$	$\mathfrak{B}$
by	$\mathfrak{A}$	a	$P\bar{1}$	$\mathfrak{re}(\mathfrak{B}, \mathfrak{v}, h, k)$

and we add to the table the following lines:

$\mathfrak{u}$		$\mathfrak{pe}(\mathfrak{u}_1, 0)$
$\mathfrak{u}_1$	$R, Pk, R, P\Theta, R, P\Theta$	$\mathfrak{u}_2$
$\mathfrak{u}_2$		$\mathfrak{re}(\mathfrak{u}_3, \mathfrak{u}_3, k, h)$
$\mathfrak{u}_3$		$\mathfrak{pe}(\mathfrak{u}_2, F)$

and similar lines with $\mathfrak{v}$ for $\mathfrak{u}$ and 1 for 0 together with the following line

$\mathfrak{c}$	$R, P\Xi, R, Ph$	$\mathfrak{b}$.

We then have the table for the machine $\mathfrak{N}'$ which computes β. The initial m-configuration is $\mathfrak{c}$, and the initial scanned symbol is the second ə.

11. *Application to the Entscheidungsproblem.*

The results of § 8 have some important applications. In particular, they can be used to show that the Hilbert Entscheidungsproblem can have no solution. For the present I shall confine myself to proving this particular theorem. For the formulation of this problem I must refer the reader to Hilbert and Ackermann's *Grundzüge der Theoretischen Logik* (Berlin, 1931), chapter 3.

I propose, therefore, to show that there can be no general process for determining whether a given formula $\mathfrak{A}$ of the functional calculus **K** is provable, *i.e.* that there can be no machine which, supplied with any one $\mathfrak{A}$ of these formulae, will eventually say whether $\mathfrak{A}$ is provable.

It should perhaps be remarked that what I shall prove is quite different from the well-known results of Gödel[†]. Gödel has shown that (in the formalism of Principia Mathematica) there are propositions $\mathfrak{A}$ such that neither $\mathfrak{A}$ nor $-\mathfrak{A}$ is provable. As a consequence of this, it is shown that no proof of consistency of Principia Mathematica (or of **K**) can be given within that formalism. On the other hand, I shall show that there is no general method which tells whether a given formula $\mathfrak{A}$ is provable in **K**, or, what comes to the same, whether the system consisting of **K** with $-\mathfrak{A}$ adjoined as an extra axiom is consistent.

If the negation of what Gödel has shown had been proved, *i.e.* if, for each $\mathfrak{A}$, either $\mathfrak{A}$ or $-\mathfrak{A}$ is provable, then we should have an immediate solution of the Entscheidungsproblem. For we can invent a machine $\mathcal{K}$ which will prove consecutively all provable formulae. Sooner or later $\mathcal{K}$ will reach either $\mathfrak{A}$ or $-\mathfrak{A}$. If it reaches $\mathfrak{A}$, then we know that $\mathfrak{A}$ is provable. If it reaches $-\mathfrak{A}$, then, since **K** is consistent (Hilbert and Ackermann, p. 65), we know that $\mathfrak{A}$ is not provable.

Owing to the absence of integers in **K** the proofs appear somewhat lengthy. The underlying ideas are quite straightforward.

Corresponding to each computing machine $\mathcal{M}$ we construct a formula Un $(\mathcal{M})$ and we show that, if there is a general method for determining whether Un $(\mathcal{M})$ is provable, then there is a general method for determining whether $\mathcal{M}$ ever prints 0.

The interpretations of the propositional functions involved are as follows :

$R_{S_l}(x, y)$ is to be interpreted as "in the complete configuration x (of $\mathcal{M}$) the symbol on the square y is S".

[†] *Loc. cit.*

$I(x, y)$ is to be interpreted as "in the complete configuration x the square y is scanned".

$K_{q_m}(x)$ is to be interpreted as "in the complete configuration x the m-configuration is q_m.

$F(x, y)$ is to be interpreted as "y is the immediate successor of x".

Inst $\{q_i S_j S_k L q_l\}$ is to be an abbreviation for

$$(x, y, x', y')\Big\{\Big(R_{S_j}(x, y) \,\&\, I(x, y) \,\&\, K_{q_i}(x) \,\&\, F(x, x') \,\&\, F(y', y)\Big) \rightarrow \Big(I(x', y') \,\&\, R_{S_k}(x', y) \,\&\, K_{q_l}(x') \,\&\, (z)\Big[F(y', z) \vee \Big(R_{S_j}(x, z) \rightarrow R_{S_k}(x', z)\Big)\Big]\Big)\Big\}.$$

Inst $\{q_i S_j S_k R q_l\}$ and Inst $\{q_i S_j S_k N q_l\}$

are to be abbreviations for other similarly constructed expressions.

Let us put the description of $\mathcal{M}$ into the first standard form of § 6. This description consists of a number of expressions such as "$q_i\, S_j\, S_k\, L q_l$" (or with R or N substituted for L). Let us form all the corresponding expressions such as Inst $\{q_i S_j S_k L q_l\}$ and take their logical sum. This we call Des $(\mathcal{M})$.

The formula Un $(\mathcal{M})$ is to be

$$(\exists u)\Big[N(u) \,\&\, (x)\Big(N(x) \rightarrow (\exists x')\, F(x, x')\Big) \,\&\, (y, z)\Big(F(y, z) \rightarrow N(y) \,\&\, N(z)\Big) \,\&\, (y)\, R_{S_0}(u, y) \,\&\, I(u, u) \,\&\, K_{q_1}(u) \,\&\, \mathrm{Des}(\mathcal{M})\Big] \rightarrow (\exists s)(\exists t)\,[N(s) \,\&\, N(t) \,\&\, R_{S_1}(s, t)].$$

$[N(u) \,\&\, \ldots \,\&\, \mathrm{Des}\,(\mathcal{M})]$ may be abbreviated to $A(\mathcal{M})$.

When we substitute the meanings suggested on p. 259–60 we find that Un $(\mathcal{M})$ has the interpretation "in some complete configuration of $\mathcal{M}$, S_1 (*i.e.* 0) appears on the tape". Corresponding to this I prove that

(*a*) If S_1 appears on the tape in some complete configuration of $\mathcal{M}$, then Un $(\mathcal{M})$ is provable.

(*b*) If Un $(\mathcal{M})$ is provable, then S_1 appears on the tape in some complete configuration of $\mathcal{M}$.

When this has been done, the remainder of the theorem is trivial.

LEMMA 1. *If S_1 appears on the tape in some complete configuration of $\mathcal{M}$, then* Un $(\mathcal{M})$ *is provable.*

We have to show how to prove Un $(\mathcal{M})$. Let us suppose that in the n-th complete configuration the sequence of symbols on the tape is $S_{r(n,0)}, S_{r(n,1)}, \ldots, S_{r(n,n)}$, followed by nothing but blanks, and that the scanned symbol is the $i(n)$-th, and that the m-configuration is $q_{k(n)}$. Then we may form the proposition

$$R_{S_{r(n,0)}}(u^{(n)}, u) \;\&\; R_{S_{r(n,1)}}(u^{(n)}, u') \;\&\; \ldots \;\&\; R_{S_{r(n,n)}}(u^{(n)}, u^{(n)})$$
$$\& \; I(u^{(n)}, u^{(i(n))}) \;\&\; K_{q_{k(n)}}(u^{(n)})$$
$$\& \; (y)F\Big((y, u') \mathrm{v} F(u, y) \mathrm{v} F(u', y) \mathrm{v} \ldots \mathrm{v} F(u^{(n-1)}, y) \mathrm{v} R_{S_0}(u^{(n)}, y)\Big),$$

which we may abbreviate to CC_n.

As before, $F(u, u') \;\&\; F(u', u'') \;\&\; \ldots \;\&\; F(u^{(r-1)}, u^{(r)})$ is abbreviated to $F^{(r)}$.

I shall show that all formulae of the form $A(\mathcal{M}) \;\&\; F^{(n)} \rightarrow CC_n$ (abbreviated to CF_n) are provable. The meaning of CF_n is "The n-th complete configuration of $\mathcal{M}$ is so and so", where "so and so" stands for the actual n-th complete configuration of $\mathcal{M}$. That CF_n should be provable is therefore to be expected.

CF_0 is certainly provable, for in the complete configuration the symbols are all blanks, the m-configuration is q_1, and the scanned square is u, *i.e.* CC_0 is

$$(y)\, R_{S_0}(u, y) \;\&\; I(u, u) \;\&\; K_{q_1}(u).$$

$A(\mathcal{M}) \rightarrow CC_0$ is then trivial.

We next show that $CF_n \rightarrow CF_{n+1}$ is provable for each n. There are three cases to consider, according as in the move from the n-th to the $(n+1)$-th configuration the machine moves to left or to right or remains stationary. We suppose that the first case applies, *i.e.* the machine moves to the left. A similar argument applies in the other cases. If $r\big(n, i(n)\big) = a$, $r\big(n+1, i(n+1)\big) = c$, $k\big(i(n)\big) = b$, and $k\big(i(n+1)\big) = d$, then Des $(\mathcal{M})$ must include Inst $\{q_a S_b S_d L q_c\}$ as one of its terms, *i.e.*

$$\text{Des}\,(\mathcal{M}) \rightarrow \text{Inst}\,\{q_a S_b S_d L q_c\}.$$

Hence $\qquad A(\mathcal{M}) \;\&\; F^{(n+1)} \rightarrow \text{Inst}\,\{q_a S_b S_d L q_c\} \;\&\; F^{(n+1)}.$

But $\qquad \text{Inst}\{q_a S_b S_d L q_c\} \;\&\; F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$

is provable, and so therefore is

$$A(\mathcal{M}) \;\&\; F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

and $$\left(A(\mathfrak{M}) \mathbin{\&} F^{(n)} \rightarrow CC_n\right) \rightarrow \left(A(\mathfrak{M}) \mathbin{\&} F^{(n+1)} \rightarrow CC_{n+1}\right),$$

i.e. $$CF_n \rightarrow CF_{n+1}.$$

CF_n is provable for each n. Now it is the assumption of this lemma that S_1 appears somewhere, in some complete configuration, in the sequence of symbols printed by $\mathfrak{M}$; that is, for some integers N, K, CC_N has $R_{S_1}(u^{(N)}, u^{(K)})$ as one of its terms, and therefore $CC_N \rightarrow R_{S_1}(u^{(N)}, u^{(K)})$ is provable. We have then

$$CC_N \rightarrow R_{S_1}(u^{(N)}, u^{(K)})$$

and $$A(\mathfrak{M}) \mathbin{\&} F^{(N)} \rightarrow CC^N.$$

We also have

$$(\exists u) A(\mathfrak{M}) \rightarrow (\exists u)(\exists u') \ldots (\exists u^{(N')}) \left(A(\mathfrak{M}) \mathbin{\&} F^{(N)}\right),$$

where $N' = \max(N, K)$. And so

$$(\exists u) A(\mathfrak{M}) \rightarrow (\exists u)\,(\exists u') \ldots (\exists u^{(N')})\, R_{S_1}(u^{(N)}, u^{(K)}),$$

$$(\exists u) A(\mathfrak{M}) \rightarrow (\exists u^{(N)})\,(\exists u^{(K)})\, R_{S_1}(u^{(N)}, u^{(K)}),$$

$$(\exists u) A(\mathfrak{M}) \rightarrow (\exists s)\,(\exists t)\, R_{S_1}(s, t),$$

i.e. $\mathrm{Un}(\mathfrak{M})$ is provable.

This completes the proof of Lemma 1.

LEMMA 2. *If* $\mathrm{Un}(\mathfrak{M})$ *is provable, then* S_1 *appears on the tape in some complete configuration of* $\mathfrak{M}$.

If we substitute any propositional functions for function variables in a provable formula, we obtain a true proposition. In particular, if we substitute the meanings tabulated on pp. 259–260 in $\mathrm{Un}(\mathfrak{M})$, we obtain a true proposition with the meaning "S_1 appears somewhere on the tape in some complete configuration of $\mathfrak{M}$".

We are now in a position to show that the Entscheidungsproblem cannot be solved. Let us suppose the contrary. Then there is a general (mechanical) process for determining whether $\mathrm{Un}(\mathfrak{M})$ is provable. By Lemmas 1 and 2, this implies that there is a process for determining whether $\mathfrak{M}$ ever prints 0, and this is impossible, by § 8. Hence the Entscheidungsproblem cannot be solved.

In view of the large number of particular cases of solutions of the Entscheidungsproblem for formulae with restricted systems of quantors, it is interesting to express $\mathrm{Un}(\mathcal{M})$ in a form in which all quantors are at the beginning. $\mathrm{Un}(\mathcal{M})$ is, in fact, expressible in the form

$$(u)(\exists x)(w)(\exists u_1)\ldots(\exists u_n)\mathfrak{B}, \tag{I}$$

where $\mathfrak{B}$ contains no quantors, and $n = 6$. By unimportant modifications we can obtain a formula, with all essential properties of $\mathrm{Un}(\mathcal{M})$, which is of form (I) with $n = 5$.

Added 28 *August*, 1936.

APPENDIX.

Computability and effective calculability

The theorem that all effectively calculable (λ-definable) sequences are computable and its converse are proved below in outline. It is assumed that the terms "well-formed formula" (W.F.F.) and "conversion" as used by Church and Kleene are understood. In the second of these proofs the existence of several formulae is assumed without proof; these formulae may be constructed straightforwardly with the help of, *e.g.*, the results of Kleene in "A theory of positive integers in formal logic", *American Journal of Math.*, 57 (1935), 153-173, 219-244.

The W.F.F. representing an integer n will be denoted by N_n. We shall say that a sequence γ whose n-th figure is $\phi_\gamma(n)$ is λ-definable or effectively calculable if $1+\phi_\gamma(u)$ is a λ-definable function of n, *i.e.* if there is a W.F.F. M_γ such that, for all integers n,

$$\{M_\gamma\}(N_n) \text{ conv } N_{\phi_\gamma(n)+1},$$

i.e. $\{M_\gamma\}(N_n)$ is convertible into $\lambda xy.x\big(x(y)\big)$ or into $\lambda xy.x(y)$ according as the n-th figure of λ is 1 or 0.

To show that every λ-definable sequence γ is computable, we have to show how to construct a machine to compute γ. For use with machines it is convenient to make a trivial modification in the calculus of conversion. This alteration consists in using $x, x', x'', \ldots$ as variables instead of $a, b, c, \ldots$. We now construct a machine $\mathcal{L}$ which, when supplied with the formula M_γ, writes down the sequence γ. The construction of $\mathcal{L}$ is somewhat similar to that of the machine $\mathcal{K}$ which proves all provable formulae of the functional calculus. We first construct a choice machine $\mathcal{L}_1$, which, if supplied with a W.F.F., M say, and suitably manipulated, obtains any formula into which M is convertible. $\mathcal{L}_1$ can then be modified so as to yield an automatic machine $\mathcal{L}_2$ which obtains successively all the formulae into which M is convertible (cf. foot-note p. 252). The machine $\mathcal{L}$ includes $\mathcal{L}_2$ as a part. The motion of the machine $\mathcal{L}$ when supplied with the formula M_γ is divided into sections of which the n-th is devoted to finding the n-th figure of γ. The first stage in this n-th section is the formation of $\{M_\gamma\}(N_n)$. This formula is then supplied to the machine $\mathcal{L}_2$, which converts it successively into various other formulae. Each formula into which it is convertible eventually appears, and each, as it is found, is compared with

$$\lambda x\left[\lambda x'\left[\{x\}\left(\{x\}(x')\right)\right]\right], \quad i.e.\ N_2,$$

and with
$$\lambda x\left[\lambda x'[\{x\}(x')]\right], \quad i.e.\ N_1.$$

If it is identical with the first of these, then the machine prints the figure 1 and the n-th section is finished. If it is identical with the second, then 0 is printed and the section is finished. If it is different from both, then the work of $\mathcal{L}_2$ is resumed. By hypothesis, $\{M_\gamma\}(N_n)$ is convertible into one of the formulae N_2 or N_1; consequently the n-th section will eventually be finished, *i.e.* the n-th figure of γ will eventually be written down.

To prove that every computable sequence γ is λ-definable, we must show how to find a formula M_γ such that, for all integers n,

$$\{M_\gamma\}(N_n) \text{ conv } N_{1+\phi_\gamma(n)}.$$

Let $\mathcal{M}$ be a machine which computes γ and let us take some description of the complete configurations of $\mathcal{M}$ by means of numbers, *e.g.* we may take the D.N of the complete configuration as described in §6. Let $\xi(n)$ be the D.N of the n-th complete configuration of $\mathcal{M}$. The table for the machine $\mathcal{M}$ gives us a relation between $\xi(n+1)$ and $\xi(n)$ of the form

$$\xi(n+1) = \rho_\gamma\left(\xi(n)\right),$$

where ρ_γ is a function of very restricted, although not usually very simple, form: it is determined by the table for $\mathcal{M}$. ρ_γ is λ-definable (I omit the proof of this), *i.e.* there is a W.F.F. A_γ such that, for all integers n,

$$\{A_\gamma\}(N_{\xi(n)}) \text{ conv } N_{\xi(n+1)}.$$

Let U stand for

$$\lambda u\left[\left\{\{u\}(A_\gamma)\right\}(N_r)\right],$$

where $r = \xi(0)$; then, for all integers n,

$$\{U_\gamma\}(N_n) \text{ conv } N_{\xi(n)}.$$

It may be proved that there is a formula V such that

$$\Big\{\{V\}(N_{\xi(n+1)})\Big\}(N_{\xi(n)}) \begin{cases} \text{conv } N_1 & \text{if, in going from the } n\text{-th to the } (n+1)\text{-th complete configuration, the figure 0 is printed.} \\ \text{conv } N_2 & \text{if the figure 1 is printed.} \\ \text{conv } N_3 & \text{otherwise.} \end{cases}$$

Let W_γ stand for

$$\lambda u\Big[\Big\{\{V\}\Big(\{A_\gamma\}\big(\{U_\gamma\}(u)\big)\Big)\Big\}\big(\{U_\gamma\}(u)\big)\Big],$$

so that, for each integer n,

$$\Big\{\{V\}(N_{\xi(n+1)})\Big\}(N_{\xi(n)}) \text{ conv } \{W_\gamma\}(N_n),$$

and let Q be a formula such that

$$\Big\{\{Q\}(W_\gamma)\Big\}(N_s) \text{ conv } N_{r(z)},$$

where $r(s)$ is the s-th integer q for which $\{W_\gamma\}(N_q)$ is convertible into either N_1 or N_2. Then, if M_γ stands for

$$\lambda w\Big[\{W_\gamma\}\Big(\Big\{\{Q\}(W_\gamma)\Big\}(w)\Big)\Big],$$

it will have the required property†.

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† In a complete proof of the λ-definability of computable sequences it would be best to modify this method by replacing the numerical description of the complete configurations by a description which can be handled more easily with our apparatus. Let us choose certain integers to represent the symbols and the m-configurations of the machine. Suppose that in a certain complete configuration the numbers representing the successive symbols on the tape are $s_1 s_2 \ldots s_n$, that the m-th symbol is scanned, and that the m-configuration has the number t; then we may represent this complete configuration by the formula

$$\Big[[N_{s_1}, N_{s_2}, \ldots, N_{s_{m-1}}],\ [N_t, N_{s_m}],\ [N_{s_{m+1}}, \ldots, N_{s_n}]\Big],$$

where

$$[a, b] \text{ stands for } \lambda u\Big[\Big\{\{u\}(a)\Big\}(b)\Big],$$

$$[a, b, c] \text{ stands for } \lambda u\Big[\Big\{\Big\{\{u\}(a)\Big\}(b)\Big\}(c)\Big],$$

etc.

THE LINEAR DERIVATES AND APPROXIMATE LINEAR DERIVATES OF A FUNCTION OF TWO VARIABLES

By A. J. Ward.

[Received 9 June, 1936.—Read 12 November, 1936.]

This note may be considered as a continuation of a previous paper* on the differential structure of real functions. Mr. A. S. Besicovitch pointed out that there remained unsolved problems concerning the possible dispositions of the linear derivates (as distinct from the Haslam-Jones† directed derivates) of a function of two variables, and gave an example which showed that the linear derivates could behave differently from the directed derivates. I now give a theorem on the disposition of the linear derivates of a function measurable B, and show that a corresponding theorem holds for the approximate linear derivates of any measurable function (not necessarily measurable B). I shall make extensive use of the methods and results of Chapter 11, paragraphs 4, 5, of Saks's *Théorie de l'Intégrale*‡. An alternative method of obtaining the present results, using those of my own former paper, will be briefly sketched.

1. For convenience we represent a point (x, y) by the complex variable $z=x+iy$. We consider a real function of (x, y), which we write, indifferently, as $f(x, y)$ or $f(z)$ $(z=x+iy)$. For the present we suppose that $f(z)$ is defined everywhere in some region. The upper linear derivate of $f(z)$ at the point z, in the direction θ, is defined as

$$\bar{\partial}^{\theta}(f, z)=\bar{\partial}^{\theta}(z)=\overline{\lim_{r\to+0}}\,\frac{f(z+re^{i\theta})-f(z)}{r},$$

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* A. J. Ward, *Proc. London Math. Soc.* (2), 39 (1935), 339–363.

† U. S. Haslam-Jones, *Quart. J. of Math.* (Oxford), 3 (1932), 120–132.

‡ Warsaw, 1933.

and the lower derivate $\partial_\theta(z)$ as the corresponding lower limit. We shall say that θ is *a Denjoy direction for* $f(z)$ *at the point* z if

either $$\partial^\theta(f, z) = -\partial^{\theta+\pi}(z), \text{ finite},$$

or $$\partial^\theta(z) = \partial^{\theta+\pi}(z) = +\infty,$$

and also either $$\partial_\theta(f, z) = -\partial_{\theta+\pi}(z), \text{ finite},$$

or $$\partial_\theta(z) = \partial_{\theta+\pi}(z) = -\infty.$$

Besicovitch's example* consists of a continuous function such that, at each point of a set of positive measure in the z-plane, the directions which are not Denjoy directions are non-enumerable.

THEOREM 1. *If* $f(z)$ *is continuous, in fact if it is measurable* B, *then, for almost all* (x, y, θ) [*a fortiori, for almost all* (x, y) *for almost all* θ], θ *is a Denjoy direction for* f *at* z $(=x+iy)$†.

Suppose, first, that $f(z)$ is continuous. Then the three-dimensional set $E(K, h)$, of points (x, y, θ), such that

$$f(z+re^{i\theta})-f(z) \leqslant Kr$$

whenever $0 \leqslant r \leqslant h$, is easily seen to be closed. Now the set of (x, y, θ) for which $\partial^\theta(z) < K$ can be expressed as $\sum_{n=1}^{\infty} E(K-1/n, 1/n)$. It follows that $\partial^\theta(z)$ is a measurable function of (x, y, θ), and, similarly, so also is $\partial_\theta(z)$. Hence the set H, of points (x, y, θ), such that θ is not a Denjoy direction for $f(z)$ at z, is measurable. We say that it is of measure zero. For if not, there exists θ_0 such that the set H_1, of values of z for which (x, y, θ_0) is in H, is measurable and has positive measure. Then there exists x_0 such that the set H_2, of values of r such that $(x_0 e^{i(\theta_0+\frac{1}{2}\pi)}+re^{i\theta_0})$ is in H_1, has positive linear measure. But this clearly contradicts the Denjoy theorem for the derivates of the function $f(x_0 e^{i(\theta_0+\frac{1}{2}\pi)}+re^{i\theta_0})$ with respect to r.

A similar proof applies if $f(z)$ is no longer continuous, but merely measurable B. We remark that the function of four variables,

$$\phi(x, y, r, \theta) = f(x+iy+re^{i\theta}),$$

is also measurable B [this is most easily seen by using the fact that $f(z)$ belongs to some Baire class, and applying induction]. It follows that

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* *Math. Zeitschrift*, 41 (1936), 402–404.

† Wherever x, y, z occur in conjunction, it is to be taken for granted that x, y are real and that $z=x+iy$.

$E(K, h)$ is the complement of an analytic set*, and therefore is measurable; the proof now proceeds as before.

If $f(z)$ is merely measurable, however, there seems to be no way of proving that $\partial^{\theta}(f, z)$ is a measurable function of (x, y, θ), although I do not know any example to the contrary. The approximate linear derivates, however, prove to be measurable with respect to (x, y, θ).

2. We now suppose only that $f(z)$ is measurable and defined in some measurable set E. Let z be a point of E, $\psi(E, z, \theta, h)$ the outer linear measure of the set of r such that $0 \leqslant r \leqslant h$ and $z+re^{i\theta}$ lies in E, and $\phi(f, z, \theta, h, K)$ the outer linear measure of the set of r such that $0 \leqslant r \leqslant h$, $z+re^{i\theta}$ lies in E, and $f(z+re^{i\theta})-f(z) \geqslant Kr$. For any λ such that

$$0 \leqslant \lambda < \overline{\lim_{h \to +0}} \frac{\psi(E, z, \theta, h)}{h},$$

we define the *upper approximate linear λ-derivate* of $f(z)$ at z in the direction θ, $A\partial^{\theta}(f, z, \lambda)$, as the lower bound of numbers K such that

$$\overline{\lim_{h \to +0}} \frac{\phi(f, z, \theta, h, K)}{h} \leqslant \lambda\dagger.$$

In particular, we write $A\partial^{\theta}(f, z)$ for $A\partial^{\theta}(f, z, 0)$. The lower approximate derivates are defined similarly. It is clear that, if

$$A\partial^{\theta}(f, z) = A\partial_{\theta}(f, z),$$

then all the λ-derivates [for fixed (z, θ)] which are defined must have the same value.

If E is any set of points z, we denote by $\mathbf{E}$ the set of (x, y, θ) such that $x+iy$ is in E and $0 \leqslant \theta \leqslant 2\pi$. We denote by $\mathcal{E}$ the set of (x, y, r, θ) such that $x+iy+re^{i\theta}$ is in E and $0 \leqslant \theta \leqslant 2\pi$. It is easily seen that, if E is open (closed), then $\mathcal{E}$ is open (closed), relative to the space limited by $0 \leqslant \theta \leqslant 2\pi$.

The theorems of this section correspond closely to Theorems 1–3 of the chapter, to which we have referred, in Saks's book.

THEOREM 2. *If E is a measurable set in two dimensions, then, for almost all points (x, y, θ) of $\mathbf{E}$, the set of values of r, such that (x, y, r, θ) is in $\mathcal{E}$, is measurable and has linear density* 1 *at $r = 0$; that is, E has linear density* 1 *at z in the direction θ.*

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* *Cf.* Hausdorff, *Mengenlehre* (1927), 274.

† *Cf.* J. C. Burkill and U. S. Haslam-Jones, *Proc. London Math. Soc.* (2), 32 (1931), 346–355. We shall call $\overline{\lim_{h \to +0}} \frac{\psi(E, z, \theta, h)}{h}$ the upper outer linear density of E at z in the direction θ.

Take any bounded region $\mathcal{G}$ in the space of (x, y, r, θ). If C is a circle of radius ρ in the z-plane, it can be shown by calculation that $m_{(4)}(\mathcal{C}\mathcal{G}) \leqslant K\rho^2$, where K is a constant depending on $\mathcal{G}$. Since E is measurable, given any $\epsilon > 0$ we can find a closed set $F \subset E$, an open set $O \supset E$, and a sequence of circles C_i, radius ρ_i, such that

$$\Sigma\, C_i \supset O - F, \quad \Sigma \rho_i^2 < \epsilon / K.$$

Then $\mathcal{O} \supset \mathcal{E} \supset \mathcal{F}$ and $\Sigma\, \mathcal{C}_i \supset \mathcal{O} - \mathcal{F}$, also $m_{(4)}(\Sigma\, \mathcal{C}_i \mathcal{G}) \leqslant K \Sigma \rho_i^2 < \epsilon$. Since $\mathcal{O}$, $\mathcal{F}$ are measurable and ϵ is arbitrary, $\mathcal{E}\mathcal{G}$ is measurable. Finally, since $\mathcal{G}$ is arbitrary, $\mathcal{E}$ is measurable. It follows that, for almost all (x, y, θ), the set of r, such that (x, y, r, θ) is in $\mathcal{E}$, is measurable.

This being so, it is sufficient to prove the second part of the theorem for any closed set F [which may be taken such that $F \subset E$, and with $m_{(2)}(E - F)$, and so also $m_{(3)}(\mathbf{E} - \mathbf{F})$, arbitrarily small]. Let $D(p, n)$ be the set of (x, y, θ) such that the set of r, with $0 \leqslant r \leqslant h$ and (x, y, r, θ) in $\mathcal{F}$, has measure at least $(1 - 1/p)h$ whenever $0 \leqslant h \leqslant 1/n$. We say that $D(p, n)$ is closed.

This may be seen by an argument similar to that used by Saks, or by the following rather more elementary method*. Suppose that (x_0, y_0, θ_0) is not in $D(p, n)$. There exists h, $0 < h \leqslant 1/n$, such that

$$m\{\underset{r}{E}[(x_0, y_0, r, \theta_0) \in \mathcal{F},\ 0 \leqslant r \leqslant h]\} < (1 - 1/p)\, h.$$

Since $\mathcal{F}$ is closed, the set in brackets is closed. Then there exists a set G of r, consisting of a finite number of closed intervals lying in $0 \leqslant r \leqslant h$, such that $mG > h/p$, and, if r is in G, (x_0, y_0, r, θ_0) is not in $\mathcal{F}$. Let δ be the distance of the closed set $\mathcal{F}$ from the closed set of points (x_0, y_0, r, θ_0), where r is in G. Then, if (x, y, θ) is distant less than δ from (x_0, y_0, θ_0), (x, y, r, θ) cannot lie in $\mathcal{F}$ for any r of G. Hence

$$m\{\underset{r}{E}[x, y, r, \theta) \in \mathcal{F},\ 0 \leqslant r \leqslant h]\} \leqslant h - mG < (1 - 1/p)\, h.$$

Thus we see that the complement of $D(p, n)$ is open.

Now the set of (x, y, θ), lying in $\mathbf{F}$, such that the statement of our theorem (density $= 1$) is not true, can be expressed as $\mathbf{F} - \prod_{p=1}^{\infty} \sum_{n=1}^{\infty} D(p, n)$, and so is measurable. Just as in Theorem 1, if it were not of measure zero we should obtain, by taking first a section $\theta = \theta_0$ and then a section of the z-plane by lines of direction θ_0, a contradiction with the one-dimensional density theorem.

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* *Loc. cit.*, 226, Theorem 1. It should be noted that $D_{m,n}$ there is a two-dimensional set, while we are dealing with a three-dimensional set.

THEOREM[*] 3. *For fixed* λ, $A\partial^\theta(f, z, \lambda)$, *taken where it exists, is a measurable function of* x, y, θ.

We can write $E = \sum_n F_n + N$, where F_n is closed and $f(z)$ is continuous over F_n, and N is of measure zero. Since F_n has linear density 1 at z in the direction θ (and therefore $E - F_n$ has density 0), for almost all (x, y, θ) of $\mathbf{F}_n$, we see that

$$A\partial^\theta(f, z, \lambda) = A\partial^\theta(f_{F_n}, z, \lambda)\dagger,$$

defined for all $\lambda < 1$, for almost all (x, y, θ) of $\mathbf{F}_n$. Hence it suffices to prove the theorem for $f_{F_n}(z)$.

Let $D(\lambda, p, n, K)$ be the set of (x, y, θ), lying in $\mathbf{F}_n$, such that, whenever $0 \leqslant h \leqslant 1/n$,

$$m\{\underset{r}{E}[(x, y, r, \theta) \in \mathfrak{F}_n,\ f(z+re^{i\theta}) - f(z) \leqslant Kr,\ 0 \leqslant r \leqslant h]\} \geqslant (1-\lambda-1/p)\,h.$$

Just as in Theorem 2, since the set of (x, y, r, θ), such that (x, y, θ) is in $\mathbf{F}_n$ and also the conditions within the square brackets are satisfied, is closed, therefore the set $D(\lambda, p, n, K)$ is closed. Now consider the set

$$\prod_{q=1}^{\infty} \prod_{p=1}^{\infty} \sum_{n=1}^{\infty} D(\lambda, p, n, K+1/q).$$

It differs from the set of (x, y, θ) for which $A\partial^\theta(f_{F_n}, z, \lambda) \leqslant K$ only by the addition or subtraction of certain points for which the linear density of F_n at $x+iy$, in the direction θ, is not 1. But we know that all such points form a set of measure zero. It follows that $A\partial^\theta(f_{F_n}, z, \lambda)$ is a measurable function of (x, y, θ).

THEOREM 4. *For almost all* z *of* E, *either*

(i) *For almost all* θ, $A\partial^\theta(f, z, \lambda) = +\infty$, *for all* $\lambda < 1$, *and there exists no approximate derivative plane*‡; *or*

(ii) *There exists an approximate derivative plane given by*

$$a(z)\cos\theta + b(z)\sin\theta,$$

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[*] *Cf. loc. cit.*, 226, Theorem 2.

† $f_{F_n}(z)$ is equal to $f(z)$ if z is in F_n, and is undefined elsewhere.

‡ That is, no approximate differential; Saks, *loc. cit.*, 225, or Ward, *loc. cit.*, 352. For brevity, we write "*AD*-plane".

say, and, for almost all θ,

$$A\partial^{\theta}(f, z, \lambda) = A\partial_{\theta}(f, z, \lambda) = a\cos\theta + b\sin\theta$$

for all $\lambda < 1$.

It is clearly sufficient to prove that

(i) for almost all (x, y, θ) of $\mathbf{E}$,

$$A\partial^{\theta}(f, z) = A\partial^{\theta}(f, z, \lambda), \quad \text{all} \quad \lambda < 1;$$

(ii) for almost all (x, y, θ) such that $A\partial^{\theta}(f, z) < \infty$, an AD-plane exists at z;

(iii) for almost all (x, y, θ) such that an AD-plane exists at z, given by $a(z)\cos\theta + b(z)\sin\theta$, say, we have

$$A\partial^{\theta}(f, z) = a(z)\cos\theta + b(z)\sin\theta.$$

By Theorem 2 we know that, for almost all (x, y, θ) of $\mathbf{E}$, $A\partial^{\theta}(f, z, \lambda)$ is defined for all $\lambda < 1$. Since it is a decreasing function of λ, if the statement (i) is not true, there exists a fixed λ such that the set

$$\{A\partial^{\theta}(f, z) > A\partial^{\theta}(f, z, \lambda)\}$$

is not of measure zero. By Theorem 3, this set is measurable. As in Theorem 1, we find a fixed θ_0 and a fixed line $\theta = \theta_0$, $z = z_0 + re^{i\theta_0}$, cutting this set in a linear set of non-zero measure. Further, we can choose z_0 so that $f(z_0 + re^{i\theta_0})$ is a measurable function of r on the line. We now have a contradiction with the known theorem for approximate derivates of a function of one variable*. Thus (i) is established.

Now suppose that (ii) is not true. Let H be the set of (x, y, θ) such that $A\partial^{\theta}(f, z) < \infty$ and no AD-plane exists at z. From Theorem 3 and known results†, we see that H is measurable. There exists a cube R of side δ, say, such that $m_{(3)}(HR) > \frac{1}{2}\delta^3$. Then there exist two values θ_1, θ_2, such that the sections of HR by $\theta = \theta_1$, $\theta = \theta_2$, are measurable and have each plane measure greater than $\frac{1}{2}\delta^2$, where $|\theta_1 - \theta_2| \leqslant \delta$, and we can suppose that $\delta < \pi$. That is, at the points of a measurable set H_1, of positive measure, in the z-plane, we have $A\partial^{\theta_1}(f, z) < \infty$, $A\partial^{\theta_2}(f, z) < \infty$, and no AD-plane. By a known theorem, almost everywhere in H_1, $f(z)$ cannot have an approximate linear derivative along both the lines of direction θ_1, θ_2; that is, one of the equalities

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* Burkill and Haslam-Jones, *loc. cit.*

† *E.g.*, Saks, *loc. cit.*, Theorems 2, 3 (pp. 226, 228).

$$A\partial^{\theta_1}(f, z) = A\partial_{\theta_1}(f, z) = -A\partial^{\theta_1+\pi}(f, z) = -A\partial_{\theta_1+\pi}(f, z)$$

(or the similar equalities for θ_2) must break down*. Just as before, by cutting H_1 by lines of direction θ_1 or θ_2 on which $f(z)$ is linearly measurable, we obtain a contradiction with the one-dimensional theorem.

In the same way, if (iii) is not true, we obtain a fixed θ_1 such that, at the points of a set of (x, y) of positive measure, there exists an AD-plane, while $A\partial^{\theta_1}(f, z)$ does not agree with the value prescribed by this plane. This contradicts the theorem already referred to*.

3. Theorem 4 gives us a certain amount of information about the linear derivates of any measurable function, by means of the trivial inequality $\partial^{\theta}(f, z) \geqslant A\partial^{\theta}(f, z)$. For instance, we see that, for almost all z of E, $\partial^{\theta}(f, z) + \partial^{\theta+\pi}(f, z) \geqslant 0$ for almost all θ. Further, we are now able to give a rather sharper form of Theorem 1, exhibiting the relation between the values of $\partial^{\theta}(f, z)$ for fixed z and variable θ.

THEOREM 1*a*. *If $f(z)$ is measurable B, then, for almost all z, either*

(i) $\partial^{\theta}(f, z) = +\infty$, $\partial_{\theta}(f, z) = -\infty$, *almost all* θ; *or*

(ii) *there exists an AD-plane, given by* $a(z)\cos\theta + b(z)\sin\theta$, *say, and, for almost all* θ, *either*

$$\partial^{\theta}(f, z) = -\partial^{\theta+\pi}(f, z) = a\cos\theta + b\sin\theta,$$

or

$$\partial^{\theta}(f, z) = \partial^{\theta+\pi}(f, z) = +\infty;$$

and similar equalities hold for the lower derivates.

4. An alternative method of proving Theorem 4 proceeds as follows. We first prove Theorem 2 (but not necessarily Theorem 3). Next we show that, for fixed z_0 and fixed λ, λ', such that $0 \leqslant \lambda \leqslant 2\lambda - \lambda^2 < \lambda' < 1$, we have $A\partial^{\theta}(f_0, \lambda) \geqslant AD^{\theta+}(f, z_0, \lambda')$ for almost all θ†. This is achieved by estimating the two-dimensional measure of the set of points, where

$$f(z) - f(z_0) \geqslant K|z - z_0|, \quad \theta_1 < \arg(z - z_0) < \theta_2, \quad 0 < |z - z_0| < \delta,$$

by integrating with respect to r and θ (where $z = z_0 + re^{i\theta}$). We now apply

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* Saks, *loc. cit.*, Theorem 3. It is necessary to make a linear transformation of the variables, using the fact that $|\theta_1 - \theta_2| < \pi$.

† For the notation, see Ward, *loc. cit.*, 355 ff.

Theorem VI of my previous paper*. We thus prove that, almost everywhere, either $A\partial^{\theta}(f, z, \lambda) = +\infty$, all λ, almost all θ, or the AD-plane exists. Now suppose, if possible, that at the points z of a set of positive outer measure, the AD-plane exists, but $A\partial^{\theta}(f, z) > AD^{\theta}(f, z)$ for a set of θ of positive outer measure. We define, for $M > 0$, $\delta > 0$, $H(M, \delta)$ as the set of points z_0 with the following property: if $0 < r < \delta$, the measure of the set of points z, such that

$$|z-z_0| < r, \quad f(z) \geqslant f(z_0) - M|z-z_0|,$$

is greater than $\frac{1}{2}\pi r^2$. $H(M, \delta)$ is shown to be measurable; using Theorem 2, we then show that, for suitable M, δ, there exist z_0, where the AD-plane exists, and θ_0 such that

$$A\partial^{\theta_0}(f, z_0) > AD^{\theta_0}(f, z_0),$$

while $H(M, \delta)$ is linearly measurable and has linear density 1 at z_0 on the line of direction θ_0. This is easily seen to be impossible.

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* *Loc. cit.*, 357.

THE MOCK THETA FUNCTIONS (2)

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[Received 3 August, 1936.—Read 12 November, 1936.]

1. *Introduction.*

Ramanujan's last letter to Hardy was concerned with mock theta functions of orders three, five, and seven. On a previous occasion* I have dealt with the functions of the third order, and I now come to the functions of the fifth order. Ramanujan defined ten such functions, connected by certain relations. I have succeeded in establishing all the relations whose existence was indicated by Ramanujan; but I have failed to construct a complete and exact transformation theory of the functions, on the lines of the transformation theory of functions of the third order, and, in view of the complexity of all the series which are involved, I am becoming somewhat sceptical concerning the existence of an exact transformation theory for functions of the fifth order. The somewhat nebulous results given in § 6 are all that I have succeeded in obtaining in this direction.

After dealing with Ramanujan's work on the functions of the fifth order, I use his results to study the properties of a transcendental function $f(x, \xi;\ q, \mathfrak{q})$ previously investigated by Lerch; some special cases of this function are readily expressible as the sum of two mock theta functions. Lerch has shown that it is possible to express his function (with $q = \mathfrak{q}$) in terms of theta functions combined with two auxiliary functions $\Psi_0(x, q)$ and $\Psi_1(x, q)$, defined by the formulae

$$\Psi_0(x, q) = \sum_{n=0}^{\infty} \frac{q^{2n^2} x^{4n}}{(1-q^2)(1-q^4)\dots(1-q^{4n})},$$

$$\Psi_1(x, q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+\frac{1}{2}} x^{4n+2}}{(1-q^2)(1-q^4)\dots(1-q^{4n+2})}.$$

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* *Journal London Math. Soc.*, 11 (1936), 55–80. In that paper I quoted the whole of Ramanujan's letter, with the exception of his statements about functions of orders five and seven; these statements had been published previously in his *Collected Papers* (1927), 354–355. No reference to mock theta functions is to be found elsewhere in Ramanujan's writings.

With the help of Ramanujan's results about mock theta functions, I have succeeded (§ 9) in establishing the rather remarkable special formulae

$$\Psi_0(1, q) = \prod_{n=1}^{\infty} (1+q^{2n}) \,.\, G(q^8), \qquad \Psi_1(1, q) = q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1+q^{2n}) \,.\, H(-q^2),$$

$$\Psi_0(q^{\frac{1}{2}}, q) = \prod_{n=1}^{\infty} (1+q^{2n}) \,.\, G(-q^2), \quad \Psi_1(q^{\frac{1}{2}}, q) = q^{\frac{3}{2}} \prod_{n=1}^{\infty} (1+q^{2n}) \,.\, H(q^8),$$

where $G(q)$ and $H(q)$ are the functions occurring in the Rogers-Ramanujan identities. It will be realised that my proof of these formulae is somewhat indirect; I intend to publish elsewhere a direct and moderately simple proof of them*.

It is now convenient to explain the special notations used in this paper and to enunciate a set of known theorems which will constantly be required.

Certain finite and infinite products are of such frequent occurrence that the use of some kind of abridged notation for them seems essential; accordingly, I shall write

$$\prod_{n=0}^{m-1} (1+xq^n) = \Pi_m(x, q); \quad \prod_{n=0}^{\infty} (1+xq^n) = \Pi_\infty(x, q).$$

In this notation, various theorems concerning rearrangements of products can be very compactly expressed, *e.g.*,

$$\Pi_\infty(x, q) = \Pi_m(x, q)\, \Pi_\infty(xq^m, q), \quad \Pi_{2m}(x, q) = \Pi_m(x, q^2)\, \Pi_m(xq, q^2),$$

$$\Pi_m(-x^2, q^2) = \Pi_m(x, q)\, \Pi_m(-x, q).$$

It is desirable to adopt the convention that empty products are to be interpreted to mean unity.

Three expansions, of which the first two are due to Euler†, will continually be required, namely

$$\text{(E 1)} \qquad \prod_{n=0}^{\infty} (1+q^n z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{(1-q)(1-q^2)\dots(1-q^n)},$$

$$\text{(E 2)} \qquad \left[\prod_{n=0}^{\infty} (1-q^n z)\right]^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{(1-q)(1-q^2)\dots(1-q^n)},$$

$$\text{(E 3)} \qquad \left[\prod_{n=0}^{m-1} (1-q^n z)\right]^{-1} = \sum_{n=0}^{\infty} \frac{z^n(1-q^m)(1-q^{m+1})\dots(1-q^{m+n-1})}{(1-q)(1-q^2)\dots(1-q^n)}.$$

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* A paper containing such a proof has been accepted for publication in the *Quart. J. of Math.*

† *Introductio in analysin infinitorum*, 1 (1748), 259, 263. The third expansion is given by Cauchy, *Comptes rendus*, 10 (1840), 178–181, though I think that it must have been known earlier. Some related expansions were discovered by Gauss in 1811; see his *Werke*, 2 (1876), 9–45.

In the notation which has just been introduced, these expansions assume the forms

(E 1) $$\Pi_\infty(z, q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{\Pi_n(-q, q)},$$

(E 2) $$[\Pi_\infty(-z, q)]^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{\Pi_n(-q, q)},$$

(E 3) $$[\Pi_m(-z, q)]^{-1} = \sum_{n=0}^{\infty} \frac{z^n \, \Pi_n(-q^m, q)}{\Pi_n(-q, q)}.$$

Use will also be made of another formula due to Euler*

(E 4) $$\Pi_\infty(q, q) \,.\, \Pi_\infty(-q, q^2) = 1,$$

and two formulae due to Jacobi†

(J 1) $$\vartheta_4(0, q) = \frac{\Pi_\infty(-q, q)}{\Pi_\infty(q, q)},$$

(J 2) $$\vartheta_2(0, q) = 2q^{\frac{1}{4}} \frac{\Pi_\infty(-q^4, q^4)}{\Pi_\infty(-q^2, q^4)}.$$

In this connection, I remark that fractional indices can often be avoided by using Ramanujan's function $\psi(q)$, which he defined by the formula

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)},$$

in place of $\vartheta_2(q)$. In this notation, the last theorem quoted assumes the form

(J 2) $$\psi(q) = \frac{\Pi_\infty(-q^2, q^2)}{\Pi_\infty(-q, q^2)}.$$

The functions $G(q)$ and $H(q)$ which occur in the Rogers-Ramanujan identities are defined by the formulae

(R 1) $$G(q) = \sum_{n=0}^{\infty} \frac{q^{n_2}}{\Pi_n(-q, q)},$$

(R 2) $$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{\Pi_n(-q, q)},$$

and the Rogers-Ramanujan identities themselves are expressed by the equations

(R 3) $$G(q) = [\Pi_\infty(-q, q^5)\, \Pi_\infty(-q^4, q^5)]^{-1},$$

(R 4) $$H(q) = [\Pi_\infty(-q^2, q^5)\, \Pi_\infty(-q^3, q^5)]^{-1}.$$

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* *Ibid.*, 271.

† *Fundamenta nova* (1829), 185.

The following formulae involving $G(q)$ and $H(q)$ will also be required:

(R 5) $$2G(q)\,G(q^4)=[\vartheta_3(0,\,q)+\vartheta_3(0,\,q^5)]/\Pi_\infty(-q^2,\,q^2),$$

(R 6) $$2qH(q)\,H(q^4)=[\vartheta_3(0,\,q)-\vartheta_3(0,\,q^5)]/\Pi_\infty(-q^2,\,q^2),$$

(R 7) $$G(q)\,H(-q)=[\psi(q^2)+q\psi(q^{10})]/\Pi_\infty(-q^2,\,q^2).$$

These formulae were discovered by Ramanujan and they were enunciated in his note book; I published proofs of them a few years ago*.

2. *Definitions of the mock theta functions of the fifth order, and the relations between them.*

Ramanujan defined ten mock theta functions of the fifth order in all, these ten functions being divisible into two sets of five. The definitions† of the first five functions are as follows:

$$f_0(q)=1+\frac{q}{1+q}+\frac{q^4}{(1+q)(1+q^2)}+\ldots,$$

$$\phi_0(q)=1+q(1+q)+q^4(1+q)(1+q^3)+q^9(1+q)(1+q^3)(1+q^5)+\ldots,$$

$$\psi_0(q)=q+q^3(1+q)+q^6(1+q)(1+q^2)+q^{10}(1+q)(1+q^2)(1+q^3)+\ldots,$$

$$F_0(q)=1+\frac{q^2}{1-q}+\frac{q^8}{(1-q)(1-q^3)}+\ldots,$$

(A$_0$) $$\chi_0(q)=1+\frac{q}{1-q^2}+\frac{q^2}{(1-q^3)(1-q^4)}+\frac{q^3}{(1-q^4)(1-q^5)(1-q^6)}+\ldots$$

$$=1+\frac{q}{1-q}+\frac{q^3}{(1-q^2)(1-q^3)}+\frac{q^5}{(1-q^3)(1-q^4)(1-q^5)}+\ldots.$$

Ramanujan asserted that these functions satisfy the following relations:

(B$_0$) $$\phi_0(-q)+\chi_0(q)=2F_0(q),$$

(C$_0$) $$f_0(-q)+2F_0(q^2)-2=\phi_0(-q^2)+\psi_0(-q)$$

$$=2\phi_0(-q^2)-f_0(q)=\frac{1-2q+2q^4-2q^9+\ldots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\ldots},$$

(D$_0$) $$\psi_0(q)-F_0(q^2)+1=q\,\frac{1+q^2+q^6+q^{12}+\ldots}{(1-q^8)(1-q^{12})(1-q^{28})(1-q^{32})\ldots}.$$

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* *Journal Indian Math. Soc.*, 20 (1934), 57–69.

† I have added the suffixes 0 and 1 to discriminate between the two sets, and I have numbered the relations; otherwise I have made no changes in Ramanujan's manuscript.

The definitions of the first four functions need no comment, save that the series may be written compactly thus:

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\Pi_n(q,\,q)},\qquad \phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}\,\Pi_n(q,\,q^2),$$

$$\psi_0(q)=\sum_{n=0}^{\infty}q^{\frac{1}{2}(n+1)(n+2)}\,\Pi_n(q,\,q),\qquad F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{\Pi_n(-q,\,q^2)}.$$

The definition of $\chi_0(q)$, namely

$$(\mathrm{A}_0)\qquad \chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{\Pi_n(-q^{n+1},\,q)}=1+\sum_{m=0}^{\infty}\frac{q^{2m+1}}{\Pi_{m+1}(-q^{m+1},\,q)},$$

however, contains an assertion of the equality of two series. I dispose of the proof of this assertion forthwith by observing that, when formula (E 3) is used to sum the double series

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{q^{mn+m+n}\,\Pi_{m+n-1}(-q,\,q)}{\Pi_{n-1}(-q,\,q)\,\Pi_m(-q,\,q)}$$

in two different ways, the results obtained are

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{q^{mn+m+n}\,\Pi_m(-q^n,\,q)}{\Pi_m(-q,\,q)}=\sum_{n=1}^{\infty}\frac{q^n}{\Pi_n(-q^{n+1},\,q)}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{q^{mn+m+n}\,\Pi_{n-1}(-q^{m+1},\,q)}{\Pi_{n-1}(-q,\,q)}=\sum_{m=0}^{\infty}\frac{q^{2m+1}}{\Pi_{m+1}(-q^{m+1},\,q)},$$

respectively; and this establishes the equality of the two series involved in (A_0).

I also remark that the expressions on the extreme right of (C_0) and on the right of (D_0) may be written in the forms

$$\vartheta_4(0,\,q)\,G(q),\qquad q\psi(q^2)\,H(q^4),$$

respectively.

It is now convenient to leave the first set of functions temporarily and to give the definitions of the five functions of the second set. It seems adequate to state these in their compact forms*, thus:

$$f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\Pi_n(q,\,q)},\qquad \phi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}\,\Pi_n(q,\,q^2),$$

$$\psi_1(q)=\sum_{n=0}^{\infty}q^{\frac{1}{2}n(n+1)}\,\Pi_n(q,\,q),\qquad F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\Pi_{n+1}(-q,\,q^2)},$$

$$\chi_1(q)=\sum_{n=0}^{\infty}\frac{q^n}{\Pi_{n+1}(-q^{n+1},\,q)}.$$

---

* There is a trivial misprint in the definition of $\phi_1(q)$ in Ramanujan's *Collected papers* (1927), 355.

Ramanujan asserted that these functions satisfy relations similar to those satisfied by the first set, but he did not state the actual relations explicitly in his letter. The relations are

$$(\mathrm{B}_1) \qquad \chi_1(q) - q^{-1}\phi_1(-q) = 2F_1(q),$$

$$(\mathrm{C}_1) \qquad f_1(-q) - 2q\,F_1(q^2) = q^{-1}\phi_1(-q^2) + \psi_1(-q)$$

$$= 2q^{-1}\phi_1(-q^2) + f_1(q) = \vartheta_4(0, q)\,H(q),$$

$$(\mathrm{D}_1) \qquad \psi_1(q) - qF_1(q^2) = \psi(q^2)\,G(q^4).$$

It is to be observed that, if $\chi_1(q)$ is treated in the same way as $\chi_0(q)$, by expanding

$$[\Pi_{n+1}(-q^{n+1}, q)]^{-1},$$

the result that we get is

$$\chi_1(q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{mn+m+n}\,\Pi_{m+n}(-q, q)}{\Pi_m(-q, q)\,\Pi_n(-q, q)}.$$

Since the general term of the double series on the right is symmetrical in m and n, there is evidently no analogue, in the case of $\chi_1(q)$, to the dual definition of $\chi_0(q)$ supplied by the formula (A_0).

Instead of establishing the various relations in the forms in which they have now been quoted, it is more convenient, after proving (B_0) and (B_1), to prove the following formulae:

$$(1_0) \qquad \psi_0(q) = q\psi(q^2)\,H(q^4) + [F_0(q^2) - 1],$$

$$(1_1) \qquad \psi_1(q) = \psi(q^2)\,G(q^4) + qF_1(q^2),$$

$$(2_0) \qquad f_0(q) + \vartheta_4(0, q)\,G(q) = 2\phi_0(-q^2),$$

$$(3_0) \qquad f_0(q) - \vartheta_4(0, q)\,G(q) = 2q\psi(q^2)\,H(q^4) - 2[F_0(q^2) - 1],$$

$$(2_1) \qquad f_1(q) - \vartheta_4(0, q)\,H(q) = -2q^{-1}\phi_1(-q^2),$$

$$(3_1) \qquad f_1(q) + \vartheta_4(0, q)\,H(q) = 2\psi(q^2)\,G(q^4) - 2qF_1(q^2).$$

When these six formulae have been proved, it is an easy matter to derive the results contained in (C_0), (D_0), (C_1), (D_1) from them. The six formulae will be proved by starting with the series on the left and transforming them with the help of various repeated series, while (B_0) and (B_1) will be proved directly by effecting similar transformations on the series for $\chi_0(q)$ and $\chi_1(q)$.

In writing out the proofs, I explain each step which involves the summation of a series or the reverse, or which involves the separation of the terms of even rank in a series from those of odd rank; a step of the latter type will be written in the form

$$\sum_{m=0}^{\infty} a_m = \sum_{n=0}^{\infty} a_{2n} + \sum_{n=0}^{\infty} a_{2n+1} \quad (m=2n \text{ or } m=2n+1).$$

The steps which involve rearrangements of products are much easier to follow (and more troublesome to explain briefly), so I decided to save space by leaving the explanation of such steps to the ingenuity of the reader.

3. *Transformations of series for mock theta functions.*

We now come to the actual work of transforming the series for mock theta functions. The transformation of the series for $\chi_0(q)$ is as follows:

$$\frac{2\chi_0(q)}{\Pi_\infty(-q,\, q)} = \frac{2}{\Pi_\infty(-q,\, q)} \sum_{n=0}^{\infty} \frac{q^n(1-q)(1-q^2)\dots(1-q^n)}{(1-q)(1-q^2)\dots(1-q^{2n})}$$

$$= 2\sum_{n=0}^{\infty} \frac{q^n}{\Pi_{2n}(-q,\, q)\,\Pi_\infty(-q^{n+1},\, q)}$$

$$= 2\sum_{n=0}^{\infty} \frac{q^n}{\Pi_{2n}(-q,\, q)} \sum_{m=0}^{\infty} \frac{q^{(n+1)m}}{\Pi_m(-q,\, q)} \quad \text{(by E 2)}$$

$$= \sum_{m=0}^{\infty} \frac{q^m}{\Pi_m(-q,\, q)} \sum_{p=0}^{\infty} \frac{(q^{\frac{1}{2}m+\frac{1}{2}})^p + (-q^{\frac{1}{2}m+\frac{1}{2}})^p}{\Pi_p(-q,\, q)}$$

$$(p=2n \quad \text{or} \quad p=2n+1)$$

$$= \sum_{m=0}^{\infty} \frac{q^m}{\Pi_m(-q,\, q)} \left[\frac{1}{\Pi_\infty(-q^{\frac{1}{2}m+\frac{1}{2}},\, q)} + \frac{1}{\Pi_\infty(q^{\frac{1}{2}m+\frac{1}{2}},\, q)}\right] \quad \text{(by E 2)}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n}}{\Pi_{2n}(-q,\, q)} \left[\frac{1}{\Pi_\infty(-q^{n+\frac{1}{2}},\, q)} + \frac{1}{\Pi_\infty(q^{n+\frac{1}{2}},\, q)}\right]$$

$$+ \sum_{n=0}^{\infty} \frac{q^{2n+1}}{\Pi_{2n+1}(-q,\, q)} \left[\frac{1}{\Pi_\infty(-q^{n+1},\, q)} + \frac{1}{\Pi_\infty(q^{n+1},\, q)}\right]$$

$$(m=2n \quad \text{or} \quad m=2n+1)$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n}}{\Pi_{2n}(-q,\, q)} \left[\frac{\Pi_\infty(q^{n+\frac{1}{2}},\, q)}{\Pi_\infty(-q^{2n+1},\, q^2)} + \frac{\Pi_\infty(-q^{n+\frac{1}{2}},\, q)}{\Pi_\infty(-q^{2n+1},\, q^2)}\right]$$

$$+ \frac{1}{\Pi_\infty(-q,\, q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}\,\Pi_n(-q,\, q)}{\Pi_{2n+1}(-q,\, q)} + \frac{1}{\Pi_\infty(q,\, q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}\,\Pi_n(q,\, q)}{\Pi_{2n+1}(-q,\, q)}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n}}{\Pi_{2n}(-q,\, q)\,\Pi_\infty(-q^{2n+1},\, q^2)} \sum_{m=0}^{\infty} \frac{\{1+(-)^m\}\, q^{\frac{1}{2}m(m-1)}\, q^{m(n+\frac{1}{2})}}{\Pi_m(-q,\, q)}$$

$$+ \frac{\chi_0(q)-1}{\Pi_\infty(-q,\, q)} + \frac{1}{\Pi_\infty(q,\, q)} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{\Pi_{n+1}(-q,\, q^2)\,\Pi_n(-q,\, q)}$$

$$\text{(by E 1)}$$

$$= 2 \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{q^{2p^2+2np+2n}}{\Pi_{2n}(-q,\, q)\, \Pi_\infty(-q^{2n+1},\, q^2)\, \Pi_{2p}(-q,\, q)}$$

$$+ \frac{\chi_0(q)-1}{\Pi_\infty(-q,\, q)} + \frac{1}{\Pi_\infty(q,\, q)\, \Pi_\infty(-q,\, q^2)} \sum_{n=0}^{\infty} \frac{q^{2n+1}\, \Pi_\infty(-q^{2n+3},\, q^2)}{\Pi_n(-q,\, q)}$$

$$(m = 2p \quad \text{or} \quad m = 2p+1)$$

$$= \frac{2}{\Pi_\infty(-q,\, q^2)} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{q^{2p^2+2np+2n}}{\Pi_n(-q^2,\, q^2)\, \Pi_{2p}(-q,\, q)}$$

$$+ \frac{\chi_0(q)-1}{\Pi_\infty(-q,\, q)} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^m\, q^{m(m-1)+m(2n+3)+2n+1}}{\Pi_n(-q,\, q)\, \Pi_m(-q^2,\, q^2)}$$

$$(\text{by E}\,4 \text{ and E}\,1)$$

$$= \frac{2}{\Pi_\infty(-q^2,\, q^2)} \sum_{p=0}^{\infty} \frac{q^{2p^2}}{\Pi_{2p}(-q,\, q)\, \Pi_\infty(-q^{2p+2},\, q^2)}$$

$$+ \frac{\chi_0(q)-1}{\Pi_\infty(-q,\, q)} + \sum_{m=0}^{\infty} \frac{(-)^m\, q^{(m+1)^2}}{\Pi_m(-q^2,\, q^2)\, \Pi_\infty(-q^{2m+2},\, q)} \quad (\text{by E}\,2)$$

$$= \frac{2}{\Pi_\infty(-q,\, q)} \sum_{p=0}^{\infty} \frac{q^{2p^2}}{\Pi_p(-q,\, q^2)} + \frac{\chi_0(q)-1}{\Pi_\infty(-q,\, q)}$$

$$+ \frac{1}{\Pi_\infty(-q,\, q)} \sum_{m=0}^{\infty} (-)^m\, q^{(m+1)^2}\, \Pi_{m+1}(-q,\, q^2)$$

and so $$2\chi_0(q) = 2F_0(q) + \chi_0(q) - 1 + [1 - \phi_0(-q)],$$

whence formula (B_0) follows immediately.

The transformation of the series for $\chi_1(q)$ is of like nature; we have

$$\frac{2\chi_1(q)}{\Pi_\infty(-q, q)} = \frac{2}{\Pi_\infty(-q,\, q)} \sum_{n=0}^{\infty} \frac{q^n (1-q)(1-q^2) \ldots (1-q^n)}{(1-q)(1-q^2) \ldots (1-q^{2n+1})}$$

$$= 2 \sum_{n=0}^{\infty} \frac{q^n}{\Pi_{2n+1}(-q,\, q)\, \Pi_\infty(-q^{n+1},\, q)}$$

$$= 2 \sum_{n=0}^{\infty} \frac{q^n}{\Pi_{2n+1}(-q,\, q)} \sum_{m=0}^{\infty} \frac{q^{(n+1)m}}{\Pi_m(-q,\, q)} \quad (\text{by E}\,2)$$

$$= \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m-\frac{1}{2}}}{\Pi_m(-q,\, q)} \sum_{p=0}^{\infty} \frac{(q^{\frac{1}{2}m+\frac{1}{2}})^p - (-q^{\frac{1}{2}m+\frac{1}{2}})^p}{\Pi_p(-q,\, q)}$$

$$(p = 2n \quad \text{or} \quad p = 2n+1)$$

$$= \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}m-\frac{1}{2}}}{\Pi_m(-q,\, q)} \left[\frac{1}{\Pi_\infty(-q^{\frac{1}{2}m+\frac{1}{2}},\, q)} - \frac{1}{\Pi_\infty(q^{\frac{1}{2}m+\frac{1}{2}},\, q)} \right] \quad (\text{by E}\,2)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n-\frac{1}{2}}}{\Pi_{2n}(-q,\, q)} \left[\frac{1}{\Pi_{\infty}(-q^{n+\frac{1}{2}},\, q)} - \frac{1}{\Pi_{\infty}(q^{n+\frac{1}{2}},\, q)} \right]$$

$$+ \sum_{n=0}^{\infty} \frac{q^{n}}{\Pi_{2n+1}(-q,\, q)} \left[\frac{1}{\Pi_{n}(-q^{n+1},\, q)} - \frac{1}{\Pi_{\infty}(q^{n+1},\, q)} \right]$$

$$(m = 2n \quad \text{or} \quad m = 2n+1)$$

$$= \sum_{n=0}^{\infty} \frac{q^{n-\frac{1}{2}}}{\Pi_{2n}(-q,\, q)} \left[\frac{\Pi_{\infty}(q^{n+\frac{1}{2}},\, q)}{\Pi_{\infty}(-q^{2n+1},\, q^{2})} - \frac{\Pi_{\infty}(-q^{n+\frac{1}{2}},\, q)}{\Pi_{\infty}(-q^{2n+1},\, q^{2})} \right]$$

$$+ \frac{1}{\Pi_{\infty}(-q,\, q)} \sum_{n=0}^{\infty} \frac{q^{n}\, \Pi_{n}(-q,\, q)}{\Pi_{2n+1}(-q,\, q)} - \frac{1}{\Pi_{\infty}(q,\, q)} \sum_{n=0}^{\infty} \frac{q^{n}\, \Pi_{n}(q,\, q)}{\Pi_{2n+1}(-q,\, q)}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n-\frac{1}{2}}}{\Pi_{2n}(-q,\, q)\, \Pi_{\infty}(-q^{2n+1},\, q^{2})} \sum_{m=0}^{\infty} \frac{\{1-(-)^{m}\}\, q^{\frac{1}{2}m(m-1)}\, q^{m(n+\frac{1}{2})}}{\Pi_{m}(-q,\, q)}$$

$$+ \frac{\chi_{1}(q)}{\Pi_{\infty}(-q,\, q)} - \frac{1}{\Pi_{\infty}(q,\, q)} \sum_{n=0}^{\infty} \frac{q^{n}}{\Pi_{n+1}(-q,\, q^{2})\, \Pi_{n}(-q,\, q)}$$

(by E 1)

$$= 2 \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{q^{2p^{2}+2np+2p+2n}}{\Pi_{2n}(-q,\, q)\, \Pi_{\infty}(-q^{2n+1},\, q^{2})\, \Pi_{2p+1}(-q,\, q)}$$

$$+ \frac{\chi_{1}(q)}{\Pi_{\infty}(-q,\, q)} - \frac{1}{\Pi_{\infty}(q,\, q)\, \Pi_{\infty}(-q,\, q^{2})} \sum_{n=0}^{\infty} \frac{q^{n}\, \Pi_{\infty}(-q^{2n+3},\, q^{2})}{\Pi_{n}(-q,\, q)}$$

$$(m = 2p \quad \text{or} \quad m = 2p+1)$$

$$= \frac{2}{\Pi_{\infty}(-q,\, q^{2})} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{q^{2p^{2}+2np+2p+2n}}{\Pi_{n}(-q^{2},\, q^{2})\, \Pi_{2p+1}(-q,\, q)}$$

$$+ \frac{\chi_{1}(q)}{\Pi_{\infty}(-q,\, q)} - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^{m}\, q^{m(m-1)+n+m(2n+3)}}{\Pi_{n}(-q,\, q)\, \Pi_{m}(-q^{2},\, q^{2})}$$

(by E 4 and E 1)

$$= \frac{2}{\Pi_{\infty}(-q,\, q^{2})} \sum_{p=0}^{\infty} \frac{q^{2p^{2}+2p}}{\Pi_{2p+1}(-q,\, q)\, \Pi_{\infty}(-q^{2p+2},\, q^{2})}$$

$$+ \frac{\chi_{1}(q)}{\Pi_{\infty}(-q,\, q)} - \sum_{m=0}^{\infty} \frac{(-)^{m}\, q^{m(m+2)}}{\Pi_{m}(-q^{2},\, q^{2})\, \Pi_{\infty}(-q^{2m+1},\, q)} \quad \text{(by E 2)}$$

$$= \frac{2}{\Pi_{\infty}(-q,\, q)} \sum_{p=0}^{\infty} \frac{q^{2p^{2}+2p}}{\Pi_{p+1}(-q,\, q^{2})} + \frac{\chi_{1}(q)}{\Pi_{\infty}(-q,\, q)}$$

$$- \frac{1}{\Pi_{\infty}(-q,\, q)} \sum_{m=0}^{\infty} (-)^{m}\, q^{m(m+2)}\, \Pi_{m}(-q,\, q^{2})$$

and so $$2\chi_{1}(q) = 2F_{1}(q) + \chi_{1}(q) + q^{-1}\phi_{1}(-q),$$

whence formula (B_1) follows immediately.

The transformations of the series for $\psi_0(q)$ and $\psi_1(q)$ are rather simpler. For $\psi_0(q)$ we have

$$
\begin{aligned}
\psi_0(q) &= \sum_{n=0}^{\infty} q^{\frac{1}{2}(n+1)(n+2)}\,\Pi_n(q,\,q) \\
&= \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n+1)(n+2)}\,\Pi_n(-q^2,\,q^2)}{\Pi_n(-q,\,q)} \\
&= \Pi_\infty(-q^2,\,q^2) \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n+1)(n+2)}}{\Pi_n(-q,\,q)\,\Pi_\infty(-q^{2n+2},\,q^2)} \\
&= \Pi_\infty(-q^2,\,q^2) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}(n+1)(n+2)+m(2n+2)}}{\Pi_n(-q,\,q)\,\Pi_m(-q^2,\,q^2)} \quad \text{(by E 2)} \\
&= \Pi_\infty(-q^2,\,q^2) \sum_{m=0}^{\infty} \frac{q^{2m+1}\,\Pi_\infty(q^{2m+2},\,q)}{\Pi_m(-q^2,\,q^2)} \quad \text{(by E 1)} \\
&= \Pi_\infty(-q^2,\,q^2) \sum_{m=0}^{\infty} \frac{q^{2m+1}\,\Pi_\infty(q^{2m+2},\,q^2)\,\Pi_\infty(q^{2m+3},\,q^2)}{\Pi_m(-q^2,\,q^2)} \\
&= \Pi_\infty(-q^2,\,q^2)\,\Pi_\infty(q^2,\,q^2) \sum_{m=0}^{\infty} \frac{q^{2m+1}\,\Pi_\infty(q^{2m+3},\,q^2)}{\Pi_m(-q^2,\,q^2)\,\Pi_m(q^2,\,q^2)} \\
&= \Pi_\infty(-q^4,\,q^4) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{n(n-1)+n(2m+3)+2m+1}}{\Pi_m(-q^4,\,q^4)\,\Pi_n(-q^2,\,q^2)} \quad \text{(by E 1)} \\
&= \Pi_\infty(-q^4,\,q^4) \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{\Pi_n(-q^2,\,q^2)\,\Pi_\infty(-q^{2n+2},\,q^4)} \quad \text{(by E 2)} \\
&= \Pi_\infty(-q^4,\,q^4) \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_{2m}(-q^2,\,q^2)\,\Pi_\infty(-q^{4m+2},\,q^4)} \\
&\qquad\qquad + \Pi_\infty(-q^4,\,q^4) \sum_{m=0}^{\infty} \frac{q^{(2m+2)^2}}{\Pi_{2m+1}(-q^2,\,q^2)\,\Pi_\infty(-q^{4m+4},\,q^4)} \\
&\qquad\qquad\qquad\qquad (n = 2m \quad \text{or} \quad n = 2m+1) \\
&= \frac{\Pi_\infty(-q^4,\,q^4)}{\Pi_\infty(-q^2,\,q^4)} \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}\,\Pi_m(-q^2,\,q^4)}{\Pi_{2m}(-q^2,\,q^2)} + \sum_{m=0}^{\infty} \frac{q^{(2m+2)^2}\,\Pi_m(-q^4,\,q^4)}{\Pi_{2m+1}(-q^2,\,q^2)} \\
&= \psi(q^2) \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_m(-q^4,\,q^4)} + \sum_{m=0}^{\infty} \frac{q^{(2m+2)^2}}{\Pi_{m+1}(-q^2,\,q^4)} \quad \text{(by J 2)} \\
&= q\psi(q^2)\,H(q^4) + \sum_{n=1}^{\infty} \frac{q^{4n^2}}{\Pi_n(-q^2,\,q^4)} \quad (m = n-1) \\
&= q\psi(q^2)\,H(q^4) + F_0(q^2) - 1,
\end{aligned}
$$

and so we have proved formula (1_0).

Next, for $\psi_1(q)$ we have

$$\psi_1(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} \Pi_n(q, q)$$

$$= \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)} \Pi_n(-q^2, q^2)}{\Pi_n(-q, q)}$$

$$= \Pi_\infty(-q^2, q^2) \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{\Pi_n(-q, q)\, \Pi_\infty(-q^{2n+2}, q^2)}$$

$$= \Pi_\infty(-q^2, q^2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)+m(2n+2)}}{\Pi_n(-q, q)\, \Pi_m(-q^2, q^2)} \quad \text{(by E 2)}$$

$$= \Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \frac{q^{2m}\, \Pi_\infty(q^{2m+1}, q)}{\Pi_m(-q^2, q^2)} \quad \text{(by E 1)}$$

$$= \Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \frac{q^{2m}\, \Pi_\infty(q^{2m+1}, q^2)\, \Pi_\infty(q^{2m+2}, q^2)}{\Pi_m(-q^2, q^2)}$$

$$= \Pi_\infty(-q^2, q^2)\, \Pi_\infty(q^2, q^2) \sum_{m=0}^{\infty} \frac{q^{2m}\, \Pi_\infty(q^{2m+1}, q^2)}{\Pi_m(-q^2, q^2)\, \Pi_m(q^2, q^2)}$$

$$= \Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^{n(n-1)+n(2m+1)+2m}}{\Pi_m(-q^4, q^4)\, \Pi_n(-q^2, q^2)} \quad \text{(by E 1)}$$

$$= \Pi_\infty(-q^4, q^4) \sum_{n=0}^{\infty} \frac{q^{n^2}}{\Pi_n(-q^2, q^2)\, \Pi_\infty(-q^{2n+2}, q^4)} \quad \text{(by E 2)}$$

$$= \Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{4m^2}}{\Pi_{2m}(-q^2, q^2)\, \Pi_\infty(-q^{4m+2}, q^4)}$$

$$+ \Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_{2m+1}(-q^2, q^2)\, \Pi_\infty(-q^{4m+4}, q^4)}$$

$$(n = 2m \quad \text{or} \quad n = 2m+1)$$

$$= \frac{\Pi_\infty(-q^4, q^4)}{\Pi_\infty(-q^2, q^4)} \sum_{m=0}^{\infty} \frac{q^{4m^2}\, \Pi_m(-q^2, q^4)}{\Pi_{2m}(-q^2, q^2)} + \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}\, \Pi_m(-q^4, q^4)}{\Pi_{2m+1}(-q^2, q^2)}$$

$$= \psi(q^2) \sum_{m=0}^{\infty} \frac{q^{4m^2}}{\Pi_m(-q^4, q^4)} + \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_{m+1}(-q^2, q^4)} \quad \text{(by J 2)}$$

$$= \psi(q^2)\, G(q^4) + q F_1(q^2),$$

and so we have proved formula (1_1).

To transform $f_0(q)$, we consider $f_0(q) \pm \vartheta_4(0, q)\, G(q)$; we have first, by (J 1),

$$
\begin{aligned}
f_0(q) \pm \vartheta_4(0, q)\, G(q) \\
&= \sum_{n=0}^{\infty} \frac{q^n}{\Pi_n(q, q)} \pm \frac{\Pi_\infty(-q, q)}{\Pi_\infty(q, q)} \sum_{n=0}^{\infty} \frac{q^{n^2}}{\Pi_n(-q, q)} \\
&= \Pi_\infty(-q, q) \left[\sum_{n=0}^{\infty} \frac{q^{n^2}}{\Pi_n(q, q)\, \Pi_n(-q, q)\, \Pi_\infty(-q^{n+1}, q)} \right. \\
&\qquad \left. \pm \sum_{n=0}^{\infty} \frac{q^{n^2}}{\Pi_n(-q, q)\, \Pi_n(q, q)\, \Pi_\infty(q^{n+1}, q)} \right] \\
&= \Pi_\infty(-q, q) \sum_{n=0}^{\infty} \frac{q^{n^2}}{\Pi_n(-q^2, q^2)} \left[\frac{1}{\Pi_\infty(-q^{n+1}, q)} \pm \frac{1}{\Pi_\infty(q^{n+1}, q)} \right] \\
&= \Pi_\infty(-q, q) \sum_{n=0}^{\infty} \sum_{p-0}^{\infty} \frac{\{1 \pm (-)^p\}\, q^{n^2+p(n+1)}}{\Pi_n(-q^2, q^2)\, \Pi_p(-q, q)} \qquad \text{(by E 2).}
\end{aligned}
$$

We now take the upper signs, so that we may write $p = 2m$; we thus get

$$
\begin{aligned}
f_0(q) + \vartheta_4(0, q)\, G(q) \\
&= 2\Pi_\infty(-q, q) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{n^2+2m(n+1)}}{\Pi_n(-q^2, q^2)\, \Pi_{2m}(-q, q)} \\
&= 2\Pi_\infty(-q, q) \sum_{m=0}^{\infty} \frac{q^{2m}\, \Pi_\infty(q^{2m+1}), q^2)}{\Pi_{2m}(-q, q)} \qquad \text{(by E 1)} \\
&= 2\Pi_\infty(-q, q^2)\, \Pi_\infty(-q^2, q^2) \\
&\qquad \times \sum_{m=0}^{\infty} \frac{q^{2m}\, \Pi_m(-q, q^2)\, \Pi_\infty(-q^{2m+1}, q^2)\, \Pi_\infty(q^{2m+1}, q^2)}{\Pi_\infty(-q, q^2)\, \Pi_m(-q, q^2)\, \Pi_m(-q^2, q^2)} \\
&= 2\Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \frac{q^{2m}\, \Pi_\infty(-q^{4m+2}, q^4)}{\Pi_m(-q^2, q^2)} \\
&= 2\Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-)^n\, q^{2n(n+2m)+2m}}{\Pi_m(-q^2, q^2)\, \Pi_n(-q^4, q^4)} \qquad \text{(by E 1)} \\
&= 2\Pi_\infty(-q^2, q^2) \sum_{n=0}^{\infty} \frac{(-)^n\, q^{2n^2}}{\Pi_n(-q^4, q^4)\, \Pi_\infty(-q^{4n+2}, q^2)} \qquad \text{(by E 2)} \\
&= 2 \sum_{n=0}^{\infty} \frac{(-)^n\, q^{2n^2}\, \Pi_{2n}(-q^2, q^2)}{\Pi_n(-q^4, q^4)} \\
&= 2 \sum_{n=0}^{\infty} (-)^n\, q^{2n^2}\, \Pi_n(-q^2, q^4) = 2\phi_0(-q^2),
\end{aligned}
$$

and now formula (2_0) is established.

We next take the lower signs, so that we may write $p = 2m+1$; we thus get

$$f_0(q) - \vartheta_4(0, q)\, G(q)$$

$$= 2\Pi_\infty(-q, q) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{n^2+(2m+1)(n+1)}}{\Pi_n(-q^2, q^2)\, \Pi_{2m+1}(-q, q)}$$

$$= 2\Pi_\infty(-q, q) \sum_{m=0}^{\infty} \frac{q^{2m+1}\, \Pi_\infty(q^{2m+2}, q^2)}{\Pi_{2m+1}(-q, q)} \quad \text{(by E 1)}$$

$$= 2\Pi_\infty(-q, q^2)\, \Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \frac{q^{2m+1}\, \Pi_\infty(q^2, q^2)}{\Pi_{m+1}(-q, q^2)\, \Pi_m(-q^2, q^2)\, \Pi_m(q^2, q^2)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{2m+1}\, \Pi_\infty(-q^{2m+3}, q^2)}{\Pi_m(-q^4, q^4)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-)^n\, q^{n(n+2m+2)+2m+1}}{\Pi_m(-q^4, q^4)\, \Pi_n(-q^2, q^2)} \quad \text{(by E 1)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{n=0}^{\infty} \frac{(-)^n\, q^{(n+1)^2}}{\Pi_n(-q^2, q^2)\, \Pi_\infty(-q^{2n+2}, q^4)} \quad \text{(by E 2)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_{2m}(-q^2, q^2)\, \Pi_\infty(-q^{4m+2}, q^4)}$$

$$- 2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{(2m+2)^2}}{\Pi_{2m+1}(-q^2, q^2)\, \Pi_\infty(-q^{4m+4}, q^4)}$$

$$(n = 2m \quad \text{or} \quad n = 2m+1)$$

$$= \frac{2\Pi_\infty(-q^4, q^4)}{\Pi_\infty(-q^2, q^4)} \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}\, \Pi_m(-q^2, q^4)}{\Pi_{2m}(-q^2, q^2)} - 2 \sum_{m=0}^{\infty} \frac{q^{(2m+2)^2}\, \Pi_m(-q^4, q^4)}{\Pi_{2m+1}(-q^2, q^2)}$$

$$= 2\psi(q^2) \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_m(-q^4, q^4)} - 2 \sum_{m=0}^{\infty} \frac{q^{(2m+2)^2}}{\Pi_{m+1}(-q^2, q^4)} \quad \text{(by J 2)}$$

$$= 2q\psi(q^2)\, H(q^4) - 2[F_0(q^2) - 1],$$

and so we have proved formula (3_0).

To transform $f_1(q)$, we consider $f_1(q) \pm \vartheta_4(0, q)\, H(q)$; by analysis similar to that used in dealing with $f_0(q)$, we have

$$f_1(q) \pm \vartheta_4(0, q)\, H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{\Pi_n(q, q)} \pm \frac{\Pi_\infty(-q, q)}{\Pi_\infty(q, q)} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{\Pi_n(-q, q)}$$

$$= \Pi_\infty(-q, q) \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\{1 \pm (-)^p\}\, q^{(n+1)(n+p)}}{\Pi_n(-q^2, q^2)\, \Pi_p(-q, q)}.$$

We now take the lower signs, so that we may write $p = 2m+1$; we thus get

$$f_1(q) - \vartheta_4(0, q)\, H(q)$$

$$= 2\Pi_\infty(-q, q) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{(n+1)(n+2m+1)}}{\Pi_n(-q^2, q^2)\,\Pi_{2m+1}(-q, q)}$$

$$= 2\Pi_\infty(-q, q) \sum_{m=0}^{\infty} \frac{q^{2m+1}\,\Pi_\infty(q^{2m+3}, q^2)}{\Pi_{2m+1}(-q, q)} \quad \text{(by E 1)}$$

$$= 2\Pi_\infty(-q, q^2)\,\Pi_\infty(-q^2, q^2)$$

$$\times \sum_{m=0}^{\infty} \frac{q^{2m+1}\,\Pi_{m+1}(-q, q^2)\,\Pi_\infty(-q^{2m+3}, q^2)\,\Pi_\infty(q^{2m+3}, q^2)}{\Pi_\infty(-q, q^2)\,\Pi_{m+1}(-q, q^2)\,\Pi_m(-q^2, q^2)}$$

$$= 2\Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \frac{q^{2m+1}\,\Pi_\infty(-q^{4m+6}, q^4)}{\Pi_m(-q^2, q^2)}$$

$$= 2\Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-)^n q^{2n(n-1)+n(4m+6)+2m+1}}{\Pi_m(-q^2, q^2)\,\Pi_n(-q^4, q^4)} \quad \text{(by E 1)}$$

$$= 2\Pi_\infty(-q^2, q^2) \sum_{n=0}^{\infty} \frac{(-)^n q^{2n^2+4n+1}}{\Pi_n(-q^4, q^4)\,\Pi_\infty(-q^{4n+2}, q^2)} \quad \text{(by E 2)}$$

$$= 2 \sum_{n=0}^{\infty} \frac{(-)^n q^{2n^2+4n+1}\,\Pi_{2n}(-q^2, q^2)}{\Pi_n(-q^4, q^4)}$$

$$= 2 \sum_{n=0}^{\infty} (-)^n q^{2n^2+4n+1}\,\Pi_n(-q^2, q^4) = -2\phi_1(-q^2)/q,$$

and now formula (2_1) is established.

We next take the upper signs, so that we may write $p = 2m$; we thus get

$$f_1(q) + \vartheta_4(0, q)\, H(q)$$

$$= 2\Pi_\infty(-q, q) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{(n+1)(n+2m)}}{\Pi_n(-q^2, q^2)\,\Pi_{2m}(-q, q)}$$

$$= 2\Pi_\infty(-q, q) \sum_{m=0}^{\infty} \frac{q^{2m}\,\Pi_\infty(q^{2m+2}, q^2)}{\Pi_{2m}(-q, q)} \quad \text{(by E 1)}$$

$$= 2\Pi_\infty(-q, q^2)\,\Pi_\infty(-q^2, q^2) \sum_{m=0}^{\infty} \frac{q^{2m}\,\Pi_\infty(q^2, q^2)}{\Pi_m(-q, q^2)\,\Pi_m(-q^2, q^2)\,\Pi_m(q^2, q^2)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{2m}\,\Pi_\infty(-q^{2m+1}, q^2)}{\Pi_m(-q^4, q^4)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-)^n q^{n(n+2m)+2m}}{\Pi_m(-q^4, q^4)\,\Pi_n(-q^2, q^2)} \quad \text{(by E 1)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{n=0}^{\infty} \frac{(-)^n q^{n^2}}{\Pi_n(-q^2, q^2)\, \Pi_\infty(-q^{2m+2}, q^4)} \quad \text{(by E 2)}$$

$$= 2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{4m^2}}{\Pi_{2m}(-q^2, q^2)\, \Pi_\infty(-q^{4m+2}, q^4)}$$

$$-2\Pi_\infty(-q^4, q^4) \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_{2m+1}(-q^2, q^2)\, \Pi_\infty(-q^{4m+4}, q^4)}$$

$$(n = 2m \quad \text{or} \quad n = 2m+1)$$

$$= \frac{2\Pi_\infty(-q^4, q^4)}{\Pi_\infty(-q^2, q^4)} \sum_{m=0}^{\infty} \frac{q^{4m^2}\, \Pi_m(-q^2, q^4)}{\Pi_{2m}(-q^2, q^2)} - 2 \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}\, \Pi_m(-q^4, q^4)}{\Pi_{2m+1}(-q^2, q^2)}$$

$$= 2\psi(q^2) \sum_{m=0}^{\infty} \frac{q^{4m^2}}{\Pi_m(-q^4, q^4)} - 2 \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{\Pi_{m+1}(-q^2, q^4)} \quad \text{(by J 2)}$$

$$= 2\psi(q^2)\, G(q^4) - 2qF_1(q^2),$$

and so we have proved formula (3_1).

4. *Proofs of Ramanujan's formulae.*

Before proving the formulae (C_0), (D_0), (C_1), (D_1), we observe that we can deduce the formulae

(4_0) $$f_0(q) = \phi_0(-q^2) + q\psi(q^2)\, H(q^4) - [F_0(q^2) - 1],$$

(5_0) $$\vartheta_4(0, q)\, G(q) = \phi_0(-q^2) - q\psi(q^2)\, H(q^4) + [F_0(q^2) - 1],$$

(4_1) $$f_1(q) = -q^{-1}\phi_1(-q^2) + \psi(q^2)\, G(q^4) - qF_1(q^2),$$

(5_1) $$\vartheta_4(0, q)\, H(q) = q^{-1}\phi_1(-q^2) + \psi(q^2)\, G(q^4) - qF_1(q^2)$$

from formulae (2_0), (3_0), (2_1), (3_1) by addition and subtraction.

It now follows at once from (4_0), (1_0), (4_0), (5_0) respectively that each of the four members of (C_0) is equal to

$$\phi_0(-q^2) - q\psi(q^2)\, H(q^4) + [F_0(q^2) - 1].$$

That is to say, (C_0) is completely proved; and (D_0) is an immediate consequence of (1_0).

It also follows from (4_1), (1_1), (4_1), (5_1) respectively that each of the four members of (C_1) is equal to

$$q^{-1}\phi_1(-q^2) + \psi(q^2)\, G(q^4) - qF_1(q^2).$$

That is to say, (C_1) is completely proved; and (D_1) is an immediate consequence of (1_1).

5. *Miscellaneous relations connecting mock theta functions.*

We have now proved all the relations connecting mock theta functions of the fifth order whose existence was indicated by Ramanujan. There are, however, various other relations, which can be derived from the relations already obtained by using with known properties of the functions $G(q)$ and $H(q)$.

We first show how to obtain a form of the modular equation of the fifth order. Replace q by $-q$ in (5_0) and (5_1); it then becomes evident that

$$\vartheta_3(0, q)\, G(-q) - \vartheta_4(0, q)\, G(q) = 2q\psi(q^2)\, H(q^4), \tag{6}$$

$$\vartheta_3(0, q)\, H(-q) - \vartheta_4(0, q)\, H(q) = 2\psi(q^2)\, G(q^4). \tag{7}$$

These formulae appear to be new. If we substitute in them the values of $G(\pm q)$ and $H(\pm q)$ which are derivable from (R 5) and (R 6), we arrive in two different ways at the relation

$$\vartheta_3(0, q)\,\vartheta_4(0, q^5) - \vartheta_4(0, q)\,\vartheta_3(0, q^5) = 4q\psi(q^2)\, G(q^4)\, H(q^4)\, \Pi_\infty(-q^2, q^2),$$

which reduces to

$$\vartheta_3(0, q)\,\vartheta_4(0, q^5) - \vartheta_4(0, q)\,\vartheta_3(0, q^5) = 4q\, \Pi_\infty(-q^4, q^4)\, \Pi_\infty(-q^{20}, q^{20})$$

when $G(q)$ and $H(q)$ are expressed as reciprocals of products by (R 3) and (R 4). To reduce this result still further, write

$$\vartheta_3(0, q) = \vartheta_3(0, q^4) + 2q\psi(q^8), \quad \vartheta_4(0, q) = \vartheta_3(0, q^4) - 2q\psi(q^8);$$

and then replace q^4 by q; we thus get

$$\psi(q^2)\,\vartheta_3(0, q^5) - q\psi(q^{10})\,\vartheta_3(0, q) = \Pi_\infty(-q, q)\, \Pi_\infty(-q^5, q^5).$$

This version of the modular equation of the fifth order assumes a more familiar form when a change is made in the sign of q; it then becomes

$$\psi(q^2)\,\vartheta_4(0, q^5) + q\psi(q^{10})\,\vartheta_4(0, q)$$
$$= \Pi_\infty(q, q^2)\, \Pi_\infty(-q^2, q^2)\, \Pi_\infty(q^5, q^{10})\, \Pi_\infty(-q^{10}, q^{10}),$$

which, in the standard notation, is*

$$\sqrt{(kl')} + \sqrt{(k'l)} = \sqrt[12]{(256\, kk'll')}.$$

It may not be out of place to remark that it is possible to establish this modular equation by means of transformations of q-series less elaborate than the transformations which have been used here.

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* R. Russell, *Proc. London Math. Soc.* (1), 21 (1891), 351–395 (379).

Equations (6) and (7) and the modular equation which has just been derived from them exhaust the potentialities of Ramanujan's formulae for giving results independent of mock theta functions.

The procedure of replacing q by $-q$ in (5_0) and (5_1), which has already been used to obtain (6) and (7), also leads to the equations

$$(8_0) \qquad \phi_0(-q^2)+F_0(q^2)-1=\tfrac{1}{2}[\vartheta_3(0,\,q)\,G(-q)+\vartheta_4(0,\,q)\,G(q)],$$

$$(8_1) \qquad \phi_1(-q^2)-q^2\,F_1(q^2)=\tfrac{1}{2}q[\vartheta_3\,(0,\,q)\,H(-q)-\vartheta_4(0,\,q)\,H(q)].$$

We now apply formulae (R 5) (R 7) to exhibit the expressions on the right of these equations as rational functions of q^2. We have, in fact,

$$\begin{aligned}
&\tfrac{1}{2}[\vartheta_3(0,\,q)\,G(-q)+\vartheta_4(0,\,q)\,G(q)]\\
&\quad=[\{\vartheta_3(0,\,q^4)+2q\psi(q^8)\}\{\vartheta_3(0,\,q^4)-2q\psi(q^8)+\vartheta_3(0,\,q^{20})-2q^5\,\psi(q^{40})\}\\
&\qquad\quad+\{\vartheta_3(0,\,q^4)-2q\psi(q^8)\}\{\vartheta_3(0,\,q^4)+2q\psi(q^8)+\vartheta_3(0,\,q^{20})+2q^5\,\psi(q^{40})\}]\\
&\qquad\quad\div[4G(q^4)\,\Pi_\infty(-q^2,\,q^2)]\\
&\quad=[\vartheta_3(0,\,q^4)\{\vartheta_3(0,\,q^4)+\vartheta_3\,(0,\,q^{20})\}-4q^2\,\psi(q^8)\{\psi(q^8)+q^4\,\psi(q^{40})\}]\\
&\qquad\quad\div[2G(q^4)\,\Pi_\infty(-q^2,\,q^2)]\\
&\quad=\frac{2\vartheta_3(0,\,q^4)\,G(q^4)\,G(q^{16})\,\Pi\,(-q^8,\,q^8)-4q^2\,\psi(q^8)\,G(q^4)\,H(-q^4)\,\Pi\,(-q^8,\,q^8)}{2G(q^4)\,\Pi_\infty(-q^2,\,q^2)}\\
&\quad=\frac{\Pi_\infty(-q^8,\,q^8)}{\Pi_\infty(-q^2,\,q^2)}\,[\vartheta_3(0,\,q^4)\,G(q^{16})-2q^2\,\psi(q^8)\,H(-q^4)].
\end{aligned}$$

In like manner, we have

$$\begin{aligned}
&\tfrac{1}{2}q[\vartheta_3(0,\,q)\,H(-q)-\vartheta_4(0,\,q)\,H(q)]\\
&\quad=[\{\vartheta_3(0,\,q^4)+2q\psi(q^8)\}\{-\vartheta_3(0,\,q^4)+2q\psi(q^8)+\vartheta_3(0,\,q^{20})-2q^5\,\psi(q^{40})\}\\
&\qquad\quad-\{\vartheta_3(0,\,q^4)-2q\psi(q^8)\}\{\vartheta_3(0,\,q^4)+2q\psi(q^8)-\vartheta_3(0,\,q^{20}-2q^5\,\psi(q^{40})\}]\\
&\qquad\quad\div[4H(q^4)\,\Pi_\infty(-q^2,\,q^2)]\\
&\quad=[-\vartheta_3(0,\,q^4)\{\vartheta_3(0,\,q^4)-\vartheta_3(0,\,q^{20})\}+4q^2\,\psi(q^8)\{\psi(q^8)-q^4\,\psi(q^{40})\}]\\
&\qquad\quad\div[2H(q^4)\,\Pi_\infty(-q^2,\,q^2)]\\
&\quad=\frac{\begin{array}{l}-2q^4\,\vartheta_3(0,\,q^4)\,H(q^4)\,H(q^{16})\,\Pi_\infty(-q^8,\,q^8)\\ \qquad\qquad\qquad+4q^2\,\psi(q^8)\,G(-q^4)\,H(q^4)\,\Pi_\infty(-q^8,\,q^8)\end{array}}{2H(q^4)\,\Pi_\infty(-q^2,\,q^2)}\\
&\quad=\frac{\Pi_\infty(-q^8,\,q^8)}{\Pi_\infty(-q^2,\,q^2)}\,[2q^2\,\psi(q^8)\,G(-q^4)-q^4\,\vartheta_3(0,\,q^4)\,H(q^{16})].
\end{aligned}$$

We have thus attained the results

$$(9_0)\quad \phi_0(q^2)+F_0(q^2)-1=\frac{\Pi_\infty(-q^8,\,q^8)}{\Pi_\infty(-q^2,\,q^2)}\,[\vartheta_3(0,\,q^4)\,G(q^{16})-2q^2\,\psi(q^8)\,H(-q^4)],$$

$$(9_1)\quad \phi_1(-q^2)-q^2\,F_1(q^2)$$

$$=-\frac{\Pi_\infty(-q^8,\,q^8)}{\Pi_\infty(-q^2,\,q^2)}\,[2q^2\,\psi(q^8)\,G(-q^4)-q^4\,\vartheta_3(0,\,q^4)\,H(q^{16})].$$

If, as seems likely to have been the case, Ramanujan made an examination of the behaviour of the mock theta functions of the fifth order near the "rational points" of the unit circle, he cannot have failed to observe that each of the seven functions $\psi_0(\pm q)$, $f_0(\pm q)$, $\phi_0(-q^2)$, $F_0(q^2)$, $\chi_0(q^2)$ is expressible in terms of any other one of the seven and of more elementary functions; and that the seven functions $\psi_1(\pm q)$, $f_1(\pm q)$, $\phi_1(-q^2)$, $F_1(q^2)$, $\chi_1(q^2)$ enjoy a like property. To prove these assertions, it is evidently sufficient to express six members of each set in terms of the seventh member of that set and more elementary functions; we shall take as the seventh members the functions $\chi_0(q^2)$ and $\chi_1(q^2)$. The results look slightly neater if equations (8_0) and (8_1) are used, rather than (9_0) and (9_1). Combining the former equations with (B_0) and (B_1), we have

$$(10_0)\qquad 3F_0(q^2)=1+\chi_0(q^2)+\tfrac{1}{2}[\vartheta_3(0,\,q)\,G(-q)+\vartheta_4(0,\,q)\,G(q)],$$

$$(11_0)\qquad 3\phi_0(-q^2)=2-\chi_0(q^2)+[\vartheta_3(0,\,q)\,G(-q)+\vartheta_4(0,\,q)\,G(q)],$$

$$(10_1)\qquad 3F_1(q^2)=\chi_1(q^2)+\tfrac{1}{2}q^{-1}[\vartheta_3(0,\,q)\,H(-q)-\vartheta_4(0,\,q)\,H(q)],$$

$$(11_1)\qquad 3\phi_1(-q^2)=q^2\chi_1(q^2)-q[\vartheta_3(0,\,q)\,H(-q)-\vartheta_4(0,\,q)\,H(q)].$$

The results of substituting these formulae into (1_0), (1_1), (4_0), (4_1) are as follows:

$$(12_0)\quad 3\psi_0(\pm q)=\chi_0(q^2)-2+\tfrac{1}{2}[\vartheta_3(0,\,q)\,G(-q)+\vartheta_4(0,\,q)\,G(q)]$$
$$\pm 3q\psi(q^2)\,H(q^4),$$

$$(12_1)\quad 3\psi_1(\pm q)=3\psi(q^2)\,G(q^4)\pm q\chi_1(q^2)\pm\tfrac{1}{2}[\vartheta_3(0,\,q)\,H(-q)-\vartheta_4(0,\,q)\,H(q)],$$

$$(13_0)\quad 3f_0(\pm q)=4-2\chi_0(q^2)+\tfrac{1}{2}[\vartheta_3(0,\,q)\,G(-q)+\vartheta_4(0,\,q)\,G(q)]$$
$$\pm 3q\psi(q^2)\,H(q^4),$$

$$(13_1)\quad 3f_1(\pm q)=3\psi(q^2)\,G(q^4)\mp 2q\chi_1(q^2)\pm\tfrac{1}{2}[\vartheta_3(0,\,q)\,H(-q)-\vartheta_4(0,\,q)\,H(q)].$$

These eight formulae embody the twelve required relations.

6. *The behaviour of mock theta functions in the neighbourhood of the unit circle.*

It appears from Ramanujan's letter that the property of a mock theta function which he regarded as characteristic is that, corresponding to each "rational point" $q = e^{M\pi i/N}$ (with M and N integers) of the unit circle $|q| = 1$, there exists a theta function* of q whose difference from the mock theta function is bounded when q approaches this rational point along a radius of the circle.

We proceed to prove that the mock theta functions of the fifth order possess this characteristic property; and we give brief indications as to how the appropriate theta functions are to be determined.

The rational numbers M/N, when expressed as fractions in their lowest terms with N positive, are divisible into three categories, according to the parities of M and N, thus:

(i) M even and N odd,

(ii) M and N both odd,

(iii) M odd and N even.

The corresponding values of $e^{M\pi i/N}$ will be described as points of the first (second or third) categories on the unit circle; and, if $q = \rho e^{M\pi i/N}$ with $0 \leqslant \rho \leqslant 1$, we shall say that q approaches the circle along a radius of the first (second or third) category when $\rho \to 1$.

We shall now prove the following theorems.

For approach to $|q| = 1$ *along a radius of the first category,*

$$f_0(q) = O(1), \quad f_1(q) = O(1).$$

For approach to $|q| = 1$ *along a radius of the second category,*

$$f_0(-q) = O(1), \quad f_1(-q) = O(1).$$

For approach to $|q| = 1$ *along a radius of the third category,*

$$\psi_0(q) = O(1), \quad \psi_1(q) = O(1).$$

It is to be noticed that we cannot obtain these results by mere substitution of $e^{M\pi i/N}$ into the corresponding series. *E.g.*, the fact that

$$f_0(1) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

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* Ramanujan used the term "theta function" in a somewhat wide sense; he included, for instance, such functions as $\Pi_\alpha(q, q^2)$ in addition to those functions to which the term is usually applied.

is not in itself a sufficient condition for $f_0(q)$ to be bounded as $q \to 1$. Actually, however, results of this kind indicate which functions should be investigated for boundedness along radii of the various categories.

First take $q = \rho e^{M\pi i/N}$ with M even and N odd. Then, evidently,

$$f_0(q) = \sum_{m=0}^{N-1} \sum_{n=0}^{\infty} \frac{q^{(nN+m)^2}}{\Pi_{nN+m}(q, q)} = \sum_{m=0}^{N-1} \sum_{n=0}^{\infty} u_{n,m},$$

say, so that

$$\frac{u_{n,m}}{u_{n+1,m}} = q^{-N(2nN+N+2m)} \Pi_N(q^{nN+m+1}, q).$$

Now

$$\begin{aligned} |\Pi_N(q^{nN+m+1}), q)| &= \prod_{p=1}^{N} |(1+q^{nN+m+p})| \\ &= \prod_{p=1}^{N} \left(1 + 2\rho^{nN+m+p} \cos \frac{(m+p)\,M\pi}{N} + \rho^{2nN+2m+2p}\right)^{\frac{1}{2}} \\ &\geqslant \prod_{p=1}^{N} \left[\rho^{nN+m+p} \left(2 + 2\cos \frac{(m+p)\,M\pi}{N}\right)\right]^{\frac{1}{2}} \\ &= \rho^{\frac{1}{2}N(2nN+N+2m)} \prod_{p=1}^{N} |(1+e^{(m+p)M\pi i/N})|. \end{aligned}$$

It is easy to see that, as p runs through the values 1, 2, 3, ..., N, the points $e^{(m+p)M\pi i/N}$ assume the positions 1, $e^{2\pi i/N}$, $e^{4\pi i/N}$, ..., $e^{2(N-1)\pi i/N}$ in some order. Hence

$$\prod_{p=1}^{N} |(1+e^{(m+p)M\pi i/N})| = \prod_{p=0}^{N-1} |(1+e^{2p\pi i/N})| = 2;$$

we thus have

$$\left|\frac{u_{n,m}}{u_{n+1,m}}\right| \geqslant 2\rho^{-\frac{1}{2}N(2nN+N+2m)} \geqslant 2,$$

and so

$$|f_0(q)| \leqslant \sum_{m=0}^{N-1} \sum_{n=0}^{\infty} \left|\frac{u_{0,m}}{2^n}\right| = 2 \sum_{m=0}^{N-1} |u_{0,m}| = 2 \sum_{m=0}^{N-1} \left|\frac{q^{m^2}}{\Pi_m(q, q)}\right|.$$

Each term of the last finite sum is obviously bounded when ρ lies in the closed interval $0 \leqslant \rho \leqslant 1$; and so $f_0(q)$ is bounded when q lies on a radius of the first category.

Since the terms of the series defining $f_1(q)$ do not exceed in absolute value the corresponding terms of the series defining $f_0(q)$, it follows *a fortiori* that $f_1(q)$ is bounded when q lies on a radius of the first category.

It is now very easy to dispose of radii of the second category. When q lies on a radius of the second category, it is evident that $-q$ lies on a radius of the first category. Hence, by the preceding results, for radial approach to points of the second category on the unit circle we have

$$f_0(-q) = O(1), \quad f_1(-q) = O(1).$$

The radii of the third category need rather different treatment. As previously, we write $q = \rho e^{M\pi i/N}$, but now M is odd and N even. We consider $\psi_1(q)$ rather than $\psi_0(q)$, because the terms of the series for $\psi_0(q)$ do not exceed in absolute value the corresponding terms of the series for $\psi_1(q)$.

We evidently have

$$\psi_1(q) = \sum_{m=0}^{2N-1} \sum_{n=0}^{\infty} q^{(2Nn+m)(2Nn+m+1)} \Pi_{2Nn+m}(q, q) = \sum_{m=0}^{2N-1} \sum_{n=0}^{\infty} v_{n,m},$$

say, so that

$$\frac{v_{n+1,m}}{v_{n,m}} = q^{2N(4Nn+2N+2m+1)} \Pi_{2N}(q^{2Nn+m+1}, q).$$

Now

$$|\Pi_{2N}(q^{2Nn+m+1}, q)| = \prod_{p=1}^{2N} |(1+q^{2Nn+m+p})|$$

$$= \prod_{p=1}^{2N} \left(1+2\rho^{2Nn+m+p} \cos\frac{(m+p)M\pi}{N} + \rho^{4Nn+2m+2p}\right)^{\frac{1}{2}}.$$

Since, when $\beta \leqslant \alpha \leqslant 1$, we have

$$\frac{1+2\alpha\cos\theta+\alpha^2}{\alpha} \leqslant \frac{1+2\beta\cos\theta+\beta^2}{\beta},$$

we find immediately that

$$|\Pi_{2N}(q^{2Nn+m+1}, q)|$$

$$\leqslant \prod_{p=1}^{2N} \left[\rho^{p-2N}\left(1+2\rho^{2Nn+2N+m} \cos\frac{(m+p)M\pi}{N} + \rho^{4Nn+4N+2m}\right)\right]^{\frac{1}{2}}$$

$$= \rho^{-\frac{1}{2}N(2N-1)} \prod_{p=1}^{2N} |(1+\rho^{2Nn+2N+m} e^{(m+p)M\pi i/N})|.$$

It is easy to see that, as p runs through the values 1, 2, 3, ..., $2N$, the points $e^{(m+p)M\pi i/N}$ assume the positions 1, $e^{\pi i/N}$, $e^{2\pi i/N}$, ..., $e^{(2N-1)\pi i/N}$ in some order. Hence

$$\prod_{p=1}^{2N} |(1+\rho^{2Nn+2N+m} e^{(m+p)M\pi i/N})| = \prod_{p=0}^{2N-1} |(1+\rho^{2Nn+2N+m} e^{p\pi i/N})|$$

$$= 1-\rho^{2N(2Nn+2N+m)}.$$

On collecting our results, we deduce that

$$\left|\frac{v_{n+1,\,m}}{v_{n,\,m}}\right| \leqslant \rho^{2N(4Nn+2N+2m+1)-\frac{1}{2}N(2N-1)}\left(1-\rho^{2N(2Nn+2N+m)}\right),$$

and so, always,
$$|v_{n+1,\,m}| \leqslant |v_{n,\,m}|,$$
while, when $n \geqslant 1$,

$$\left|\frac{v_{n+1,\,m}}{v_{n,\,m}}\right| \leqslant \rho^{2N(2Nn+2N+m)}\left(1-\rho^{2N(2Nn+2N+m)}\right) \leqslant \tfrac{1}{4}.$$

Therefore

$$|\psi_1(q)| \leqslant \sum_{m=0}^{2N-1}\left[|v_{0,\,m}| + \sum_{n=1}^{\infty}\frac{|v_{0,\,m}|}{4^{n-1}}\right] = \tfrac{7}{3}\sum_{m=0}^{2N-1}\left|q^{m(m+1)}\,\Pi_m(q,\,q)\right|.$$

Each term of the last finite sum is obviously bounded when ρ lies in the closed interval $0 \leqslant \rho \leqslant 1$; consequently $\psi_1(q)$ and, *a fortiori*, $\psi_0(q)$ are bounded when q lies on a radius of the third category. We have therefore established the three theorems enunciated at the beginning of this section.

With the help of the formulae (10_0)–(13_1) it is now possible to determine an approximation to any one of the ten mock theta functions of the fifth order as q makes a radial approach to any "rational" point of the unit circle. Each approximation takes the form of a combination of theta functions (in the wide sense) together with an error-term which remains bounded when $|q| \to 1$.

7. *Complete series derived from mock theta functions.*

It is evident from the definition of $f_0(q)$ that

$$f_0(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}\,\Pi_\infty(q^{n+1},\,q)}{\Pi_\infty(q,\,q)}.$$

Now the general term of this series, when written in this form, exists for negative as well as for positive values of n; and this suggests the study of the complete series

$$\sum_{n=-\infty}^{\infty}\frac{q^{n^2}\,\Pi_\infty(q^{n+1},\,q)}{\Pi_\infty(q,\,q)}.$$

Obviously,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{q^{n^2}\,\Pi_\infty(q^{n+1},\,q)}{\Pi_\infty(q,\,q)} &= f_0(q) + \sum_{n=1}^{\infty}\frac{q^{n^2}\,\Pi_\infty(q^{1-n},\,q)}{\Pi_\infty(q,\,q)}\\ &= f_0(q)+2q+2(1+q^{-1})\,q^4+2(1+q^{-1})(1+q^{-2})\,q^9+\dots\\ &= f_0(q)+2[q+q^3(1+q)+q^6(1+q)(1+q^2)+\dots]\\ &= f_0(q)+2\psi_0(q).\end{aligned}$$

We thus have

$$(14_0) \quad \sum_{n=-\infty}^{\infty} \frac{q^{n^2}\Pi_\infty(q^{n+1}, q)}{\Pi_\infty(q, q)} = f_0(q) + 2\psi_0(q)$$

$$= \phi_0(-q^2) + [F_0(q^2) - 1] + 3q\psi(q^2)H(q^4);$$

and, in like manner,

$$(15_0) \quad \sum_{n=-\infty}^{\infty} \frac{q^{4n^2}\Pi_\infty(-q^{4n+2}, q^4)}{\Pi_\infty(-q^2, q^4)} = \phi_0(-q^2) + [F_0(q^2) - 1],$$

$$(14_1) \quad \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}\Pi_\infty(q^{n+1}, q)}{\Pi_\infty(q, q)} = f_1(q) + 2\psi_1(q)$$

$$= 3\psi(q^2)G(q^4) + qF_1(q^2) - q^{-1}\phi_1(-q^2),$$

$$(15_1) \quad \sum_{n=-\infty}^{\infty} \frac{q^{4n(n+1)}\Pi_\infty(-q^{4n+6}, q^4)}{\Pi_\infty(-q^2, q^4)} = F_1(q^2) - q^{-2}\phi_1(-q^2).$$

It follows from the results of § 5 that the four complete series are expressible in terms of theta functions (in the wide sense). The most curious feature of the complete series, however, is the phenomenon that, when we take the constituent parts, $f_0(q)$ and $2\psi_0(q)$, of the first complete series, the odd parts of the constituents are respectively $q\psi(q^2)H(q^4)$ and $2q\psi(q^2)H(q^4)$; that is to say, not only are these odd parts expressible in terms of theta functions, but the latter is exactly double of the former. In like manner, the constituents of the third complete series, $f_1(q)$ and $2\psi_1(q)$, have for their even parts $\psi(q^2)G(q^4)$ and $2\psi(q^2)G(q^4)$. This duplicity of the constituents of the two complete series in question caused me to spend a little time in barking up the wrong tree when I was endeavouring to construct proofs of Ramanujan's relations.

8. *Lerch's transcendant.*

The results obtained in § 7, concerning the expression of complete series in terms of theta functions, naturally suggest the advisability of studying Lerch's* more general transcendant

$$(16) \quad f(x, \xi;\, q, \mathfrak{q}) \equiv 1 + \sum_{n=1}^{\infty} q^{n^2} x^{2n} (1+\mathfrak{q}\xi^2)(1+\mathfrak{q}^3\xi^2)\dots(1+\mathfrak{q}^{2n-1}\xi^2)$$

$$+ \sum_{n=1}^{\infty} \frac{q^{n^2} x^{-2n}}{(1+\mathfrak{q}^{-1}\xi^2)(1+\mathfrak{q}^{-3}\xi^2)\dots(1+\mathfrak{q}^{1-2n}\xi^2)}.$$

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* *Rozpravy České Akademie cisaře Františka Josefa,* 3 (1893), Čislo 5.

The complete series are obviously special cases of this function, and Lerch's work does not raise any suspicions that there are any special cases of his transcendant which are completely expressible in terms of theta functions. The most general result which I have obtained is that $f(x, \xi;\ q, \mathfrak{q})$ is expressible in terms of theta functions and more elementary functions when $\mathfrak{q} = q$ and x is of the form ξq^r, where r is any integer.

Since copies of Lerch's paper are hard to come by*, I give an epitome of the relevant parts of his work, for the sake of completeness, abbreviating his proofs whenever possible.

Write

$$(17) \qquad \phi(x, \xi;\ q, \mathfrak{q}) = \prod_{m=1}^{\infty} (1+\mathfrak{q}^{2m-1}\xi^{-2}) f(x, \xi;\ q, \mathfrak{q}),$$

so that

$$(18) \qquad \phi(x, \xi;\ q, \mathfrak{q}) = \sum_{n=-\infty}^{\infty} q^{n^2} \mathfrak{q}^{n^2} x^{2n} \xi^{2n} \Pi_\infty(\mathfrak{q}^{1-2n}\xi^{-2}, \mathfrak{q}^2).$$

Hence, by (E 1), we have

$$(19) \qquad \phi(x, \xi;\ q, \mathfrak{q}) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{q^{n^2} \mathfrak{q}^{(n-m)^2} x^{2n} \xi^{2n-2m}}{\Pi_m(-\mathfrak{q}^2, \mathfrak{q}^2)}.$$

Now consider $\phi(e^{i(z+\alpha)}, e^{i(z+\beta)};\ q, q)$, *qua* function of z; calling it $\Phi(z)$, we see at a glance that

$$\Phi(z+\pi) = \Phi(z);$$

also, with the standard notation $q = e^{\pi i \tau}$, we have

$$\begin{aligned} \Phi(z+\pi\tau) &= \sum_{n=-\infty}^{\infty} q^{2n^2+4n} e^{2ni(2z+\alpha+\beta)} \Pi_\infty(q^{-2n-1} e^{-2i(z+\beta)}, q^2) \\ &= \sum_{m=-\infty}^{\infty} q^{2m^2-2} e^{2(m-1)i(2z+\alpha+\beta)} \Pi_\infty(q^{1-2m} e^{-2i(z+\beta)}, q^2) \qquad (m = n+1) \\ &= q^{-2} e^{-2i(2z+\alpha+\beta)} \Phi(z). \end{aligned}$$

Consequently $\Phi(z)$ is quasi-periodic in the periods π and $\pi\tau$, its periodicity factors being 1 and $q^{-2} e^{-2i(2z+\alpha+\beta)}$; and these are exactly the periodicity factors which are possessed by each of the functions†

$$\vartheta_1{}^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, q), \quad \vartheta_2{}^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, q).$$

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* I must express my gratitude to Prof. E. Čech for his generosity in giving me separata of this and of many other papers by Lerch.

† I use here Jacobi's notation for theta functions; Lerch used the Tannery-Molk modified notation for which the periods are 1 and τ.

Consider, therefore, the function

$$\frac{\vartheta_2^2(0,\,q)\,\Phi(z)-\vartheta_1^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,\,q)\,\Phi(\tfrac{1}{2}\pi-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta)-\vartheta_2^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,\,q)\,\Phi(-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta)}{\vartheta_1^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,\,q)}.$$

Qua function of z, it is doubly periodic, with periods π and $\pi\tau$; and, since $\Phi(z)$ is an integral function of z, its only possible singularities are simple poles at the zeros of $\vartheta_1(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,\,q)$; that is to say, it is an elliptic function whose only singularities are, at most, one simple pole in any cell. By a well-known theorem[*], such a function must be a constant; and, by taking $z=\tfrac{1}{2}\pi-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta$, we see that the constant value of the function under consideration is zero.

We have therefore attained Lerch's result that

$$\vartheta_2^2(0,\,q)\,\Phi(z)$$
$$=\vartheta_1^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,\,q)\,\Phi(\tfrac{1}{2}\pi-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta)+\vartheta_2^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,q)\,\Phi(-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta).$$

This result, when written in full, assumes the form

$$(20)\quad \vartheta_2^2(0,\,q)\,\phi(e^{i(z+\alpha)},\,e^{i(z+\beta)};\;q,\,q)$$
$$=\vartheta_1^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,\,q)\,\phi(ie^{\frac{1}{2}i(\alpha-\beta)},\,ie^{\frac{1}{2}i(\beta-\alpha)};\;q,\,q)$$
$$+\vartheta_2^2(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta,\,q)\,\phi(e^{\frac{1}{2}i(\alpha-\beta)},\,e^{\frac{1}{2}i(\beta-\alpha)};\;q,\,q).$$

Both of Lerch's transcendants which appear on the right in this formula are of the form $\phi(x,\,\pm x^{-1};\;q,\,q)$; we now show how to simplify them by means of (19). We have

$$\phi(x,\,\pm x^{-1};\;q,\,q)$$
$$=\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{q^{n^2+(n-m)^2}x^{2m}}{\Pi_m(-q^2,\,q^2)}$$
$$=\sum_{m=0}^{\infty}\frac{q^{m^2}x^{2m}\vartheta_3(m\pi\tau,\,q^2)}{\Pi_m(-q^2,\,q^2)}$$
$$=\sum_{n=0}^{\infty}\frac{q^{4n^2}x^{4n}\vartheta_3(2n\pi\tau,\,q^2)}{\Pi_{2n}(-q^2,\,q^2)}+\sum_{n=0}^{\infty}\frac{q^{(2n+1)^2}x^{4n+2}\vartheta_3(2n\pi\tau+\pi\tau,\,q^2)}{\Pi_{2n+1}(-q^2,\,q^2)}$$
$$=\sum_{n=0}^{\infty}\frac{q^{2n^2}x^{4n}\vartheta_3(0,\,q^2)}{\Pi_{2n}(-q^2,\,q^2)}+\sum_{n=0}^{\infty}\frac{q^{2n(n+1)+\frac{1}{2}}x^{4n+2}\vartheta_2(0,\,q^2)}{\Pi_{2n+1}(-q^2,\,q^2)}.$$

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[*] See, *e.g.*, Whittaker and Watson, *Modern analysis* (1927), 432.

This result may be written in the form

$$(21) \qquad \phi(x, \pm x^{-1};\ q, q) = \vartheta_3(0, q^2)\,\Psi_0(x, q) + \vartheta_2(0, q^2)\,\Psi_1(x, q),$$

where

$$(22) \quad \Psi_0(x, q) = \sum_{n=0}^{\infty} \frac{q^{2n^2} x^{4n}}{\Pi_{2n}(-q^2, q^2)}, \qquad \Psi_1(x, q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+\frac{1}{2}} x^{4n+2}}{\Pi_{2n+1}(-q^2, q^2)}.$$

Substituting these results into (20), and using the formulae

$$\Psi_0(ix, q) = \Psi_0(x, q), \quad \Psi_1(ix, q) = -\Psi_1(x, q)$$

(the truth of which is obvious), we get, without difficulty,

$$\begin{aligned}(23) \quad & \vartheta_2^{\,2}(0, q)\,\phi(e^{i(z+\alpha)}, e^{i(z+\beta)};\ q, q) \\ & = [\vartheta_3(0, q^2)\,\Psi_0(e^{\frac{1}{2}i(\alpha-\beta)}, q) - \vartheta_2(0, q^2)\,\Psi_1(e^{\frac{1}{2}i(\alpha-\beta)}, q)]\,\vartheta_1^{\,2}(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, q) \\ & \quad + [\vartheta_3(0, q^2)\,\Psi_0(e^{\frac{1}{2}i(\alpha-\beta)}, q) + \vartheta_2(0, q^2)\,\Psi_1(e^{\frac{1}{2}i(\alpha-\beta)}, q)]\,\vartheta_2^{\,2}(z+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, q),\end{aligned}$$

which is Lerch's final result. The point of this result, as Lerch remarked, is that it expresses $\phi(e^{i(z+\alpha)}, e^{i(z+\beta)};\ q, q)$ in terms of theta functions and constants when z is regarded as the only variable.

Independent though they may be of z, the functions of the type $\Psi_0(x, q)$ and $\Psi_1(x, q)$ which occur in Lerch's result are functions of x and q which are not without interest. So far as I know, these functions have never occurred elsewhere in analysis. It seems likely that, when x is unrestricted, they are not reducible in any way. The results of the earlier part of this paper, however, make it possible to prove that, whenever x is of the form $q^{\frac{1}{2}r}$ (with r an integer), both of the functions $\Psi_0(x, q)$ and $\Psi_1(x, q)$ are expressible in terms of theta functions and elementary functions. We now proceed to investigate this somewhat surprising result.

9. *Reduction of Lerch's special transcendants.*

It is easy to see that the four complete series which were introduced in §7 are expressible in terms of Lerch's transcendant $f(x, \xi;\ q, q)$ by the following formulae:

$$(24_0) \qquad f_0(q^2) + 2\psi_0(q^2) = f(q^{\frac{1}{2}}, q^{-\frac{1}{2}};\ q, q),$$

$$(25_0) \qquad F_0(q) - 1 + \phi_0(-q) = f(i, i;\ q, q),$$

$$(24_1) \qquad f_1(q^2) + 2\psi_1(q^2) = 2f(q^{\frac{1}{2}}, q^{\frac{1}{2}};\ q, q),$$

$$(25_1) \qquad F_1(q) - q^{-1}\phi_1(-q) = f(iq, i;\ q, q).$$

By using formula (23) to express the functions on the right in terms of functions of the types Ψ_0 and Ψ_1, we obtain equations to determine combinations of such functions in terms of the functions on the left, and hence (by § 7) in terms of theta functions.

First take formula (25_0); this gives

$$\Pi_\infty(-q, q^2)[F_0(q)-1+\phi_0(-q)] = \phi(i, i;\ q, q)$$
$$= \vartheta_3(0, q^2)\Psi_0(1, q)-\vartheta_2(0, q^2)\Psi_1(1, q),$$

and hence, by (9_0),

$$\vartheta_3(0, q^2)\Psi_0(1, q)-\vartheta_2(0, q^2)\Psi_1(1, q)$$
$$= \Pi_\infty(q^2, q^2)[\vartheta_3(0, q^2)\,G(q^8)-\vartheta_2(0, q^2)\,q^{\frac{1}{2}}H(-q^2)].$$

Compare this result with the result obtained from it by replacing q by $-q$; it is then evident that

$$(26) \qquad \Psi_0(1, q) = \Pi_\infty(q^2, q^2)\,G(q^8), \quad \Psi_1(1, q) = \Pi_\infty(q^2, q^2)\,H(-q^2).$$

We get no more than this from (24_1); for this equation gives

$$\vartheta_2{}^2(0, q)\,\Pi_\infty(q^2, q^2)[f_1(q^2)+2\psi_1(q^2)]$$
$$= \vartheta_2{}^2(0, q)\,\phi(q^{\frac{1}{2}}, q^{\frac{1}{2}};\ q, q)$$
$$= [\vartheta_3(0, q^2)\Psi_0(1, q)-\vartheta_2(0, q^2)\Psi_1(1, q)]\,\vartheta_1{}^2(\tfrac{1}{2}\pi\tau, q)$$
$$+[\vartheta_3(0, q^2)\Psi_0(1, q)+\vartheta_2(0, q^2)\Psi_1(1, q)]\,\vartheta_2{}^2(\tfrac{1}{2}\pi\tau, q).$$

Observing that, by (1_1), (4_1), and (5_1), we have

$$f_1(q)+2\psi_1(q) = 3\psi(q^2)\,G(q^4)-q^{-1}\phi_1(-q^2)+qF_1(q^2)$$
$$= 2\psi(q^2)\,G(q^4)+\vartheta_3(0, q)\,H(-q),$$

we can reduce the last result to

$$4q\psi^2(q^2)\,\Pi_\infty(q^2, q^2)[2\psi(q^4)\,G(q^8)+\vartheta_3(0, q^2)\,H(-q^2)]$$
$$= \vartheta_3(0, q^2)[\vartheta_3{}^2(0, q)-\vartheta_4{}^2(0, q)]\,\Psi_0(1, q)$$
$$+\vartheta_2(0, q^2)[\vartheta_3{}^2(0, q)+\vartheta_4{}^2(0, q)]\,\Psi_1(1, q).$$

This equation is compatible with the two results contained in (26); for it follows from elementary properties of theta functions that

$$8q\psi^2(q^2)\,\psi(q^4) = \vartheta_3(0, q^2)\,[\vartheta_3{}^2(0, q)-\vartheta_4{}^2(0, q)],$$
$$4q^{\frac{1}{2}}\,\psi^2(q^2)\,\vartheta_3(0, q^2) = \vartheta_2(0, q^2)\,[\vartheta_3{}^2(0, q)+\vartheta_4{}^2(0, q)],$$

both of these equations being immediate consequences of the quadratic transformation of theta functions*.

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* *Cf.* J. Tannery et J. Molk, *Théorie des fonctions elliptiques*, 2 (1896), 268, formulae XLVII (4).

Next, taking (24_0), we have

$$\Pi_\infty(q^2, q^2)[f_0(q^2)+2\psi_0(q^2)] = \phi(q^{\frac12}, q^{-\frac12};\ q, q)$$
$$= \vartheta_3(0, q^2)\,\Psi_0(q^{\frac12}, q)+\vartheta_2(0, q^2)\,\Psi_1(q^{\frac12}, q).$$

Now, by (1_0), (4_0), and (5_0), we have

$$f_0(q)+2\psi_0(q) = \phi_0(-q^2)+F_0(q^2)-1+3q\psi(q^2)\,H(q^4)$$
$$= \vartheta_3(0, q)\,G(-q)+q^{\frac34}\vartheta_2(0, q)\,H(q^4),$$

and so we can reduce the last result to

$$(27)\quad \Pi_\infty(q^2, q^2)[\vartheta_3(0, q^2)\,G(-q^2)+q^{\frac32}\vartheta_2(0, q^2)\,H(q^8)]$$
$$= \vartheta_3(0, q^2)\,\Psi_0(q^{\frac12}, q)+\vartheta_2(0, q^2)\,\Psi_1(q^{\frac12}, q).$$

We leave equation (27) for the moment, and consider (25_1); this gives

$$\vartheta_2{}^2(0, q)\,\Pi_\infty(-q, q^2)[F_1(q)-q^{-1}\phi_1(-q)]$$
$$= \phi(iq, i;\ q, q)$$
$$= [\vartheta_3(0, q^2)\,\Psi_0(q^{\frac12}, q)-\vartheta_2(0, q^2)\,\Psi_1(q^{\frac12}, q)]\,\vartheta_1{}^2(\tfrac12\pi+\tfrac12\pi\tau, q)$$
$$+[\vartheta_3(0, q^2)\,\Psi_0(q^{\frac12}, q)+\vartheta_2(0, q^2)\,\Psi_1(q^{\frac12}, q)]\,\vartheta_2{}^2(\tfrac12\pi+\tfrac12\pi\tau, q),$$

and hence, by (9_1), since

$$\Pi_\infty(-q, q^2)\,\frac{\Pi_\infty(-q^4, q^4)}{\Pi_\infty(-q, q)} = \Pi_\infty(q^2, q^2),$$

we have

$$q^{\frac12}\vartheta_2{}^2(0, q)\,\Pi_\infty(q^2, q^2)[2\psi(q^4)\,G(-q^2)-q\,\vartheta_3(0, q^2)\,H(q^8)]$$
$$= \vartheta_3{}^2(0, q)[\vartheta_3(0, q^2)\,\Psi_0(q^{\frac12}, q)-\vartheta_2(0, q^2)\,\Psi_1(q^{\frac12}, q)]$$
$$-\vartheta_4{}^2(0, q)[\vartheta_3(0, q^2)\,\Psi_0(q^{\frac12}, q)+\vartheta_2(0, q^2)\,\Psi_1(q^{\frac12}, q)].$$

Now it follows from the properties of theta functions which were quoted in the proof of the compatibility of the results derived from (24_1) and (25_0) that the expression on the left of the last equation can be written in the form

$$\vartheta_3{}^2(0, q)\,\Pi_\infty(q^2, q^2)[\vartheta_3(0, q^2)\,G(-q^2)-q^{\frac32}\vartheta_2(0, q^2)\,H(q)^8]$$
$$-\vartheta_4{}^2(0, q)\,\Pi_\infty(q^2, q^2)[\vartheta_3(0, q^2)\,G(-q^2)+q^{\frac32}\vartheta_2(0, q^2)\,H(q^8)].$$

When this expression is inserted into the left-hand side of the last equation and (27) is used to simplify the result, we see that, since $\vartheta_3{}^2(0, q) \neq 0$, we must have

$$\Pi_\infty(q^2, q^2)[\vartheta_3(0, q^2)\, G(-q^2) - q^{\frac{3}{2}}\vartheta_2(0, q^2)\, H(q^8)]$$
$$= \vartheta_3(0, q^2)\, \Psi_0(q^{\frac{1}{2}}, q) - \vartheta_2(0, q^2)\, \Psi_1(q^{\frac{1}{2}}, q).$$

When this result is compared with (27), it is evident that

$$(28) \quad \Psi_0(q^{\frac{1}{2}}, q) = \Pi_\infty(q^2, q^2)\, G(-q^2), \quad \Psi_1(q^{\frac{1}{2}}, q) = q^{\frac{3}{2}}\, \Pi_\infty(q^2, q^2)\, H(q^8).$$

Formulae (26) and (28) give the simplest cases in which functions of the types $\Psi_0(x, q)$ and $\Psi_1(x, q)$ are expressible in terms of theta functions. From these formulae we can derive formulae for $\Psi_0(q^{\frac{1}{2}r}, q)$ and $\Psi_1(q^{\frac{1}{2}r}, q)$, when r is an integer, with the help of recurrence formulae.

The recurrence formulae, which the reader will easily verify, are

$$(29_0) \qquad \Psi_0(x, q) - \Psi_0(qx, q) = q^{\frac{1}{2}} x^2\, \Psi_1(q^{\frac{1}{2}} x, q),$$

$$(29_1) \qquad \Psi_1(x, q) - \Psi_1(qx, q) = q^{\frac{1}{2}} x^2\, \Psi_0(q^{\frac{1}{2}} x, q).$$

In these formulae write

$$u_{2m} = \Psi_0(q^m, q), \quad u_{2m+1} = \Psi_1(q^{m+\frac{1}{2}}, q),$$
$$v_{2m} = \Psi_1(q^m, q), \quad v_{2m+1} = \Psi_0(q^{m+\frac{1}{2}}, q);$$

and then we have

$$u_{n+2} + q^{n+\frac{1}{2}}\, u_{n+1} - u_n = 0,$$
$$v_{n+2} + q^{n+\frac{1}{2}}\, v_{n+1} - v_n = 0.$$

When we eliminate $u_2, u_3, \ldots, u_n$ from the equations

$$u_{m+2} + q^{m+\frac{1}{2}}\, u_{m+1} - u_m = 0 \quad (m = 0, 1, 2, \ldots, n-1),$$

we find that, when $n = 2, 3, 4, \ldots,$

$$(30) \qquad u_{n+1} = (-)^{n+1}[u_0\, \Delta_{n-1}(q^{\frac{3}{2}}, q) - u_1\, \Delta_n(q^{\frac{1}{2}}, q)],$$

where $\Delta_n(x, q)$ denotes the n-rowed continuant

$$\begin{vmatrix} x, & 1, & 0, & \ldots & \ldots & \ldots & \ldots \\ -1, & qx, & 1, & \ldots & \ldots & \ldots & \ldots \\ 0, & -1, & q^2 x, & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & q^{n-3} x, & 1, & 0 \\ \ldots & \ldots & \ldots & \ldots & -1, & q^{n-2} x, & 1 \\ \ldots & \ldots & \ldots & \ldots & 0, & -1, & q^{n-1} x \end{vmatrix}.$$

It is evident that

$$\Delta_n(x, q) = q^{n-1} x \Delta_{n-1}(x, q) + \Delta_{n-2}(x, q),$$

and hence we can easily prove by induction that

$$(31) \qquad \Delta_n(x, q) = \sum_{m=0}^{[\frac{1}{2}n]} \frac{\Pi_m(-q^{2n-4m+2}, q^2)}{\Pi_m(-q^2, q^2)} q^{\frac{1}{2}(n-2m)(n-2m-1)} x^{n-2m},$$

where the symbol $[\frac{1}{2}n]$ in the upper limit of the summation means $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd.

There is no difficulty in verifying that formula (30) is valid for all integral values of n, provided that the following conventions are adopted in interpreting $\Delta_n(x, q)$:

$$(32) \qquad \Delta_0(x, q) = 1, \quad \Delta_{-1}(x, q) = 0,$$

$$(33) \qquad \Delta_{-n-2}(x, q) = (-)^n \Delta_n(xq^{-n-1}, q) \quad (n = 0, \pm 1, \pm 2, \ldots).$$

We treat the second recurrence formula in the same manner, and we deduce that, with n any integer,

$$(34_0) \qquad \Psi_0(q^n, q) = \Psi_0(1, q)\Delta_{2n-2}(q^{\frac{3}{2}}, q) - \Psi_1(q^{\frac{1}{2}}, q)\Delta_{2n-1}(q^{\frac{1}{2}}, q),$$

$$(35_0) \qquad \Psi_0(q^{n+\frac{1}{2}}, q) = \Psi_0(q^{\frac{1}{2}}, q)\Delta_{2n}(q^{\frac{1}{2}}, q) - \Psi_1(1, q)\,\Delta_{2n-1}(q^{\frac{3}{2}}, q),$$

$$(34_1) \qquad \Psi_1(q^n, q) = \Psi_1(1, q)\,\Delta_{2n-2}(q^{\frac{3}{2}}, q) - \Psi_0(q^{\frac{1}{2}}, q)\,\Delta_{2n-1}(q^{\frac{1}{2}}, q),$$

$$(35_1) \qquad \Psi_1(q^{n+\frac{1}{2}}, q) = \Psi_1(q^{\frac{1}{2}}, q)\Delta_{2n}(q^{\frac{1}{2}}, q) - \Psi_0(1, q)\,\Delta_{2n-1}(q^{\frac{3}{2}}, q).$$

The expressions on the right can be evaluated in terms of theta functions and elementary functions by means of formulae (26) and (28); and the continuants on the right are defined by formula (31) coupled with the conventions specified by (32) and (33).

10. *Cases of reducibility of Lerch's transcendant* $f(x, \xi;\ q, \mathfrak{q})$.

The results of § 9 make it possible to express the transcendant $f(\xi q^r, \xi;\ q, q)$ in terms of theta functions when r is an integer. The simplest cases are obtained by taking r equal to 0 or 1, and I give these results separately.

For the case $r = 0$, we apply the formulae (26) to Lerch's formula (23). There is no loss of generality in taking $\alpha = \beta = 0$, and we get

$$(36) \qquad \frac{\vartheta_2{}^2(0, q)\,\Pi_\infty(qe^{-2iz}, q^2)}{\Pi_\infty(q^2, q^2)} f(e^{iz}, e^{iz};\ q, q)$$
$$= [\vartheta_3(0, q^2)\,G(q^8) - 2q\psi(q^4)\,H(-q^2)]\,\vartheta_1{}^2(z, q)$$
$$+ [\vartheta_3(0, q^2)\,G(q^8) + 2q\psi(q^4)\,H(-q^2)]\,\vartheta_2{}^2(z, q).$$

Similarly, by using (28) in place of (26), we get

$$(37)\quad \frac{q^{\frac{1}{2}}e^{2iz}\vartheta_2{}^2(0,\,q)\,\Pi_\infty(qe^{-2iz},\,q^2)}{\Pi_\infty(q^2,\,q^2)}\,f(qe^{iz},\,e^{iz};\;q,\,q)$$

$$=[\vartheta_3(0,\,q^2)\,G(-q^2)+2q^2\psi(q^4)\,H(q^8)]\,\vartheta_3{}^2(z,\,q)$$

$$-[\vartheta_3(0,\,q^2)\,G(-q^2)-2q^2\psi(q^4)\,H(q^8)]\,\vartheta_4{}^2(z,\,q).$$

For general integral values of r, we use the formulae (34_0), (34_1), (35_0), (35_1). The results assume different forms according as r is even $(r=2n)$ or odd $(r=2n+1)$. The two formulae are as follows:

$$(38)\quad \frac{q^{2n^2}e^{4niz}\vartheta_2{}^2(0,\,q)\,\Pi_\infty(qe^{-2iz},\,q^2)}{\Pi_\infty(q^2,\,q^2)}\,f(q^{2n}e^{iz},\,e^{iz};\;q,\,q)$$

$$=\vartheta_3(0,\,q^2)[\vartheta_2{}^2(z,\,q)+\vartheta_1{}^2(z,\,q)]$$

$$\times[G(q^8)\,\Delta_{2n-2}(q^{\frac{3}{2}},\,q)-q^{\frac{3}{2}}H(q^8)\,\Delta_{2n-1}(q^{\frac{1}{2}},\,q)]$$

$$+\vartheta_2(0,\,q^2)[\vartheta_2{}^2(z,\,q)-\vartheta_1{}^2(z,\,q)]$$

$$\times[q^{\frac{1}{2}}H(-q^2)\,\Delta_{2n-2}(q^{\frac{3}{2}},\,q)-G(-q^2)\,\Delta_{2n-1}(q^{\frac{1}{2}},\,q)],$$

$$(39)\quad \frac{q^{2(n+\frac{1}{2})^2}e^{(4n+2)iz}\vartheta_2{}^2(0,\,q)\,\Pi_\infty(qe^{-2iz},\,q^2)}{\Pi_\infty(q^2,\,q^2)}\,f(q^{2n+1}e^{iz},\,e^{iz};\;q,\,q)$$

$$=\vartheta_3(0,\,q^2)[\vartheta_3{}^2(z,\,q)-\vartheta_4{}^2(z,\,q)]$$

$$\times[G(-q^2)\,\Delta_{2n}(q^{\frac{1}{2}},\,q)-q^{\frac{1}{2}}H(-q^2)\,\Delta_{2n-1}(q^{\frac{3}{2}},\,q)]$$

$$+\vartheta_2(0,\,q^2)[\vartheta_3{}^2(z,\,q)+\vartheta_4{}^2(z,\,q)]$$

$$\times[q^{\frac{3}{2}}H(q^8)\,\Delta_{2n}(q^{\frac{1}{2}},\,q)-G(q^8)\,\Delta_{2n-1}(q^{\frac{3}{2}},\,q)].$$

As has already been remarked, it seems unlikely that any cases of reducibility of $f(x,\,\xi;\;q,\,\mathfrak{q})$ exist with $q=\mathfrak{q}$ and x not of the form ξq^r with r an integer. There may, however, exist cases of reducibility with $\mathfrak{q}$ a rational power of q.

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ON ANALYTIC FUNCTIONS BOUNDED AT THE LATTICE POINTS

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[Received June 18, 1936.—Read 12 November, 1936.]

Introduction.

The theorems which I shall prove in the first and second parts (Theorems I, II, III) of this paper are generalizations of earlier theorems of J. M. Whittaker and G. Pólya. Even the method which I use in the first part is not very different from theirs; to put it shortly, I pass from $f(z)$ to $f(z+k)$, where the above named authors pass from $f(z)$ to $f(kz)$.

Instead of the method of J. M. Whittaker which is based on a representation of the function $f(z)$ by a contour integral, I apply in the second part (Theorems II, III) a method similar to that of Phragmén-Lindelöf.

In the third part two theorems (Theorems IV, V) of M. L. Cartwright on functions of order one, published in a recent paper, are proved again in a way similar to that used in Theorem I.

Integral functions of order two.

1. THEOREM I A. *If the integral function $f(z)$ satisfies the conditions*

$$\limsup_{r\to\infty} \frac{\log M(r)}{r^2} < \frac{\pi}{2} \quad [M(r) = \max|f(re^{i\phi})|], \tag{1.1}$$

and

$$|f(m+in)| \leqslant \kappa \quad (m, n = 0, \pm 1, \pm 2, \ldots), \tag{1.2}$$

then $f(z)$ is necessarily a constant[†].

[†] The Weierstrass σ-function with the fundamental periods 1 and i, which vanishes at all lattice points, shows that in the hypothesis (1.1) the sign $<$ may not be replaced by $\geqslant$. Theorem I A destroys an analogy which might be expected between integral functions of orders one and two. For in the theorem of L. Tschakaloff (5): *An integral function of order one and minimum type bounded at the points* 0, ± 1, ± 2, ± 3, ... *is a constant*, the first hypothesis cannot be weakened, as the function $\sin \epsilon z$ $(\epsilon > 0)$ shows; see also Theorem IV. After handing in the MS., I was told by Miss M. L. Cartwright that Theorem 1 A had just been proved by another method by V. Ganapathy Iyer. His proof has now been published in *Journal London Math. Soc.*, 11 (1936), 247–249.

THEOREM I B. *If the integral function* $f(z)$ *satisfies* (1.2), *and, instead of* (1.1), *the condition*

$$\liminf_{r\to\infty}\frac{\log M(r)}{r^2}<\frac{\pi}{4}, \tag{1.3}$$

then $f(z)$ *is, just as before, a constant.*

A first theorem in the direction of Theorem I A was conjectured by J. E. Littlewood and proved by J. M. Whittaker[†]. Then G. Pólya proved a theorem, which differs from Theorem I A in the fact that the left-hand side of (1.1) is supposed to be zero[‡]. Later, J. M. Whittaker proved a theorem which differs from Theorem I B in just the same way[§].

Proof of Theorem I. Let $f(z)$ satisfy the hypothesis (1.1) and denote by $\sigma(z)$ the Weierstrass σ-function with the fundamental periods 1 and i; then we have the formula of interpolation[||]

$$f(z)=\sigma(z)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{m+n+mn}f(m+in)\,e^{-\frac{1}{2}\pi(m^2+n^2)}}{z-m-in}. \tag{1.4}$$

Replacing $f(z)$ in (1.4) by $f(z+M+iN)$, which also satisfies the hypothesis (1.1), we get[¶]

$$f(z+M+iN) \tag{1.5}$$
$$=\sigma(z)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{m+n+mn}f(m+in+M+iN)\,e^{-\frac{1}{2}\pi(m^2+n^2)}}{z-m-in},$$

where we may suppose $M+iN$ to be any lattice-point.

Let now z be a point on the boundary of the square with the corners $\frac{1}{2}(\pm 1\pm i)$. We have

$$|z-m-in|\geqslant\tfrac{1}{2};$$

[†] Whittaker, 6.

[‡] Pólya, 3.

[§] Whittaker, 7, Theorem 3.

[||] Pólya, 3; the formula was originally given by J. M. Whittaker, 6, with another notation.

[¶] In Pólya's proof the function $f(z)$ in formula (1.4) was replaced by $f(\kappa z)$ and z by z/κ, where κ is any integer; this is the difference that I pointed out at the beginning.

and, since $\sigma(z)$ is regular along this boundary,

$$|\sigma(z)| < C.$$

Using (1.2), we obtain from (1.5)

$$|f(z+M+iN)| < C \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2\kappa\, e^{-\frac{1}{2}\pi(m^2+n^2)} < C_1;$$

it is to be observed that C_1 is independent of $M+iN$. Thus, on the boundary of all squares with the corners $M+iN+\frac{1}{2}(\pm 1 \pm i)$, and therefore in the whole z-plane, we have $|f(z)| < C_1$. We have now proved Theorem I A.

Whittaker also established the formula (1.4) for functions which satisfy the hypothesis (1.3)†. Therefore we can also obtain (1.5) on hypothesis (1.3), and Theorem I B follows in the same way as Theorem I A.

Regular functions of order two.

2. THEOREM II. *Let D denote the domain $|\arg z| \leqslant \frac{1}{4}\pi$. If the function $f(z)$ satisfies the conditions:*

(2.1) $$f(z) \text{ is regular in } D;$$

(2.2) $$|f(z)| < Ae^{B|z|^2}$$

in D for A and B positive and sufficiently large;

(2.3) $$|f(re^{\pm\frac{1}{4}\pi i})| < Ae^{(\frac{1}{2}\pi-\gamma)r^2} \quad (r>0, \quad \gamma>0, \quad \gamma \text{ fixed});$$

(2.4) $$f(\tfrac{1}{2}+m+in) = 0$$

for every point $\frac{1}{2}+m+in$ in D; then $f(z) \equiv 0$.

This theorem differs from a former theorem proved by J. M. Whittaker‡ in the fact that, in the condition (2.2), the number B on the right-hand side is there supposed to be smaller than $\frac{1}{2}\pi$.

Proof of Theorem II. Consider the function

$$\phi(z) = \frac{f(z)}{\sigma(\frac{1}{2}+z)}.$$

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† Whittaker, 7, 398.

‡ Whittaker, 6, Theorem 2.

This function satisfies conditions corresponding to those of $f(z)$ [(2.1), (2.2), (2.3)]:

$$\text{(2.1}'\text{)} \qquad \phi(z) \text{ is regular in } D;$$

$$\text{(2.2}'\text{)} \qquad |\phi(z)| < A' e^{B'|z|^2} \quad \text{in } D;$$

$$\text{(2.3}'\text{)} \qquad |\phi(re^{\pm i\frac{1}{4}\pi})| < A' e^{-\gamma' r^2} \quad (\gamma > \gamma' > 0,\ r > 0).$$

For (2.1′) follows from (2.1) and (2.4), since the poles arising from $1/\sigma(\frac{1}{2}+z)$ are balanced in D by the zeros of $f(z)$. To prove (2.2′), we describe a circle with radius $\frac{1}{2}$ around every point $\frac{1}{2}+m+in$ in D. Then, by (2.2) and†

$$\text{(2.5)} \qquad \sigma(z+m+in) = (-1)^{m+n+mn}\, \sigma(z)\, e^{\frac{1}{2}\pi(m^2+n^2)+\pi z(m-in)},$$

we first obtain (2.2′) for all z in D outside these circles, and then, by (2.1′) and the maximum-modulus principle, in the whole domain D. To prove (2.3′) we put

$$r = (m+t)\sqrt{2}, \quad m = \left[\frac{x}{\sqrt{2}}\right], \quad 0 \leqslant t < 1,$$

and obtain, by (2.5),

$$\text{(2.6)} \qquad |\sigma(\tfrac{1}{2}+re^{\pm\frac{1}{4}\pi i})| \geqslant e^{\frac{1}{2}\pi(m\sqrt{2})^2-2\pi(m\sqrt{2})}\, |\sigma(\tfrac{1}{2}+\sqrt{2}\,te^{\pm\frac{1}{4}\pi i})|.$$

Since $|\sigma(\frac{1}{2}+\sqrt{2}\,te^{\pm\frac{1}{4}\pi i})| > a > 0$, (2.3′) follows from (2.3) and (2.6). Thus (2.1′), (2.2′) and (2.3′) are satisfied. In accordance with the method of Phragmén-Lindelöf‡ we conclude that $\phi(z) \equiv 0$.

Indeed, suppose that $\phi(z) \not\equiv 0$; since $\phi(z)$ is of order two and mean type, Lindelöf's function

$$h(\phi) = \limsup_{r\to\infty} \frac{\log|\phi(re^{i\phi})|}{r^2} \quad (|\phi| \leqslant \tfrac{1}{4}\pi)$$

satisfies the inequality§

$$h(\phi_1)\sin 2(\phi_3-\phi_2) + h(\phi_2)\sin 2(\phi_1-\phi_3) + h(\phi_3)\sin 2(\phi_2-\phi_1) \geqslant 0,$$

---

† See, *e.g.*, Pólya, 3.

‡ Phragmén-Lindelöf, 2.

§ Phragmén-Lindelöf, 2.

provided that

$$\phi_1 < \phi_2 < \phi_3, \quad \phi_3 - \phi_2 < \tfrac{1}{2}\pi, \quad \phi_2 - \phi_1 < \tfrac{1}{2}\pi.$$

We put

$$\phi_1 = -\tfrac{1}{4}\pi, \quad \phi_2 = 0, \quad \phi_3 = \tfrac{1}{4}\pi,$$

and obtain

$$h(-\tfrac{1}{4}\pi) + h(\tfrac{1}{4}\pi) \geqslant 0. \tag{2.7}$$

But we infer from (2.3′) that

$$h(-\tfrac{1}{4}\pi) \leqslant -\gamma' < 0, \quad h(\tfrac{1}{4}\pi) \leqslant -\gamma' < 0,$$

which contradicts (2.7). Thus $\phi(z) \equiv 0$ and hence $f(z) \equiv 0$.

3. THEOREM III. *Let T denote any domain such that the point $z = \infty$ belongs to its boundary R. Suppose further that the distance of R from any lattice point is not smaller than a fixed positive number $0 < \rho < \frac{1}{4}$*[†]*. The arcs of the circle $|z| = r$, which are contained in T for r sufficiently large, are denoted by Γ_r. If the function $f(z)$ satisfies the conditions:*

$$f(z) \textit{ is regular in } T; \tag{3.1}$$

$$|f(z)| < \kappa_1 \tag{3.2}$$

on the boundary R except at the point $z = \infty$;

$$|f(z)| < Ae^{(\frac{1}{2}\pi - \gamma)r_n^{\,2}} \quad (\gamma > 0) \tag{3.3}$$

on a sequence of arcs Γ_{r_n} tending to infinity;

$$|f(m+in)| < \kappa_2 \tag{3.4}$$

for $m+in$ in T; then $|f(z)| < \kappa_1$ for z in T.

A theorem in this direction was proved by J. M. Whittaker[‡]. In his work the function $f(z)$ is supposed to be of order two and *minimum type*.

[†] It seems that this condition imposed on the boundary of the domain T can be dispensed with by modifying the Weierstrass σ-function used in the following proof. I have not attempted to do this here because it would make the proof rather long.

[‡] Whittaker, **6**.

The domain, which corresponds to T, is an angle $|\arg z| \leqslant \alpha$. The method of J. M. Whittaker is based on a formula which represents $f(z)$ by its values at the lattice points in $|\arg z| \leqslant \alpha$ and by its values on $|\arg z| = \alpha$.

Proof of Theorem III. Put

$$\phi_{M,N}(z) = \sigma(z-M-iN) \sum_{(T)} \frac{f(m+in)(-1)^{m-M+n\ \ N+(m-M)(n-N)}}{(z-m-in)\, e^{\frac{1}{2}\pi\{(m-M)^2+(n-N)^2\}}},$$

where M and N are any integers and the summation is extended over the lattice points $m+in$ in T. We consider the function

$$\Phi_{M,N}(z) = \frac{f(z)-\phi_{M,N}(z)}{\sigma(z-M-iN)}, \tag{3.5}$$

which satisfies conditions corresponding to those of $f(z)$ [(3.1), (3.2), (3.3)]:

$$\Phi_{M,N}(z) \text{ is regular in } T; \tag{3.1$'$}$$

$$|\Phi_{M,N}(z)| < \kappa_1 A(\rho) + \kappa_2 C(\rho), \quad z \text{ on } R, \tag{3.2$'$}$$

$$|\Phi_{M,N}(z)| < A'(M, N) \exp(-\gamma' r_n^2) + \kappa_2 C(\rho) \tag{3.3$'$}$$

on curves $\Gamma^*_{r_n}$ which join the boundaries of T in the ring $r_n - 1 < |z| < r_n$.

In fact, by the formula†

$$\sigma'(m+in) = (-1)^{m+n+mn}\, e^{\frac{1}{2}\pi(m^2+n^2)}$$

we find that $$\phi_{M,N}(m+in) = f(m+in)$$

for every lattice point $m+in$ in T. Therefore the poles arising from $1/\sigma(z-M-iN)$ are balanced in T by the zeros of $f(z)-\phi_{M,N}(z)$ and so (3.1′) follows from (3.1). To show that (3.2′) and (3.3′) are satisfied, we describe a circle with radius $\rho < \frac{1}{4}$ round every lattice point in T. By (2.5) we obtain

$$\left|\frac{1}{\sigma(z-M-iN)}\right| < A(\rho)\, e^{-(\frac{1}{2}\pi-\eta)|z-M-iN|^2} < A(\rho) \tag{3.6}$$

$$[\eta > 0, \quad A(\rho) = A(\rho, \eta)],$$

---

† See, *e.g.*, Pólya, 3.

and by (3.4)

$$(3.7)\qquad \left|\frac{\phi_{M,N}(z)}{\sigma(z-M-iN)}\right| \leqslant \left|\sum_{(T)} \frac{f(m+in)(-1)^{m-M+n-N+(m-M)(n-N)}}{(z-m-in)\,e^{\frac{1}{2}\pi\{(m-M)^2+(n-N)^2\}}}\right|$$

$$\leqslant \frac{\kappa_2}{\rho} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}\pi(m^2+n^2)}$$

$$\leqslant \kappa_2 C(\rho)$$

for z in T (including the boundary R) outside those circles. Since the boundary R avoids those circles, (3.2′) follows from (3.2), (3.6), and (3.7). Taking n sufficiently large in (3.3), we conclude from (3.2) that the inequality (3.3) is valid for z on the arcs Γ_{r_n} and on the boundary R, and therefore in the domain T_{r_n} included by Γ_{r_n} and a finite part of R. Since $\rho < \frac{1}{4}$, we can join each boundary of T in the ring $r_n - 1 < |z| < r_n$ by a curve $\Gamma^*_{r_n}$ which lies in this ring and in the domain T_{r_n}, but which lies outside the circles $|z-m-in| \leqslant \rho$. By (3.3) and (3.6) we find

$$(3.8)\qquad \left|\frac{f(z)}{\sigma(z-M-iN)}\right| < Ae^{|\frac{1}{2}\pi-\gamma|r_n} A(\rho)\, e^{-(\frac{1}{2}\pi-\eta)|r_n-1-|M+iN||^2}$$

$$< A'(M, N) \exp(-\gamma' r_n{}^2) \quad (0 < \gamma' < \gamma)$$

for z on $\Gamma^*_{r_n}$. Hence (3.3′) follows from (3.8), (3.7), and (3.5). Thus (3.1′), (3.2′), and (3.3′) are fulfilled. Continuing, we conclude by a method similar to that of Phragmén-Lindelöf, and we find that

$$(3.9)\qquad |\Phi_{M,N}(z)| < \kappa_1 A(\rho) + \kappa_2 C(\rho) = C_1, \quad z \text{ in } T.$$

For, in whatever way we choose the integers M and N, (3.9) is obtained by (3.3′) on $\Gamma^*_{r_n}$ for all sufficiently large n, and by (3.2′) on the boundary R, and consequently by means of the maximum-modulus principle for all z in T. Observe that the right-hand of (3.9) is independent of M and N. Picking out any square with some of its corners

$$M + iN + \tfrac{1}{2}(\pm 1 \pm i)$$

inside the domain T, and supposing z to be a point on the boundary of the complete or incomplete square in question, then we see that $z-M-iN$ is a point in the square with the corners $\frac{1}{2}(\pm 1 \pm i)$, but outside the circle $|z| < \rho$. In consequence of this, we have

$$|\sigma(z-M-iN)| < m,$$

where m is independent of $M+iN$, and by (3.7)

$$|\phi_{M,N}(z)| < \kappa_2 m C(\rho).$$

From this and (3.9) and (3.5) we obtain

$$\begin{aligned}|f(z)| &\leqslant |\phi_{M,N}(z)| + C_1 |\sigma(z-M-iN)| \\ &\leqslant \kappa_2 m\, C(\rho) + mC_1\end{aligned}$$

on the boundaries of the squares dealt with above, and so in the whole domain T. Since $|f(z)|$ is bounded in T and smaller than κ_1 on its boundary, we conclude from a well-known result of the Phragmén-Lindelöf method† that $|f(z)| < \kappa_1$ in T. Thus we have proved Theorem III.

We easily infer from the preceding demonstration that in Theorem I B the condition (1.3) may be replaced by

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^2} < \tfrac{1}{2}\pi.$$

Integral functions of order one.

4. In what follows I shall prove two theorems of M. L. Cartwright‡ by a method which is almost the same as that used in the proof of Theorem I.

THEOREM IV. *Suppose that $f(z)$ is an integral function such that*

$$|f(z)| < Ae^{k|z|} \quad (k < \pi), \tag{4.1}$$

and

$$|f(\pm n)| < B \quad (n = 0, 1, 2, \ldots), \tag{4.2}$$

then

$$|f(x)| < KB \quad [K = K(k)], \tag{4.3}$$

for all real x.

Proof of Theorem IV. If $f(z)$ satisfies the hypotheses (4.1) and (4.2), then§

$$f(z) = \frac{\sin \pi z}{\pi}\left[\frac{f(0)}{z} + f'(0) + \sum_{n=-\infty}^{\infty}{}' (-1)^n f(n)\left\{\frac{1}{z-n} + \frac{1}{n}\right\}\right],$$

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† Phragmén-Lindelöf, 2.

‡ Cartwright, 1.

§ Tschakaloff, 5; Cartwright, 1.

where Σ' denotes that the term $n=0$ is to be omitted. Let N be any integer and p an integer greater than $\pi/(\pi-k)$. Then $f(z+N)\sin\pi z/p$ also satisfies the hypotheses (4.1) and (4.2), and we have[†]

$$(4.4)\quad f(z+N)=\frac{\sin\pi z}{\pi\sin\pi z/p}\Big[\frac{\pi}{p}f(N)$$

$$+\sum_{n=-\infty}^{\infty}{}'(-1)^n f(n+N)\sin\pi n/p\,.\Big\{\frac{1}{z-n}+\frac{1}{n}\Big\}\Big].$$

Now let z be a point on the boundary of the square with the corners $\frac{1}{2}(\pm1\pm i)$. We have

$$\left|\frac{1}{z-n}+\frac{1}{n}\right|<\frac{2}{(|n|-\frac{1}{2})|n|};$$

and, since $\dfrac{\sin\pi z}{\sin\pi z/p}$ is regular along this boundary,

$$\left|\frac{\sin\pi z}{\pi\sin\pi z/p}\right|<K_1(k).$$

Using (4.2), we obtain from (4.4)

$$|f(z+N)|<K_1(k)\Big[\frac{\pi}{p}B+2\sum_{n=1}^{\infty}\frac{2B}{n(n-\frac{1}{2})}\Big]=BK(k).$$

Since $BK(k)$ is independent of N, we have (4.3) on the boundary of all squares with the corners $N+\frac{1}{2}(\pm1\pm i)$ and therefore on the whole real axis. Thus we have proved Theorem IV[‡].

I add two remarks:

1. It is well known[§] that (4.1) and (4.3) implies that

$$|f(re^{i\phi})|<Ce^{kr|\sin\phi|}.$$

2. *If* $f(z)$ *satisfies* (4.1) *and if the values* $f(n)$ $(n=0,\pm1,\pm2,\ldots)$ *form an almost periodic set, then* $f(z)$ *is almost periodic on the real axis.*

[†] M. L. Cartwright replaces $f(z)$ by $f(z)\sin\pi z/p$, and so obtains (4.4) with $N=0$. It is then necessary to evaluate the expression on the right for all z in the strip $z=x+iy$, $|y|\leqslant 1$.

[‡] If $f(z)$ is an integral function of order one and minimum type, we infer from the Phragmén-Lindelöf method that $f(z)\equiv 0$, a result proved by L. Tschakaloff, 5.

[§] Phragmén-Lindelöf, 2; Pólya and Szegö, 4, No. 325.

In fact, having chosen $\epsilon > 0$, we find in every interval of length $L(\epsilon)$ an integer ν such that

$$|f(n+\nu)-f(n)| < \epsilon \quad (n=0, \pm 1, \pm 2, \ldots).$$

Therefore $F(z)=f(z+\nu)-f(z)$ satisfies (4.1) and (4.2) with ϵ in place of B and hence we have, by (4.3),

$$|F(x)| = |f(x+\nu)-f(x)| < \epsilon K$$

for all real x. Thus $f(x)$ is an almost periodic function on the real axis.

5. THEOREM V. *If $f(z)$ satisfies the hypotheses of Theorem IV and, in addition,*

$$\lim_{n\to\infty} f(n) = l \quad (n=1, 2, 3, \ldots),$$

then

$$\lim_{x\to\infty} f(x) = l \quad (x > 0).$$

Proof of Theorem V. We suppose, as is obviously permissible, that $l=0$. Then, given any $\epsilon > 0$, we can choose $N(\epsilon)$ so that

$$|f(n)| < \epsilon \quad \text{for} \quad n > N(\epsilon) \tag{5.1}$$

and

$$\sum_{n=N(\epsilon)}^{\infty} \frac{1}{n(n-\frac{1}{2})} < \epsilon. \tag{5.2}$$

Now z may be a point on the boundary of the square with the corners $\frac{1}{2}(\pm 1 \pm i)$. If now we divide the sum Σ' in the formula (4.4) between the terms $n=-N(\epsilon)$ and $n=-N(\epsilon)+1$ into two parts, we find, by (4.2), (5.1), and (5.2),

$$f(z+N) = \frac{\sin \pi z}{\pi \sin \pi z/p}\left[\frac{\pi}{p} f(N) + \left(\sum_{n=-\infty}^{-N(\epsilon)} + \sum_{n=-N(\epsilon)+1}^{\infty}\right)(-1)^n f(n+N) \times \sin\frac{\pi n}{p}\left(\frac{1}{z-n}+\frac{1}{n}\right)\right],$$

$$\begin{aligned}|f(z+N)| &\leqslant K_1'(k)\left[\frac{\pi}{p}\epsilon + B\sum_{n=N(\epsilon)}^{\infty}\frac{1}{n(n-\frac{1}{2})} + 4\epsilon\sum_{n=1}^{\infty}\frac{1}{n(n-\frac{1}{2})}\right]\\ &< \epsilon K_1(k)\left[\frac{\pi}{p} + B + 4\sum_{1}^{\infty}\frac{1}{n(n-\frac{1}{2})}\right]\\ &< \epsilon C\end{aligned}$$

for all integers $N > 2N(\epsilon)$. Thus $|f(x)| < \epsilon C$ for all real $x > 2N(\epsilon)$. Since C is independent of ϵ, and ϵ is any positive number, we have proved Theorem V.

References.

1. M. L. Cartwright, "On certain integral functions of order one", *Quart. J. of Math.* (Oxford series), 7 (1936), 46–55.
2. E. Phragmén and E. Lindelöf, "Sur une extension d'un principe classique de l'analyse", *Acta Math.*, 31 (1908), 381–406.
3. G. Pólya, "Bemerkung zu der Aufgabe 105", *Jahresber. d. Deutsch. Math. Ver.*, 43 (1933), 67–69.
4. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925).
5. L. Tschakaloff, "Zweite Lösung der Aufgabe 105", *Jahresber. d. Deutsch. Math. Ver.*, 43 (1933), 11–13.
6. J. M. Whittaker, "Flat regions of integral functions", *Proc. Edinburgh Math. Soc.* (2), 2 (1930), 111–128.
7. J. M. Whittaker, "On the fluctuation of integral and meromorphic functions", *Proc. London Math. Soc.* (2), 37 (1934), 383–401.

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SOME TYPES OF RATIONAL QUARTIC PRIMAL IN FOUR DIMENSIONS

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[Received 6 August, 1936.—Read 12 November, 1936.]

1. *Introduction.*

Although many types of rational quartic primal in [4] have been described, most of these have been discovered by means of their representations on ordinary space, and there seems in many cases to be no clear intrinsic reason for their rationality. In the present paper we consider some primals whose rationality is an immediate consequence of a geometrical property, and we obtain their representations. Some of the loci which arise are already known, while others appear to be new.

Suppose that in [4] we have a surface ϕ, reducible or not, with the property that just one of its trisecants passes through an arbitrary point of the space. Any quartic primal V which passes through ϕ must necessarily be rational, since, if Σ is an arbitrary prime, we can set up a (1, 1) correspondence between the points of V and Σ by drawing through a point of Σ the unique trisecant of ϕ and taking its fourth intersection with V. It is the primals arising in this way which we consider here.

2. *Surfaces in* [4] *with one apparent triple point.*

The irreducible surfaces in [4] such that one of their trisecants passes through an arbitrary point were enumerated by Ascione*. They are:

(i) The rational sextic surface with prime sections of genus three.

(ii) The projected quintic Del Pezzo surface.

(iii) The rational quintic scroll.

(iv) The projected quartic surface of Veronese.

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* Ascione, *Rend. Lincei* (5), 6 (1897), 162.

To these we must add the following reducible surfaces*:

(v) The cubic scroll and an arbitrary plane.

(vi) Three arbitrary planes.

If the surface ϕ is of any one of the six types just mentioned, it has one trisecant through an arbitrary point of space. There are, however, a double infinity of points through which pass a cone of trisecants, these points forming the *singular surface* of the congruence Γ formed by the trisecants of ϕ. Of this surface ϕ itself evidently forms a part, but not, in general, the whole part. If two trisecants of ϕ pass through a point P which does not lie on ϕ, then, since the order of ϕ does not exceed six, the plane containing these two trisecants must meet ϕ in a curve of order three (at least) and belong to the singular surface; conversely, the singular surface other than ϕ must consist of a number of planes. In fact, in case (i) it consists of the ten planes which meet ϕ in plane cubics; in case (iii) of the plane joining the three nodes of the quintic scroll, which meets the surface in a plane quartic; this plane counts *quadruply* in the singular surface since any line in it is a quadrisecant of ϕ; in case (v) the three conic planes of the cubic scroll which pass through pairs of its intersections with the plane are singular; and in case (vi) the plane meeting the three planes in lines is singular. In the two remaining cases the congruence Γ has no singular surface save ϕ.

3. *General features of the representation.*

The representations of the primals V passing through one or other of the surfaces ϕ, obtained in the manner described in § 1, have certain common features which may be discussed here.

We consider the cases in which ϕ is irreducible; for we shall see that, when ϕ consists of a cubic scroll and a plane, the representation of V can be obtained by other considerations, and, when ϕ consists of three planes, the representation has already been given by Roth†.

Suppose, then, that ϕ is of order n, and irreducible. Let h be the number of apparent double points of a prime section of ϕ as seen from a point on the section, and let t be the order of the scroll of trisecants of ϕ lying in an arbitrary prime. These last two characters can be readily calculated in terms of the order and section genus of ϕ.

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* Other reducible surfaces (in which the components intersect along curves) exist which possess congruences of trisecants of the first order, but these surfaces are degenerate forms of surfaces belonging to classes (i)–(v) above. A detailed account of the associated line-systems will be found in the paper by Marletta, *Rend. di Palermo*, 28 (1909), 353.

† Roth, *Proc. London Math. Soc.* (2), 35 (1933), 249.

Through a point P of ϕ there passes a cone K of trisecants of ϕ whose order is h. Let l be the order of the curve in which K meets ϕ. Then it is easily seen that the curve has a multiple point at P of multiplicity $l-2h$. But this multiplicity is the number of tangents of ϕ at P which meet ϕ again, and is thus equal to $n-3$ or $n-4$ according as ϕ is ruled or not. Thus, if ϕ is ruled, $l = 2h+n-3$, while, if ϕ is not ruled, $l = 2h+n-4$.

An arbitrary prime Σ_1 meets ϕ in a curve, and the trisecants of ϕ passing through points of this curve fill up a primal of order ξ. By considering the number of intersections of this primal with a line lying in Σ_1 it is easy to see that $\xi = 3t$, and that each point of ϕ is l-ple for the primal.

The trisecants of ϕ meeting an arbitrary line form a scroll of order $t+1$, which meets ϕ in a curve of order ξ. The trisecants of ϕ which meet an arbitrary plane form a primal of order $t+1$ on which ϕ is h-ple.

Now consider a quartic primal V through ϕ, and its representation on a prime Σ obtained in the manner indicated above. The trisecants of ϕ meeting a prime section of V form a primal of order $4(t+1)-\xi = t+4$ on which ϕ is of multiplicity $4h-l$; and this meets Σ in a surface F of order $t+4$ passing $(4h-l)$-ply through the section γ of ϕ by Σ. To the points of a plane section of V will correspond in the same way a curve f in Σ of order $4(t+1)-nh$ which meets γ in $4\xi-nl$ points. The curve f is of genus 3, and its residual intersection with a surface F consists of four points. Thus the system of surfaces $|F|$ has further base curves which arise (i) as the section g by Σ of the singular surface (other than ϕ) of the congruence of trisecants of ϕ, and (ii) from the section k of a scroll of trisecants of ϕ lying on V.

The curve g consists of a number of skew lines meeting γ in three or more points, and having no intersections with the curves f. If the plane of which one of the lines g is a section meets ϕ in a cubic curve, it meets V residually in a line; thus g is a simple base-curve of the system $|F|$ and k meets g in one point. On the other hand, if the plane meets ϕ in a quartic curve, g is a quadruple line for F, and does not meet k since the plane has no further intersection with V. We can now determine the genus of k and the numbers of its intersections with the other curves by using the formulae for residual intersections of surfaces with multiple curves. As applied here, these formulae are the following*:

(i) Two surfaces of order m passing s times through a curve of order n and rank r have a residual intersection meeting this curve in

$$2mns-2ns^2-rs^2$$

points.

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* See Pascal, *Repertorium der höheren Mathematik* II, 2 (Leipzig, 1922), 904–5.

(ii) The equivalence (reduction in grade) of the curve for three surfaces of order m passing s times through it is $3mns^2-2ns^3-rs^3$.

(iii) The equivalence of two curves, one simple and one s-ple, differs from the sum of the equivalences of the simple and s-ple curves taken separately by $i(3s-1)$, where i is the number of intersections of the two curves.

4. *Quartic primals containing a Bordiga sextic surface.*

Suppose that ϕ is a sextic surface of section genus 3, so that $n=6$, $h=3$, $t=8$. Then, using the formula of the last section, the representation of a primal V through ϕ is by surfaces of order 12 passing quadruply through the section γ of ϕ by the prime Σ, containing simply the ten trisecant lines g (section of the ten planes meeting ϕ in cubic curves) and another base-curve k, and meeting residually in curves f, of order 18, meeting γ in forty-eight points. From the intersection formulae it follows that the residual base-curve k must be of order 20 and genus 11, and that it meets each line g in one point, the curve γ in fifty, and the residual intersection f in twenty.

By means of the cubic Cremona transformation defined by the curve γ this linear system $|F|$ is transformed into a system of quartic surfaces passing simply through a curve of order 10 and genus 11 (the transform of k), and we thus identify the primal V with the general determinantal primal in [4]*.

5. *Quartic primals containing a quintic Del Pezzo surface.*

If ϕ is a quintic surface of section genus 1, not ruled, we have $n=5$, $h=2$, $t=5$. Thus the surfaces F are of order 9 and pass triply through the elliptic quintic γ in which ϕ meets Σ, and the curves f are of order 14 and meet γ in thirty-five points. There being no curves g in this case, it follows that the residual base-curve k is of order 22 and genus 9, meeting γ in fifty-five points and f in seventeen.

Cubic surfaces passing through the elliptic quintic and an arbitrary point of k form a homaloidal system, and after transformation the linear system $|F|$ becomes a *system of quintic surfaces having in common a double conic, and a* ${}_9C^{10}$ *meeting this in nine points.* This appears to be the simplest representation of V.

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* See Roth, *Proc. London Math. Soc.* (2), 30 (1930), 118.

By using the formula quoted by Baker*, we find that V, in addition to having a node at the improper double point of ϕ, has twenty-one other nodes at simple points of ϕ. In the representation by means of quintic surfaces, a plane corresponds to a surface of order 7 and section genus 4 on V. This surface is the residual intersection of two cubic primals which have two planes in common, has an improper node at the point of intersection of the planes, and meets each of the planes in a quartic curve of genus 2. The pencil of cubic primals through the surface cut out on V a single infinity of Del Pezzo quintic surfaces, all having the same improper node and passing through the other twenty-one nodes of V.

6. *Quartic primals containing a rational quintic scroll.*

In this case ϕ is ruled, and we have $n = 5$, $h = 3$, $t = 8$. Thus the surfaces F are of order 12 and pass quadruply through the rational quintic γ, section of ϕ by Σ; the curves f are of order 21 and meet γ in fifty-six points. The plane which meets ϕ in a quartic curve meets Σ in the unique quadrisecant g of γ, and this is also 4-ple for F, since the plane in question counts four times as part of the singular surface. The residual base-curve k is therefore of order 27 and genus 7, meeting f in twenty-four points and γ in seventy-two; k does not meet g. The curve γ and its quadrisecant g make up a degenerate sextic of genus 3, the cubic surfaces through which form a homaloidal system, and under the corresponding Cremona transformation the linear system $|F|$ becomes a system of quartic surfaces passing simply through a curve of order 9 and genus 7.

The primal is thus the one investigated recently by Babbage†, and it may be verified by using the formula, referred to in the last section, that it contains twenty-four nodes, three at the improper double points of ϕ and the remaining twenty-one at simple points of ϕ. We may easily verify directly that the general V_3^4 mapped by quartic surfaces through a ${}_7C^9$ contains a rational quintic scroll. For, in the representation, a plane maps a septimic surface F^7 of section genus 3; this lies on a single cubic primal, and the residual intersection of this with V_3^4 is easily verified to be the rational quintic scroll in question. In the representation, ϕ corresponds therefore to a surface of order 11 on which ${}_7C^9$ is triple (making up with an arbitrary plane the map of a cubic section of V_3^4); this is, in fact, the locus of the rational quartic curves which meet ${}_7C^9$ in fifteen points, and contains simply the twenty-one quadrisecant lines of this curve and doubly its three 8-secant conics. This verifies the fact that ϕ has nodes at three of the nodes of V and passes simply through the remainder, and shows incidentally that the twenty-four nodes of V are not all equivalent (since there is no reason whatever for believing that V contains *other* quintic scrolls having nodes at three other nodes of V).

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* Baker, *Principles of geometry*, 6 (Cambridge, 1933), 262.

† Babbage, *Proc. Camb. Phil. Soc.*, 32 (1936), 13.

7. *Quartic primals containing a projected Veronese surface.*

If ϕ is a projected Veronese surface, we have $n = 4$, $h = 1$, $t = 2$. Thus a quartic primal V through ϕ is represented by surfaces of order 6 passing doubly through the rational quartic curve γ in which Σ meets ϕ; and the curves f are of order 8 and meet γ in sixteen points. There being no curves g in this case, the residual base-curve k must be of order 12 and genus 7, meeting γ in twenty-four points and f in twelve. It does not appear to be possible to reduce the order of this representing system by Cremona transformation.

It may be noted that the system of quartic surfaces which have a common ${}_7C^8$ and a fixed node at a point P not lying on this likewise represents the sections of a V_3^4 of [4] containing a projected Veronese surface (corresponding to the neighbourhood of P). But this is a *special case* of the foregoing primal, since the Veronese surface has a double line which is double on the primal, and is mapped by the cubic surface drawn to contain ${}_7C^8$ and P.

In general, the V_3^4 through ϕ possesses twenty-four nodes lying on the surface.

8. *Quartic primals containing a cubic scroll and a plane.*

If ϕ consists of a cubic scroll R^3 and a plane π, we can obtain a representation of a quartic primal through ϕ by methods analogous to those used above, and are led to a rather complicated representing system of octavic surfaces. We do not enter into the details of this representation, however, since a much simpler one can be derived independently.

We first remark that *any quartic primal in* [4] *that contains a cubic scroll is the projection, from a line lying on it, of a* V_3^8 *of* [6], *the intersection of three quadric primals.* In fact, if the equations of the scroll are

$$\left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right\| = 0,$$

where a_i, b_i are linear functions of the coordinates $(x_0, \ldots, x_4)$, then any quartic primal through it has an equation of the form

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ R_1 & R_2 & R_3 \end{vmatrix} = 0,$$

where the R_i are quadratic in $(x_0, \ldots, x_4)$; and this is the projection of the intersection of the three quadrics in [6] whose equations are

$$x_5 a_1 + x_6 b_1 + R_1 = 0,$$

$$x_5 a_2 + x_6 b_2 + R_2 = 0,$$

$$x_5 a_3 + x_6 b_3 + R_3 = 0,$$

from the line $x_0 = x_1 = \ldots = x_4 = 0$. Conversely, any $V_3{}^8$ in [6], the intersection of three quadrics, contains a single infinity of lines from any one of which it projects into a quartic primal containing a cubic scroll.

In particular, the primal V containing R^3 and π is the projection of a $V_3{}^8$ of [6], and, since π is perfectly general in its relation to R^3, it is easy to see that on the $V_3{}^8$ there is a plane π_1, not meeting the line λ of projection, which projects into π from λ. Now any $V_3{}^8$ of [6], the intersection of three quadrics, which contains a plane π_1 is necessarily rational, a representation on [3] being obtained immediately by projecting from the plane. It is easily verified that prime sections of the $V_3{}^8$ become quartic surfaces passing through a curve of order 9 and genus 9, the intersection of a cubic and quartic surface residual to a twisted cubic. The unique cubic surface through ${}_9C^9$ maps the plane π_1. The trisecants of ${}_9C^9$ map lines on $V_3{}^8$ which do not meet π_1, and the projection into [4] of $V_3{}^8$ from one of these lines is mapped by quartic surfaces containing a ${}_9C^9$ and one of its trisecants, the R^3 on such a primal corresponding to the neighbourhood of the points of the trisecant. Thus V can be represented by quartic surfaces passing through a ${}_9C^9$ and a trisecant line; these together form a degenerate ${}_{11}C^{10}$, so that the primal is determinantal. It has twenty-three nodes, three (mapped by the neighbourhoods of the intersections of ${}_9C^9$ and the trisecant) at the intersections of π and R^3, six more in π, and fourteen others on R^3.

The determinantal character of V may be seen directly by noticing that it contains two systems of Bordiga sextic surfaces, residual with respect to cubic sections. In fact, a quadric through R^3 meets V again in a quintic surface F^5 of section genus 2, meeting π in a conic; F^5 and π thus make up a degenerate Bordiga sextic belonging to one of the families in question. A cubic primal through F^5 and π cuts out a surface of the residual system.

9. *Quartic primals containing three planes.*

When ϕ consists of three planes the representation of a V_3^4 through it has been obtained by Roth (*loc. cit.* in § 3), prime sections of V corresponding to sextic surfaces having in common three double lines, a transversal line g of these, and a ${}_9C^{14}$ meeting g in one point and each double line in nine. By means of the Cremona transformation, defined by cubic scrolls having g as a double line and containing the other three lines, we can reduce this system to a set of quintics with three triple points A, B, C (hence containing the lines joining pairs of these) and containing a curve of order 13 and genus 9 having quadruple points at A, B, C. It cannot, apparently, be reduced to a system of quartics. The primal has nodes at the three intersections of the planes in pairs, and at twenty-one other points, seven in each plane.

10. It is worthy of note that all the primals here described are general of their type, *i.e.* their *Konstantzahl* is $69-d$, d being the number of nodes that they possess. This may easily be verified from their definitions, bearing in mind the fact that in case (i) each primal contains ∞^3 surfaces ϕ, and in case (ii) ∞^1.

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RATIONAL QUARTIC PRIMALS AND ASSOCIATED CREMONA TRANSFORMATIONS OF FOUR-DIMENSIONAL SPACE

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[Received 19 June, 1936.—Read 12 November, 1936.]

Introduction.

It has been known for a long time that the general quartic primal in [4] is irrational, although its genera and plurigenera all vanish*. On the other hand, many *rational* quartic primals are known whose singularities present no conditions to the adjoint system. An extensive literature has recently accumulated concerning rational quartic primals whose only singularities are a finite number of nodes. The majority of these papers deal with the determinantal primal† and with its special cases, the exceptions being the work of d'Amico and Roth on the primal containing three planes‡, and the recent paper by Babbage on a primal with 24 nodes§. These investigations inevitably raise the question how many essentially distinct types of rational quartic primals exist whose only singularities are nodes, disregarding all loci which may be considered as special cases of a more general rational type. It is rather surprising to find that this number is apparently quite large. In the present paper we indicate some of the possibilities, and describe some primals which seem to be new. A complete catalogue of rational quartic primals does not appear, at the present time, to be a feasible undertaking.

If a V_3^4 of the type considered is rational, it must be capable of a representation on [3] by a linear system $|F|$ of surfaces of freedom and grade 4, the residual intersection of two surfaces being of genus 3, and since these surfaces map the prime sections of V_3^4 they must be regular and of genus 1, with canonical curve of order zero. It does not by any means follow that $|F|$ can be transformed by Cremona transformation into a linear system of quartic surfaces. For instance, the general quartic primal containing three planes is rational but is not so representable*, and some other examples are given in the preceding paper.

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* Fano, 1.

† The general determinantal primal was described by Roth, 1. Special cases occur in Roth, 2, and Todd, 1, 2, 3, 4, 5.

‡ d'Amico, 1; Roth, 3.

§ Babbage, 1.

We shall, however, confine our attention in this paper to the case in which $|F|$ is a system of quartic surfaces, and we shall further suppose that the general surface of $|F|$ is free from singularities. By making this latter restriction we exclude a large number of cases, but in fact this restriction is convenient, for the following reason. If the surfaces $|F|$ have a fixed double point (they cannot have a triple point or multiple curve, being of genus 1) this may be isolated or lie at a simple or double point of the base-curve. (If the base-curve has a triple point the system $|F|$ necessarily has a node there, but the system $|F|$ is then a special case of a system passing through a base curve of the same order without singularities.) If the node is at a double point of the base-curve we consider the larger system $|F_1|$ of surfaces which have a simple point there but otherwise satisfy the same conditions as $|F|$; this system is easily seen to be of freedom 5 and grade 6, with section genus 4, and to map the prime sections of a rational V_3^6 in [5], the intersection of a quadric and cubic primal, having a double point mapped by the double point of the base curve, and projecting from this point into V_3^4. Similarly, if the node of $|F|$ is at a simple point of the base curve, the system $|F_1|$ without a node there maps a V_3^8 in [6], the intersection of three quadrics, containing a line from which it projects into V_3^4. Finally, if the node of $|F|$ is isolated, V_3^4 arises as the projection, from a tangent [3], of a rational V_3^{12} in [8] whose curve sections are canonical curves of genus 7. In all these cases the primal arises as a fairly simple projection from a locus in higher space, but if $|F|$ has no fixed node the V_3^4 is no such simple projection. This distinction may seem arbitrary, but it is forced on us by the further practical consideration that, if we include systems where the general surface is singular, the number of systems which satisfy the conditions of having grade 4 and section genus 3 is unmanageably large.

In this paper, then, we consider linear systems of quartic surfaces of grade 4 and section genus 3, which are defined by their base curve C. C may itself be reducible or irreducible, but since we are confining our attention to general types of V_3^4 we may clearly suppose that each connected part of C is irreducible. The first task of the present paper is to enumerate such systems of surfaces, and describe the primals that they represent.

In the case of the determinantal primal it is known that the base-curve can be specialised in such a way that the quartic surfaces determine an involution and V_3^4 becomes a double quadric†. We show that this can happen in other cases, too; all that is necessary is that the residual intersection of two of the surfaces $|F|$ should be hyperelliptic. The conditions for this are determined in § 2. These involutions define certain involutory Cremona transformations of [3] which seem, in most cases, to be new.

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* Roth, 3.

† Montesano, 1.

Another point which emerges is that in certain of the non-involutory cases these Cremona transformations may still exist; it is known that they do so in the determinantal case*. These transformations have the property that all their fundamental curves of the first kind have equal multiplicities, and are free from isolated fundamental points; they afford excellent examples of fundamental curves of the second kind†.

We also discuss some allied Cremona transformations in [4], of which those of Godeaux‡ and Babbage§ are examples.

1. *Linear systems of quartic surfaces of grade* 4.

Consider the linear system $|F|$ in [3] formed by quartic surfaces passing through a curve C which has t irreducible components $C_1, \ldots, C_t$, no two of which intersect. Let n_i, p_i be the order and genus of C_i. It is easy to see that, as far as the residual intersection of two of the surfaces of $|F|$ is concerned, C may be regarded as a curve of order $n = \Sigma n_i$ and of genus $p = \Sigma p_i - t + 1$. It follows from the usual formula for residual intersections that the genus p' of the residual intersection of two surfaces of the system is given by $p' = p - 4n + 32$ and that the grade of $|F|$ is $62 - 8n + 2p$. Hence, if $|F|$ is of grade 4, we have $4n - p = 29$, and this automatically makes $p' = 3$. It also makes the postulation of C for $|F|$, namely $4n - p + 1$, equal to 30, so that the virtual dimension of $|F|$ is 4.

If the intersection of two of the surfaces of $|F|$, residual to C, is an irreducible curve of genus 3, its order is either four or it is not less than six. Also $p \geqslant 1 - n$, since the least virtual genus of a reducible curve of order n is $1 - n$, when the curve consists of n skew lines. Thus the relevant solutions (n, p) of the equation $4n - p = 29$ are

$$(12, 19), (10, 11), (9, 7), (8, 3), (7, -1), (6, -5).$$

We have now to consider how C may be made up of non-intersecting curves $C_1, \ldots, C_t$ so that its order and virtual genus belong to one of these classes, and so that it constitutes the entire base of a linear system of F^4. Since a C^{11} in [3] cannot have genus greater than 20, if ${}_{19}C^{12}$ consists of two parts these must be ${}_{20}C^{11}$ and a skew line, but ${}_{20}C^{11}$, being of maximum genus for its order, lies on a quadric, which forms part of every F^4 through the curve. Again, if ${}_{11}C^{10}$ is reducible, it must consist of ${}_{12}C^9$ and a skew line, and all the quartic surfaces through it contain the quadric on which ${}_{12}C^9$ lies. If ${}_7C^9$ breaks up it consists of a ${}_8C^8$ and a skew line, but ${}_8C^8$ is a (4, 4) curve on a quadric, with an actual double point, and since the line meets the quadric in two points the quadric again forms part of every F^4 through ${}_7C^9$. If ${}_3C^8$ is reducible it must consist either of ${}_4C^7$ and a line, or of ${}_4C^6$ and a conic; but in the latter case the F^4 drawn through the curve all contain a second conic, lying in the plane of the first, and passing through the intersections of this plane with ${}_4C^6$; thus ${}_3C^8$ is not the complete base. A ${}_{-1}C^7$, having virtual genus negative, is necessarily reducible. If it has two components these must be a rational sextic and a line, a rational quintic and a conic, or a rational quartic and a rational cubic. In the second of these cases the F^4 through ${}_{-1}C^7$ necessarily contain a second conic, and ${}_{-1}C^7$ is not the whole base. If ${}_{-1}C^7$ has more than two components the only possibility which does not give rise to further base curves for the system of quartics through ${}_{-1}C^7$ is an elliptic quintic and two skew lines. Finally, a ${}_{-5}C^6$ consists necessarily of six skew lines.

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* Roth, 1.

† Hudson, 1.

‡ Godeaux, 1.

§ Babbage, 1.

Thus the following nine cases arise as *a priori* possible base-curves for a system of quartic surfaces which map a $V_3{}^4$ in [4][*]:

(i) ${}_{19}C^{12}$, (ii) ${}_{11}C^{10}$, (iii) ${}_7C^9$, (iv) ${}_3C^8$, (v) ${}_4C^7+{}_0C^1$, (vi) ${}_0C^6+{}_0C^1$, (vii) ${}_0C^4+{}_0C^3$, (viii) ${}_1C^5+2{}_0C^1$, (ix) $6{}_0C^1$.

2. *Hyperelliptic curves of genus* 3.

In this section we inquire into the conditions under which the residual curve of intersection of two quartic surfaces drawn through one of the curves C enumerated above is hyperelliptic. The case (i) does not arise, since the only hyperelliptic ${}_3C^4$ is a double conic with eight branch points. The cases (ii) and (ix) have been solved already; in case (ii) it is necessary (and sufficient) that the ten points on any plane section of ${}_{11}C^{10}$ lie on a cubic curve[†], while in case (ix) the condition is that the six lines belong to a linear complex[‡]. We therefore consider cases (iii)–(viii).

We make the following preliminary remarks. Since on a non-singular quartic surface of [3] any linear system of curves is self-adjoint, the canonical series on any curve of the surface is the characteristic series of the linear system determined by the curve. If a curve is hyperelliptic, then any two points forming a pair of its g_2^1 present a single condition to all the curves of the same linear system. But this implies that *all* curves of the system are hyperelliptic, otherwise the canonical curve, the projective model of the canonical series, would have a double point, which is impossible.

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[*] *Cf.* Babbage, 1.

[†] Montesano, 1.

[‡] Todd, 2; Godeaux, 2.

We shall require to know the order of the (necessarily rational) scroll formed by the lines joining pairs of the g_2^1 on a hyperelliptic ${}_pC^n$. To obtain this, consider the correspondence on the curve between two points P, P', such that the join of P' to the companion of P in the g_2^1 meets an arbitrary line l. The indices of this correspondence are $(n-1, n-1)$ and its valency is -1; there are thus $2(n-1-p)$ coincidences. These coincidences occur at the intersections of ${}_pC^n$ with the generators of the scroll which meet l; thus the order of the scroll is $n-1-p$.

Now consider the cases (iii)–(viii) above in turn.

(iii) Suppose that the residual intersection of two F^4 through ${}_7C^9$ is a hyperelliptic ${}_3C^7$. The joins of pairs of the g_2^1 on this form a cubic scroll R^3, and it is easily seen that ${}_3C^7$ meets the double line l of R^3 in five points, the g_2^1 being cut out by planes through l. Hence any F^4 through ${}_3C^7$ contains l, and this line must form part of the base-curve. Since it meets ${}_3C^7$ in five points it must meet the rest of the base in a single point; this residual portion is thus a ${}_7C^8$. Conversely, if ${}_7C^9$ consists of a ${}_7C^8$ and a secant line, the residual ${}_3C^7$ of intersection of two F^4 through ${}_7C^9$ meets the line in five points, and is thus hyperelliptic. Hence ${}_3C^7$ *is hyperelliptic provided that* ${}_7C^9$ *breaks up into a* ${}_7C^8$ *and a secant line.*

(iv) Next suppose that the intersection of two F^4 through a ${}_3C^8$ is hyperelliptic. On any one of these F^4 we have two systems of ${}_3C^8$, $|C|$ and $|C'|$ say, of which $|C'|$ consists of hyperelliptic curves, such that $|C+C'|$ is the system of intersections of the surface with other quartic surfaces. Through each curve C' passes a rational quartic scroll R^4 generated by the joins of the g_2^1 on C', and this meets the F^4 in a ${}_3C^8$ belonging to the system $|C|$, which is hyperelliptic because it meets every generator of R^4 in two points. Hence *both* systems of ${}_3C^8$ on F^4 are hyperelliptic if one of them is, *i.e. the residual intersection of two* F^4 *through a* ${}_3C^8$ *is hyperelliptic if the original* ${}_3C^8$ *is hyperelliptic.*

(v) The ${}_3C^8$ of intersection of two F^4 passing through a ${}_4C^7$ and a line skew to this meets the line in six points and is therefore *necessarily* hyperelliptic, the g_2^1 being cut out by planes through the line.

(vi) If a ${}_3C^9$, the residual intersection of two F^4 passing through a ${}_0C^6$ and a skew line, is hyperelliptic, the ${}_3C^9$ cannot be irreducible. For if it were, the planes through the line would cut out a $g_3{}^1$ on the curve, and if the curve is hyperelliptic this would necessarily have a fixed point. Thus the ${}_3C^9$ must consist of a hyperelliptic ${}_3C^8$ and a line meeting this in one point, this line not meeting the original base line and hence being a 5-secant line of ${}_0C^6$. In this case the ${}_0C^6$ and its 5-secant form a degenerate ${}_4C^7$, and the figure reduces to a special case of the one considered in (v).

(vii) A ${}_3C^9$, the residual intersection of two F^4 through a ${}_0C^4$ and a ${}_0C^3$, meets ${}_0C^4$ in eighteen points and ${}_0C^3$ in fourteen, and its canonical series is cut out by quadrics through the ${}_0C^3$. If the ${}_3C^9$ is hyperelliptic the join of any pair of its $g_2{}^1$ must present a single condition to quadrics through the ${}_0C^3$, and must therefore be a chord of ${}_0C^3$. If ${}_3C^9$ were irreducible the joins of pairs of its $g_2{}^1$ would lie on an R^5, but no R^5 exists whose generators are chords of a twisted cubic. (All scrolls formed of chords of a twisted cubic have even order.) Hence the ${}_3C^9$ breaks up into a ${}_3C^8$ and a secant line which is a chord of the cubic and (since it meets the complete intersection of the two F^4 in six points) a trisecant of ${}_0C^4$. It therefore constitutes part of the base of $|F|$, which is a degenerate ${}_3C^8$, and the figure is a special case of (iv).

(viii) In a similar way, if the residual intersection of two F^4 through an elliptic quintic and two skew lines is hyperelliptic, it is easily seen that one of the trisecants of the quintic must meet both the lines, and once again the base is a degenerate ${}_3C^8$.

We may exhibit the results of this section in tabular form as follows.

Base-curve.	Condition for hyperelliptic residual section.
(ii) ${}_{11}C^{10}$	Points of any plane section lie on C^3.
(iii) ${}_7C^9$	${}_7C^9$ degenerates into ${}_7C^8$ and secant line.
(iv) ${}_3C^8$	${}_3C^8$ hyperelliptic.
(v) ${}_4C^7+{}_0C^1$	Always.
(vi) ${}_0C^6+{}_0C^1$	${}_0C^6$ has a 5-secant line.
(vii) ${}_0C^4+{}_0C^3$	Trisecant of ${}_0C^4$ is chord of ${}_0C^3$.
(viii) ${}_1C^5+2{}_0C^1$	Trisecant of ${}_1C^5$ meets both lines.
(ix) $6{}_0C^1$	Lines belong to linear complex.

In the cases (vi), (vii), (viii) the condition for a hyperelliptic curve of residual intersection implies the existence of base-curves other than those prescribed; and the linear systems $|F|$ so obtained are special cases of those arising under (iv) or (v). Since the system (ix) is equivalent under Cremona transformation to a special case of (ii)*, the essentially distinct cases are (ii), (iii), (iv), and (v). The involution determined in case (ii) is well known†; the other three appear to be new and are discussed below in § 4.

3. *The quartic primals in* [4].

Of the eight cases in which the system $|F|$ of quartic surfaces through one of the base-curves enumerated in § 1 maps the sections of a proper V_3^4 in [4], three‡ have already been described in detail and need no further mention here. The case (i) merely gives a primal with a triple point, the representation arising immediately by projection. The other four cases lead to new primals which are discussed here for the first time. The base-curves in question are (iv), (vi), (vii), (viii).

The primals under discussion have a finite number of nodes, represented by rational curves of order k which meet the base curve in $4k$ points. The number of these curves, in the cases $k = 1, 2$, may be deduced from known formulae§; in the cases where larger values of k arise the numbers are determined directly or by a degeneration argument.

We consider the four cases in order.

(iv) A ${}_3C^8$ in [3] possesses twenty-three quadrisecant lines and four 8-secant conics. There is also a unique rational cubic meeting the curve in twelve points. To see this, suppose that the ${}_3C^8$ degenerates into a ${}_3C^6$ and two unisecant lines l, l'; it is then easily seen that the degenerate curve possesses a unique 12-secant ${}_0C^3$, namely the curve of intersection of the two cubic surfaces containing ${}_3C^6$ and l, ${}_3C^6$ and l' respectively. It follows that the V_3^4 mapped by the F^4 through ${}_3C^8$ has twenty-eight nodes.

The neighbourhoods of points of ${}_3C^8$ represent the lines of a scroll on V_3^4 whose order is 28, since the residual intersection of two F^4 through ${}_3C^8$ meets ${}_3C^8$ in twenty-eight points. The trisecants of ${}_3C^8$ form a scroll of order 52 on which ${}_3C^8$ is 12-ple, and this maps a scroll on V_3^4 of order $8 . 52 - 28 . 12 = 80$. The 7-secant conics of ${}_3C^8$ form a surface of order 104 on which ${}_3C^8$ is 26-ple, mapping a scroll on V_3^4 of order $8 . 104 - 28 . 26 = 104$. The remaining lines on V_3^4 correspond to 11-secant cubics and 15-secant quartics of ${}_3C^8$, and we shall see later that these correspond to scrolls of orders 80, 28 respectively.

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* Todd, 2.

† Montesano, 1.

‡ (ii) gives the determinantal primal of Roth, 1; (iii) forms the subject of Babbage, 1, and (ix) of Todd, 2, wherein the primal is shown to be determinantal.

§ See Pascal, 1; Kap. 36, 8 (905 seq.); also W. E. Baker, 1.

To the points of a plane of [3] correspond on $V_3{}^4$ the points of an F^8 of section genus 3, which has a triple point at the node of $V_3{}^4$ mapped by the cubic, and (improper) nodes at the four nodes of $V_3{}^4$ mapped by conics*.

(vi) The composite curve ${}_0C^6+{}_0C^1$ possesses twenty-six quadrisecant lines, six being quadrisecants of ${}_0C^6$ and the other twenty being trisecants of ${}_0C^6$ which meet the line. The ${}_0C^6$ lies on a unique F^3, its quadrisecants forming one-half of a double-six. This F^3 maps a plane on $V_3{}^4$, and the pencil of planes through the line maps the F^3 residual to this plane with respect to prime sections. There are three conics meeting the line in two points and ${}_0C^6$ in six, and three twisted cubics meeting ${}_0C^6$ in ten points and having the line as a chord; these cubics lie on F^3 and pass through two of its intersections with ${}_0C^1$, and have the six lines of the other half of the double-six determined by the quadrisecants of ${}_0C^6$ as chords. Thus the $V_3{}^4$ has thirty-two nodes in all and contains a plane, nine of the nodes lying in the plane.

(vii) We pass now to the case ${}_0C^4+{}_0C^3$; there are six trisecants of ${}_0C^4$ which meet ${}_0C^3$, and twenty-one lines which are chords of both curves. There are no 8-secant conics of the combined curve (since its order is only 7 and no part of it lies in a plane). The residual ${}_3C^9$ of intersection of two F^4 through the combined curve meets ${}_0C^4$ in eighteen points and ${}_0C^3$ in fourteen. Thus the unique F^2 through ${}_0C^4$ has no variable intersection with ${}_3C^9$ and so represents a *line* of $V_3{}^4$, the sections of $V_3{}^4$ by primes through the line being mapped by quadrics through ${}_0C^3$. This line is easily seen to be double for $V_3{}^4$. The primal has therefore a double line and, in addition, twenty-one nodes at points not lying on this.

(viii) Finally, consider the case in which the base-curve consists of an elliptic quintic and two skew lines; the $V_3{}^4$ once more contains thirty-two nodes. Ten of these are mapped by trisecants of ${}_1C^5$ which meet one or other of the lines; fifteen by chords of ${}_1C^5$ which meet both lines, six by 5-secant conics of ${}_1C^5$ which meet one line and have the other as chord†, and one by a rational quartic curve having the lines as trisecants and meeting the quintic in ten points; this curve is the residual intersection of the two cubic surfaces through the quintic which contain one or other of the lines. These F^3 represent two planes on V_3^4 which meet in the node in question.

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* A special case of this primal is mentioned in Todd, 3; the octavic curve consists of six lines with a common transversal, the transversal, and another line meeting four of the six lines.

† See W. E. Baker, 1, formula 175 (originally due to Berzolari). Here and in the other cases we have omitted the verification that there are no ${}_0C^k$ of orders other than those mentioned meeting the base curve in $4k$ points.

4. *Involutions and associated Cremona transformations in* [3].

Consider now the involutions which arise when the base-curve is specialized in the manner indicated in § 2. From what was said there, it follows that the essentially new types of involution (other than the known one defined by a special ${}_{11}C^{10}$) arise in cases (iii), (iv), and (v). We proceed to consider these three involutions.

(iii) We consider the system of quartic surfaces passing through a ${}_7C^8$, γ say, and a line l meeting γ in one point. The curve γ, together with its unique quadrisecant line, forms the base of a pencil of cubic surfaces, any one of which, together with a plane through l, makes up a degenerate F^4 through the base-curve. The residual intersection of two F^4 through the base-curve meets the line in five points and γ in nineteen, and so has two variable intersections with any plane through l or any F^3 through γ. Thus the system of F^4 maps the prime sections of a double *quadric cone* in [4] whose two systems of generating planes (each covered twice) are mapped respectively on the planes through l and the F^3 through γ. Call these systems of planes the α and β systems respectively. An arbitrary plane of [3] maps an F^7 on the double cone, meeting each plane of the α system in a C^3 and each plane of the β system in a C^4. Such an F^7, on a quadric cone, is the residual, with regard to a quartic section of the cone, of a plane β. Thus the surface companion in the involution in [3] to a plane, which is mapped on the double cone by a surface occupying the same position in space as F^7 but lying on the other sheet, must be a surface of order 12, passing quadruply through l and triply through γ, this together with the plane and a cubic surface through γ mapping a quartic section of the double cone. Thus the base-system of the involution in [3] is composed of F^{12} with l quadruple and γ triple. These surfaces necessarily contain the twenty trisecants of γ which meet l, and contain doubly the three conics which meet γ in six points and have l as a chord. These latter curves are thus fundamental of the second kind. The residual intersection of two of the surfaces has order $12.12-9.8-16.1-20-3.2.4=12$, as it ought to.

Thus *surfaces of order* 12, *drawn to contain a triple* ${}_7C^8$ *and a quadruple line meeting it, determine an involutory Cremona transformation in* [3].

(iv) Now consider the system of quartic surfaces passing through a hyperelliptic ${}_3C^8$, γ. Each F^4 of the system is met by the other F^4 in a system of ∞^3 hyperelliptic ${}_3C^8$ of grade 4, whose projective model is a double quadric surface; thus the linear system of F^4 maps the prime sections of a double quadric in [4], this time not a cone since there are in general no reducible F^4 belonging to the system. Since a plane of [3] maps a surface of order 8, this surface, together with its companion on the double quadric, forms a quartic section. Thus in [3] the homaloidal system determining the involution consists of surfaces of order 15 passing quadruply through γ, and *ipso facto* containing its twenty-three quadrisecant lines, its four 8-secant conics (doubly), and its 12-secant cubic (triply); the residual intersection is, as it should be, of order 15.

If the ${}_3C^8$ is not hyperelliptic the involution no longer exists, but it is still true that the surfaces of order 15 which pass quadruply through it (and hence contain the other curves just mentioned) form a homaloidal system. (The existence of ∞^4 such F^{15} may be verified by evaluating the total postulation of the base-curves, as in Babbage, **1**, p. 17.) In the Cremona transformation defined by this system (which is a symmetrical one) the neighbourhoods of points of ${}_3C^8$ are transformed into rational quartics meeting the base of the reverse transformation in fifteen points, and trisecant lines of ${}_3C^8$ become 11-secant cubics of the base. We thus verify that the line-systems on $V_3{}^4$ corresponding to these quartics and cubics are of orders 28 and 80, as stated above.

(v) Consider finally the system of quartic surfaces passing through a ${}_4C^7$, γ, and a line l which does not meet it. ${}_4C^7$ possesses two quadrisecants, which together with the curve make up the base of a pencil of cubic surfaces, and any one of these together with a plane through l forms a degenerate F^4 passing through the base-curve. Thus the planes through l and the cubic surfaces through γ map the two systems of double planes, α, β, say, on a double quadric cone. An arbitrary plane maps a surface of order 8 on the double cone meeting planes of the system α in cubic curves and planes of the system β in quintics; together with its companion surface this must be the residual, with respect to quintic sections, of two planes β. Hence the involution in [3] is determined by surfaces of order 13, passing triply through γ and quintuply through l. These surfaces necessarily contain the twenty trisecants of γ which meet l, and the six 6-secant conics of γ which have l as a chord, each of these being double on F^{13}. The order of the residual intersection is then easily verified to be 13.

5. *Some associated Cremona transformations in* [4].

It has been known for some time that a symmetrical Cremona transformation exists in [4] in which the homaloidal primals are all determinantal*; this is, in fact, the bilinear transformation. Recently Babbage† has shown that a similar transformation exists whose homaloids are quartic primals represented by quartic surfaces through ${}_7C^9$. We consider now the possibility of deriving similar transformations from the primals obtained above.

5.1. We consider first a symmetrical Cremona $T_{4,4}$ in which the homaloids are 28-nodal primals V_3^4 of the type mapped in [3] by quartic surfaces through a general ${}_3C^8$. Let $|F|$ be the mapping system in [3] for one such V_3^4, γ the base curve, l_i $(i=1, \dots, 23)$ its quadrisecant lines, c_i $(i=1, \dots, 4)$ its 8-secant conics, and g its 12-secant cubic. Denote by L_i, C_i, G the corresponding nodes of V_3^4.

A plane π of [3] corresponds to a rational F^8 on V_3^4 whose prime sections are mapped in π by a particular system of ∞^4 quartic curves with eight base-points. This surface has improper nodes at C_i and an improper triple point at G; it passes simply through L_i. A quartic primal through F^8 must necessarily have nodes at C_i and G, and the tangent cone at G is fixed, since it must contain the three tangent planes to the three sheets of F^8 there. Thus the residual intersection of any such primal with V_3^4 is a surface ϕ^8, of sectional genus 3, with improper double points at C_i and an improper triple point at G. Its image in [3] is a surface of order 15, passing quadruply through γ, simply through l_i, doubly through c_i, and triply through g; in fact, it is one of the homaloids in the Cremona transformation of [3] obtained above (§4, iv).

On V_3^4 we thus have two ∞^3 systems of rational octavic surfaces F^8, ϕ^8, each of grade 1, the surfaces being of the same type, and two surfaces of opposite systems forming the complete intersection of V_3^4 and a quartic primal.

We now consider two [4]'s, Σ_1, Σ_2, and make the primes of Σ_2 correspond to the ∞^4 quartic primals in Σ_1 through a surface $\phi^8_{(1)}$ of the type described above, having an improper triple point $G^{(1)}$ and four improper nodes $C_i^{(1)}$ $(i=1, \dots, 4)$. Two quartic primals through $\phi^8_{(1)}$ meet residually in a surface $F^8_{(1)}$, with the same singularities as $\phi^8_{(1)}$, which meets $\phi^8_{(1)}$ in a curve of order 28 having double points on each sheet at $C_i^{(1)}$ and a triple point on each sheet at $G^{(1)}$. A third quartic primal through $\phi^8_{(1)}$ meets $F^8_{(1)}$ residually in a rational quartic curve which has fifteen intersections with $\phi^8_{(1)}$. A prime in Σ_1 corresponds to a 28-nodal quartic primal in Σ_2, and a plane to an octavic surface $F^8_{(2)}$ of type F^8. The homaloidal quartic primals in Σ_2 have therefore a surface $\phi^8_{(2)}$ of type F^8 as base, and the transformation is completely symmetrical. $\phi^8_{(2)}$ has an improper triple point $G^{(2)}$ and four improper double points $C_i^{(2)}$.

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* See Godeaux, 1, and Todd, 1, for the general case.

† Babbage, 1.

A general point of the surface $\phi^8_{(1)}$ gives rise to a quadrisecant line of $\phi^8_{(2)}$. The quartic primals through $\phi^8_{(1)}$ have a fixed tangent cone at $G^{(1)}$, and it is only one further condition for them to have a triple point there. The neighbourhood of $G^{(1)}$ corresponds to the point $G^{(2)}$, and the generators of the tangent cone correspond to the generators of the fixed tangent cone at $G^{(2)}$.

Consider the neighbourhood of $G^{(1)}$ on a particular member $V_3{}^4$ of the homaloidal system in Σ_1. It may conveniently be regarded as an infinitesimal quadric surface ρ. The other primals of the system contain three fixed generators of one system of ρ and meet it residually in three variable generators of the second system. The generators of the second system correspond therefore to the points of a twisted cubic curve in Σ_2. The twisted cubics which arise by varying $V_3{}^4$ are the prime sections of a rational cubic cone $R^3_{(2)}$ of lines through $G^{(2)}$ which is fundamental for the transformation. $\phi^8_{(2)}$ projects from $G^{(2)}$ into a rational quintic surface with a double twisted cubic, and $R^3_{(2)}$ is the cone of lines through $G^{(2)}$ which meet $\phi^8_{(2)}$ again in two points and give rise to the points of this cubic. $R^3_{(2)}$ meets $\phi^8_{(2)}$ in a 12-ic curve of genus 4 having a double point on each sheet at $G^{(2)}$. The points of the 12-ic curve correspond to the generators of the tangent quadric cone at $G^{(1)}$ which meet $\phi^8_{(1)}$ again; these generators form a septimic conical surface of genus 4 which meets $\phi^8_{(1)}$ in a 16-ic of genus 4 having triple points on each sheet at $G^{(1)}$. The neighbourhood of $C_1^{(1)}$ corresponds to a fundamental quadric surface $q_1^{(2)}$ lying in the prime containing $G^{(2)}$, $C_2^{(2)}$, $C_3^{(2)}$, $C_4^{(2)}$, and containing the ${}_3C^8$ with a triple point (with coplanar tangents) and three double points in which this prime meets $\phi^8_{(2)}$.

5.2. We consider next a symmetrical Cremona $T_{4,4}$ of [4] in which the general homaloid $V_3{}^4$ is 32-nodal and is represented by the system $|F|$ of quartic surfaces passing through a rational sextic γ and a line l. γ has six quadrisecants $l_1, \ldots, l_6$ and twenty trisecants $p_1, \ldots, p_{20}$ which meet l. There are three conics c_1, c_2, c_3 meeting γ in six points and l in two, and three twisted cubics g_1, g_2, g_3 10-secant to γ and having l as chord. The unique cubic surface F^3 through γ contains $l_1, \ldots, l_6$ and g_1, g_2, g_3, and represents a plane ϖ of $V_3{}^4$ through the nine nodes which correspond to these.

A plane π of [3] corresponds to an F^9 mapped by a particular system of ∞^4 quartics with seven base-points. It has three improper double points C_1, C_2, C_3 and three improper triple points G_1, G_2, G_3 corresponding to the intersections of π with $c_1, c_2, c_3, g_1, g_2, g_3$ respectively. F^9 passes simply through the remaining nodes L_i, P_i of $V_3{}^4$ which arise from the lines l_i, p_i; it meets the plane ϖ in a ${}_1C^6$ having triple points at G_1, G_2, G_3 and passing through $L_1, \ldots, L_6$.

A second quartic primal through F^9 must necessarily contain ϖ. Its residual intersection with $V_3{}^4$ is a sextic surface R^6 with rational prime sections, hence necessarily a scroll; such an R^6 has six improper nodes, which in this case are at the points G_i, C_i. R^6 passes simply through P_i. Its image in [3] is an F^{12} passing triply through γ and quadruply through l, and, as a consequence, doubly through g_i, c_i, and simply through p_i. The R^6 lies on a unique cubic primal which has nodes at the double points of R^6 and is generated by its quadrisecant lines; this primal meets $V_3{}^4$ again in a second sextic scroll H^6 mapped by the neighbourhood of the line l. A general point X of H^6 corresponds to the neighbourhood of a particular point x of l in a particular plane ξ through l; and as ξ varies, x remaining fixed, X describes a generator of H^6. There are six pairs of neighbourhoods (x, ξ) each offering only one condition to a quartic surface of the system $|F|$, and corresponding therefore to the six improper nodes of H^6. One such pair consists of the neighbourhoods of the two points (c_i, l) in the plane of c_i, and another of the neighbourhoods of the two points (g_i, l) in the planes which contain l and the tangents to g_i at these points.

We are thus led as before to a symmetrical $T_{4,4}$ in [4]. In one space, Σ_1, the homaloids are quartic primals whose base consists of a rational sextic scroll $R^6_{(1)}$, and the plane $\varpi_{(1)}$ containing three of its improper double points $G_1^{(1)}$, $G_2^{(1)}$, $G_3^{(1)}$. The quartic primals are all 32-nodal, with fixed nodes at the other improper nodes $C_i^{(1)}$ of $R^6_{(1)}$ and at $G_i^{(1)}$, and at these latter points the primals have fixed tangent cones; they have six nodes variable on $\varpi_{(1)}$ and twenty variable on $R^6_{(1)}$. In the reverse system we have a similar base $R^6_{(2)}$ and $\varpi_{(2)}$. The points of $\varpi_{(2)}$ arise from the quadrisecants of $R^6_{(1)}$, which generate a determinantal cubic primal; and the points of $R^6_{(2)}$ arise from trisecants of $R^6_{(1)}$ which meet $\varpi_{(1)}$. The neighbourhood of $G_i^{(1)}$ gives the point $G_i^{(2)}$. The cubic cone of lines through $G_i^{(2)}$ which meet $R^6_{(2)}$ in two further points is fundamental and arises from $G_i^{(1)}$ in a way similar to that described in 5.1. The neighbourhood of $C_1^{(1)}$ corresponds to a fundamental quadric surface lying in the prime through $\varpi_{(1)}$ and $C_1^{(2)}$, and containing the 4-nodal ${}_0C^6$ in which this prime meets $R^6_{(2)}$.

We may, as a check, observe that the calculated freedom of quartic primals through $R^6_{(1)}$ and $\varpi_{(1)}$ reduces to four. The postulation of a rational sextic scroll to quartic primals is 59, and the plane imposes a further six conditions. Since the total freedom of quartic primals in [4] is 69 this leaves 4 as the virtual freedom of primals through $R^6_{(1)}$ and $\varpi_{(1)}$. Since the system in question has grade 1, the actual postulation must be equal to the virtual.

5.3. We cannot obtain a $T_{4,4}$ in the same way from the primal mapped by quartic surfaces through a rational quartic and a twisted cubic. For this primal has a double line λ, mapped by the quadric through ${}_0C^4$, and twenty-one other nodes. A plane of [3] maps a surface of order 9 having λ as quadruple line and passing simply through the other nodes of $V_3{}^4$. A second quartic primal through F^9 must have λ as a double line, and must have the same tangent cone as $V_3{}^4$ at every point of λ. The residual intersection of this primal with $V_3{}^4$ is thus mapped by the quadric through ${}_0C^4$ counted four times, together with a septimic surface on which ${}_0C^3$ is quadruple. No such surface, however, can exist, since it would have to contain every chord of ${}_0C^3$.

Alternatively, we may note that there cannot exist a homaloidal system of quartic primals, including $V_3{}^4$, which have λ as a double line; for the intersection of $V_3{}^4$ with another primal of the system would be mapped on [3] by the F^2 through ${}_0C^4$ counted four times and an octavic surface on which ${}_0C^3$ was quadruple, hence necessarily ruled. No linear system of such surfaces can be homaloidal, since they are all compounded of chords of ${}_0C^3$. If, then, $V_3{}^4$ is one of a homaloidal system of quartic primals, the double lines of these must be variable on a base surface of the system, and in any case we do not obtain the representation of a primal of the system on [3] by taking a prime section of the homaloidal system, as in the cases considered above.

5.4. Consider finally the case of the 32-nodal quartic $V_3{}^4$ mapped by the system $|F|$ of quartics passing through an elliptic quintic γ and two lines l, l'. There are five trisecants p_i of γ which meet l, and five p_i' which meet l'; fifteen chords l_i of γ which meet l and l'; three 5-secant conics c_i of γ which meet l, and three c_i' which meet l'; and a rational quartic g meeting γ in ten points and having l, l' as trisecants. Denote by capital letters the corresponding nodes of $V_3{}^4$. The F^3 through γ and l maps a plane ϖ of $V_3{}^4$ passing through G, P_i, C_i; and the F^3 through γ and l' maps a plane ϖ' passing through G, P_i', C_i'.

A plane π of [3] corresponds to a rational F^9 of section genus 3, having double points at C_i, C_i' and a quadruple point at G, and passing simply through the other nodes of $V_3{}^4$. The tangent planes to the four sheets at G are generating planes of the tangent cone to $V_3{}^4$ at this point. F^9 meets ϖ in a ${}_1C^6$ with a quadruple point at G, double points at C_i, and simple points at P_i; and it has a similar intersection with ϖ'. A quartic primal through F^9 necessarily contains ϖ and ϖ'; the residual intersection with $V_3{}^4$ is a quintic surface ϕ^5 with elliptic sections, having an improper node at G and passing simply through C_i, C_i', L_i. It is, therefore, a projected del Pezzo surface (since the elliptic quintic scroll has no improper nodes). Its image in [3] is an F^9 passing doubly through γ, triply through l and l', doubly through g, and simply through c_i, c_i', l_i.

There is a unique cubic primal Ψ^3 through ϕ^5 and ϖ. It is generated by trisecants of ϕ^5 which meet ϖ and it has nodes at C_i, G. At G it has the same tangent quadric cone, Ω say, as $V_3{}^4$; ϖ, ϖ', and the tangent planes at G to the two sheets of ϕ^5 are planes of one system on Ω. Ω meets ϕ^5 in a rational decimic curve with triple points on each sheet at G, and this curve is projected from G by a rational quartic conical surface R^4, of which the lines GC_i, GC_i' are generators. R^4 must necessarily lie on Ψ^3, and the sextic cone of genus 4 generated by the lines of Ψ^3 which pass through G must break up into R^4, ϖ, and some other plane p through G which contains three generators of R^4. In the representation in [3] of Ψ^3 obtained by direct projection from G prime sections of Ψ^3 correspond to cubic surfaces containing a rational quartic curve and two of its trisecants. ϕ^5 itself projects into a cubic surface passing only through the quartic. In addition to the nodes at G, C_i, Ψ^3 also has nodes at the three further intersections of p with ϕ^5. Thus Ψ^3 has nodes at six simple points of ϕ^5, a number which is in agreement with the general formula for the number of nodes of a primal in [4] at simple points of a surface which it contains*.

There is a similar 7-nodal cubic primal Ψ'^3 containing ϕ^5 and ϖ', and the complete intersection of Ψ^3 and Ψ'^3 consists of ϕ^5 and R^4.

For the corresponding symmetrical Cremona $T_{4,4}$ the homaloidal quartic primals in Σ_1 are all of the above 32-nodal type. The base surface consists of a projected del Pezzo quintic surface $\phi^5_{(1)}$ with an improper double point $G^{(1)}$, a plane $\varpi_{(1)}$ through $G^{(1)}$ meeting $\phi^5_{(1)}$ again in $C_1^{(1)}$, $C_2^{(1)}$, $C_3^{(1)}$, and a plane $\varpi'_{(1)}$ through $G^{(1)}$ meeting $\phi^5_{(1)}$ again in $C_1'^{(1)}$, $C_2'^{(1)}$, $C_3'^{(1)}$; $\varpi_{(1)}$, $\varpi'_{(1)}$ and the tangent planes to the two sheets of $\phi^5_{(1)}$ at $G^{(1)}$ being generating planes of the same system of a quadric cone. The homaloids have double points at $G^{(1)}$, $C_i^{(1)}$, $C_i'^{(1)}$, with a fixed tangent quadric cone at $G^{(1)}$. The other twenty-five nodes are variable on the base surfaces, fifteen on $\phi^5_{(1)}$ and five on each of $\varpi_{(1)}$, $\varpi'_{(1)}$. The neighbourhood of $C_1^{(1)}$ corresponds to a fundamental quadric surface in Σ_2 lying in the prime through $\varpi'_{(2)}$ and $C_1^{(2)}$; this prime cuts $\phi^5_{(2)}$ in an elliptic quintic curve with a node, through which the quadric surface in question passes.

The rational quartic conical surface $R^4_{(2)}$ consisting of the generators of the fixed tangent quadric cone at $G^{(2)}$ which meet $\phi^5_{(2)}$ again is fundamental. The points of a prime section of $R^4_{(2)}$ correspond to the generators of one system of the infinitesimal quadric surface which constitutes the neighbourhood of $G^{(1)}$ on the corresponding prime in Σ_1.

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* H. F. Baker, *Principles of geometry* (Cambridge, 1933), 6, 263–5.

The points of $\varpi_{(2)}$ arise from the trisecants of $\phi^5_{(1)}$ which meet $\varpi_{(1)}$, generators of a 7-nodal cubic primal $\Psi^3_{(1)}$.

We may check the freedom of the homaloidal system of quartic primals in Σ_1. The postulation of $\phi^5_{(1)}$ is 50, and three more conditions are necessary to fix the tangent cone at $G^{(1)}$. The primals then meets the generating planes of the cone in quartic curves with a fixed triple point at $G^{(1)}$, and three fixed simple points at the further intersections of the plane with $\phi^5_{(1)}$; $\varpi_{(1)}$ and $\varpi'_{(1)}$ thus impose six more conditions each on the system. The total postulation is thus 65, and the virtual freedom of the system is 4; and since the grade is 1 the actual freedom must be exactly four.

References.

d'Amico, F., **1.** "Sulla varietà quartica con tre piani simplici nello spazio a quattro dimensioni", *Atti Acc. Gioenia in Catania* (4), 18 (1906), no. 11.

Babbage, D. W., **1.** "Some quartic primals and associated Cremona transformations of four-dimensional space", *Proc. Camb. Phil. Soc.*, 32 (1936), 12–22.

Baker, H. F., **1.** *Principles of geometry*, 6 (Cambridge, 1933).

Baker, W. E., **1.** "Conics satisfying eight conditions", *Proc. London Math. Soc.* (2), 36 (1934), 96–116.

Fano, G., **1.** "Sopra alcune varietà algebriche a tre dimensioni aventi tutti i generi nulli", *Atti Acc. Torino*, 43 (1908), 973–984.

Godeaux, L., **1.** "Sur une correspondance crémonienne entre deux espaces à n dimensions", *Rend. Ist. Lomb.* (2), 43 (1910), 116–119.

————, **2.** "Sur les surfaces du quatrième ordre passant par six droites", *Bull. Soc. Roy. Sci. Liége*, 4 (1935), 37–39.

Hudson, H. P., **1.** *Cremona transformations* (Cambridge, 1927).

Montesano, D., **1.** "Sulle trasformazioni involutorie dello spazio che determinano un complesso lineare di rette", *Rend. Acc. Lincei* (4), 4 (1888), 207–215.

Pascal, E., **1.** *Repertorium der höheren Mathematik*, II 2 (Leipzig, 1922).

Roth, L., **1.** "A quartic form in four dimensions", *Proc. London Math. Soc.* (2), 30 (1930), 118–126.

————, **2.** "A class of quartic forms in four dimensions", *ibid.*, 30 (1930), 305–318.

————, **3.** "Some quartic forms containing planes", *ibid.*, 35 (1933), 249–256.

Todd, J. A., **1.** "The quarto-quartic transformation of four-dimensional space associated with certain projectively generated loci", *Proc. Camb. Phil. Soc.*, 26 (1930), 323–333.

————, **2.** "Configurations defined by six lines in space of three dimensions", *ibid.*, 29 (1933), 52–68.

————, **3.** "The figure formed by six lines in space with a common transversal", *Journal London Math. Soc.*, 10 (1935), 194–199.

————, **4.** "A note on two special primals in four dimensions", *Quart. J. of Math.* (Oxford), 6 (1935), 128–136.

————, **5.** "On a quartic primal with forty-five nodes, in space of four dimensions", *ibid.*, 7 (1936), 168–174.

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THE CHARACTERS OF THE SYMMETRIC GROUP OF DEGREES 12 AND 13

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[Received 12 March, 1936.—Read 19 March, 1936.]

Several methods have been developed by Littlewood and Richardson for the calculation of characters of the symmetric group in their papers (1) "Group characters and algebra", *Phil. Royal Soc.* (A), 233 (1934), 99–141; (2) "Immanants of some special matrices", *Quart. J. of. Math.* (Oxford), 5 (1934), 269–282.

The characters of the symmetric group of order 9! inclusive have been computed by them. The table of characters for the group of degree 10 has been constructed by Littlewood* and the table of 11th degree by the author†.

Following the methods of Littlewood and Richardson, I have constructed the tables of 12th and 13th degrees; the first half of each table given here and the other halves are easily completed as indicated below.

For the 12th table, the characteristics corresponding to the partitions

$[43^2 2]$, $[3^4]$, $[4^2 21^2]$, $[43^2 1^2]$, $[52^3 1]$, $[432^2 1]$, $[3^3 21]$, $[42^4]$, $[3^2 2^3]$,

$[531^4]$, $[4^2 1^4]$, $[52^2 1^3]$, $[4321^3]$, $[3^3 1^3]$, $[42^3 1^2]$, $[3^2 2^2 1^2]$, $[32^4 1]$, $[2^6]$,

$[61^6]$, $[521^5]$, $[431^5]$, $[42^2 1^4]$, $[3^2 21^4]$, $[32^3 1^3]$, $[2^5 1^2]$, $[51^7]$, $[421^6]$,

$[3^2 1^6]$, $[32^2 1^5]$, $[2^4 1^4]$, $[41^8]$, $[321^7]$, $[2^3 1^6]$, $[31^9]$, $[2^2 1^8]$, $[21^{10}]$, of the other half are obtained by multiplying the corresponding characteristics for the conjugate partitions, namely

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* *Proc. London Math. Soc.* (2), 39 (1935), 150–199 (182).

† *Proc. London Math. Soc.* (2), 39 (1935), 200–204.

$[4^2 31]$, $[4^3]$, $[532^2]$, $[53^2 1]$, $[541^3]$, $[5421]$, $[543]$, $[5^2 1^2]$, $[5^2 2]$, $[62^2 1^2]$,
$[62^3]$, $[631^3]$, $[6321]$, $[63^2]$, $[641^2]$, $[642]$, $[651]$, $[6^2]$, $[71^5]$, $[721^3]$, $[72^2 1]$,
$[731^2]$, $[732]$, $[741]$, $[75]$, $[81^4]$, $[821^2]$, $[82^2]$, $[831]$, $[84]$, $[91^3]$, $[921]$,
$[93]$, $[10, 1^2]$, $[10, 2]$, $[11, 1]$,

respectively, by the characteristics of $[1^{12}]$ given in the last row of the table.

For the 13th table the characteristics corresponding to the partitions,

$[43^3]$, $[4^2 31^2]$, $[532^2 1]$, $[4^2 2^2 1]$, $[43^2 21]$, $[3^4 1]$, $[52^4]$, $[432^3]$, $[3^3 2^2]$,
$[541^4]$, $[62^2 1^3]$, $[5321^3]$, $[4^2 21^3]$, $[43^2 1^3]$, $[52^3 1^2]$, $[432^2 1^2]$, $[3^3 21^2]$,
$[42^4 1]$, $[3^2 2^3 1]$, $[32^5]$, $[621^5]$, $[531^5]$, $[4^2 1^5]$, $[52^2 1^4]$, $[432 1^4]$, $[3^3 1^4]$,
$[42^3 1^3]$, $[3^2 2^2 1^3]$, $[32^4 1^2]$, $[2^6 1]$, $[61^7]$, $[521^6]$, $[431^6]$, $[42^2 1^5]$, $[3^2 21^5]$,
$[32^3 1^4]$, $[2^5 1^3]$, $[51^8]$, $[421^7]$, $[3^2 1^7]$, $[32^2 1^6]$, $[2^4 1^5]$, $[41^9]$, $[321^8]$,
$[2^3 1^7]$, $[31^{10}]$, $[2^2 1^9]$, $[21^{11}]$,

of the other half are obtained by multiplying the corresponding characteristics for the conjugate partitions, namely

$[4^3 1]$, $[53^2 2]$, $[5421^2]$, $[542^2]$, $[5431]$, $[54^2]$, $[5^2 1^3]$, $[5^2 21]$, $[5^2 3]$, $[62^3 1]$,
$[631^4]$, $[6321^2]$, $[632^2]$, $[63^2 1]$, $[641^3]$, $[6421]$, $[643]$, $[651^2]$, $[652]$, $[6^2 1]$,
$[721^4]$, $[72^2 1^2]$, $[72^3]$, $[731^3]$, $[7321]$, $[73^2]$, $[741^2]$, $[742]$, $[751]$, $[76]$,
$[81^5]$, $[821^3]$, $[82^2 1]$, $[831^2]$, $[832]$, $[841]$, $[85]$, $[91^4]$, $[921^2]$, $[92^2]$, $[931]$,
$[94]$, $[10, 1^3]$, $[10, 21]$, $[10, 3]$, $[11, 1^2]$, $[11, 2]$, $[12, 1]$,

respectively, by the characteristics of $[1^{13}]$, given in the last row of the table.

Hence both the tables are complete.

The figures have been checked by using the orthogonal properties of the characters

$$\begin{aligned} \Sigma \psi_{\rho}^{(\mu)} \psi_{\rho'}^{(\mu)} &= 0 \qquad (\rho \neq \rho') \\ &= h/h_{\rho} \quad (\rho = \rho'), \end{aligned}$$

h denoting the order of the symmetric group, and h_{ρ} the order of the class ρ.

Table of characters of order 12 !

Class	1^{12}	$1^{10}\,2$	$1^{9}\,3$	$1^{8}\,4$	$1^{8}\,2^{2}$	$1^{7}\,23$	$1^{7}\,5$	$1^{6}\,6$	$1^{6}\,24$	$1^{6}\,2^{3}$	$1^{6}\,3^{2}$
Order	1	66	440	2970	1485	15840	19008	1 10880	83160	13860	36960
[12]	1	1	1	1	1	1	1	1	1	1	1
[11, 1]	11	9	8	7	7	6	6	5	5	5	5
[10, 2]	54	36	27	20	22	15	14	9	10	12	9
$[10, 1^2]$	55	35	28	21	19	14	15	10	9	7	10
[93]	154	84	49	28	42	21	14	5	10	20	7
[921]	320	160	104	64	64	34	35	16	16	16	14
$[91^3]$	165	75	57	35	21	15	20	10	5	−5	12
[84]	275	125	50	15	55	20	0	−5	5	25	5
[831]	891	351	162	69	111	36	21	0	9	27	0
$[82^2]$	616	224	112	56	56	14	21	5	4	8	−5
$[821^2]$	945	315	189	91	49	21	35	9	1	−21	9
$[81^4]$	330	90	78	34	−6	6	15	5	−6	−22	15
[75]	297	117	27	−7	49	15	−13	−9	1	21	9
[741]	1408	448	112	0	128	28	−22	−16	0	32	4
[732]	1925	525	140	35	105	0	0	−5	−5	25	−25
$[731^2]$	2376	576	216	56	56	6	1	−9	−4	−24	−9
$[72^2\,1]$	2079	441	189	77	7	−21	14	0	−5	−15	−18
$[721^3]$	1728	288	216	64	−64	−6	13	0	−16	−48	18
$[71^5]$	462	42	84	14	−42	0	7	1	−14	−14	21
$[6^2]$	132	48	6	−8	20	6	−8	−4	0	8	6
[651]	1155	315	21	−49	91	21	−35	−11	−1	19	21
[642]	2673	567	0	−63	117	0	−27	0	−9	27	0
$[641^2]$	3080	560	56	−56	56	14	−35	−9	4	−24	11
$[63^2]$	1650	300	−15	−20	50	−15	0	5	−10	20	−15
[6321]	5632	768	64	0	0	−48	−8	0	0	0	−32
$[631^3]$	3696	336	168	0	−112	0	−14	−6	0	−48	6
$[62^3]$	1925	175	35	35	−35	−35	0	0	5	15	−10
$[62^2\,1^2]$	3564	216	162	48	−132	−36	−6	0	0	0	0
$[5^2\,2]$	1320	240	−48	−56	56	6	−15	5	−4	8	15
$[5^2\,1^2]$	1485	225	−27	−55	29	15	−20	0	5	−15	18
[543]	2112	288	−120	−64	64	−6	7	16	−16	16	6
[5421]	5775	525	−210	−105	35	0	0	15	5	−15	15
$[541^3]$	3520	160	−8	−64	−64	34	−15	0	16	−48	10
$[53^2\,1]$	4158	252	−189	−28	14	−21	28	9	−10	12	−9
$[532^2]$	4455	135	−162	21	−21	−36	15	0	9	27	0
$[4^3]$	462	42	−42	−14	14	0	7	5	−6	2	3
$[4^2\,31]$	2970	90	−216	−34	34	6	35	9	−14	−6	9
*$[621^4]$	2100	0	210	0	−140	0	0	0	−20	0	30
*$[5321^2]$	7700	0	−70	0	−140	0	0	0	20	0	−10
*$[4^2\,2^2]$	2640	0	−168	0	16	0	20	0	0	0	12
$[1^{12}]$	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

Table of characters of order 12 !—*continued.*

Class	$1^5\,25$	$1^5\,2^2\,3$	$1^5\,34$	$1^5\,7$	$1^4\,8$	$1^4\,4^2$	$1^4\,2^2\,4$	$1^4\,26$	$1^4\,23^2$	$1^4\,35$	$1^4\,2^4$	$1^3\,45$	$1^3\,3^3$
Order	3 99168	1 66320	3 32640	5 70240	24 94800	6 23700	6 23700	16 63200	5 54400	13 30560	51975	39 91680	4 92800
$[12]$	1	1	1	1	1	1	1	1	1	1	1	1	1
$[11, 1]$	4	4	4	4	3	3	3	3	3	3	3	2	2
$[10, 2]$	6	7	5	5	2	2	4	3	3	2	6	0	0
$[10, 1^2]$	5	4	6	6	3	3	1	2	2	3	−1	1	1
$[93]$	4	9	1	0	−2	−2	4	1	3	−1	10	−2	1
$[921]$	5	4	4	5	0	0	0	0	−2	−1	0	−1	−4
$[91^3]$	0	−3	5	4	1	1	−5	−2	0	2	−11	0	3
$[84]$	0	10	0	−5	−3	−1	3	−1	5	0	11	0	5
$[831]$	1	6	−6	−5	−3	−5	1	0	0	−3	3	−1	0
$[82^2]$	−1	−4	−4	0	0	0	0	−1	−7	−3	8	1	−5
$[821^2]$	−5	−11	1	0	−1	1	−5	−3	−3	−1	−15	1	0
$[81^4]$	−5	−6	4	1	0	−2	−6	−3	3	3	−6	−1	6
$[75]$	−3	7	5	−4	−1	5	1	−3	3	2	9	3	0
$[741]$	−2	8	0	−6	0	0	0	0	4	2	0	0	4
$[732]$	0	0	−10	0	3	−3	−1	1	−3	0	13	0	−1
$[731^2]$	1	−4	−4	−4	0	0	0	3	3	1	−24	1	0
$[72^2\,1]$	−4	−11	−1	0	1	7	5	0	−6	−1	15	2	0
$[721^3]$	−7	−4	4	−1	0	0	0	0	6	1	0	−1	0
$[71^5]$	−3	0	2	0	0	−6	−2	−1	3	4	14	−1	3
$[6^2]$	−2	2	4	−1	0	4	0	−2	0	1	4	2	−3
$[651]$	−5	1	11	0	1	7	−1	−3	−3	1	3	1	−6
$[642]$	−3	0	0	6	3	−3	−3	0	0	0	9	−3	0
$[641^2]$	5	−4	4	0	0	0	0	3	−1	1	−24	−1	2
$[63^2]$	0	5	−5	+5	2	−2	−4	−1	3	0	2	0	3
$[6321]$	8	0	0	4	0	0	0	0	0	4	0	0	−2
$[631^3]$	6	8	0	0	0	0	0	2	6	−2	−16	0	−3
$[62^3]$	0	−5	5	0	−1	5	3	0	−2	0	21	0	8
$[62^2\,1^2]$	−4	6	6	1	0	4	8	0	0	−3	12	−2	0
$[5^2\,2]$	−5	−4	4	4	0	0	0	−1	−3	−3	8	−1	−3
$[5^2\,1^2]$	0	−7	5	1	−1	1	1	0	−6	−2	−3	0	0
$[543]$	−7	4	−4	5	0	0	0	0	6	−5	0	1	6
$[5421]$	0	−10	0	0	−3	−5	3	1	−3	0	7	0	−3
$[541^3]$	5	−4	−4	−1	0	0	0	0	−2	−3	0	1	10
$[53^2\,1]$	2	11	−1	0	−2	2	−4	−3	3	1	−18	2	0
$[532^2]$	5	−6	6	−4	−3	−1	−7	0	0	3	15	1	0
$[4^3]$	−3	2	−2	0	0	2	2	1	3	−2	−2	1	3
$[4^2\,31]$	−5	4	−4	−5	0	6	6	3	3	−1	−6	1	0
$*[621^4]$	0	10	0	0	0	−4	0	2	0	0	20	0	−6
$*[5321^2]$	0	10	0	0	0	−4	0	−6	0	0	−12	0	−4
$*[4^2\,2^2]$	0	−8	0	−6	0	0	0	4	0	2	16	0	−6
$[1^{12}]$	−1	1	−1	1	−1	1	−1	1	−1	1	1	−1	1

* Denotes a self-associated partition and character.

Table of characters of order 12 !—*continued.*

Class	$1^3\,36$	$1^3\,234$	$1^3\,27$	$1^3\,2^2\,5$	$1^3\,2^3\,3$	$1^3\,9$	$1^2\,10$	$1^2\,82$	$1^2\,62^2$	$1^2\,42^3$	$1^2\,2^5$	$1^2\,73$	$1^2\,532$
Order	44 35200	33 26400	57 02400	19 95840	5 54400	88 70400	239 50080	149 68800	49 89600	12 47400	62370	114 04800	79 83360
[12]	1	1	1	1	1	1	1	1	1	1	1	1	1
[11, 1]	2	2	2	2	2	2	1	1	1	1	1	1	1
[10, 2]	0	1	1	2	3	0	−1	0	1	2	4	−1	0
$[10, 1^2]$	1	0	0	−1	−2	1	0	−1	−2	−3	−5	0	−1
[93]	−1	1	0	2	5	−2	−1	0	1	2	4	0	1
[921]	−2	−2	−1	−1	−2	−1	0	0	0	0	0	−1	−1
$[91^3]$	1	−1	−2	−4	−5	0	0	−1	−2	−3	−5	1	0
[84]	1	2	−1	0	4	−1	0	−1	−1	1	5	1	0
[831]	0	0	1	1	0	0	1	1	0	−3	−9	1	1
$[82^2]$	−1	−2	0	1	2	1	1	0	1	4	16	0	−1
$[821^2]$	0	1	0	−1	−3	0	0	1	1	1	−5	0	1
$[81^4]$	2	0	−1	−1	2	0	0	0	1	2	10	1	1
[75]	0	1	−2	−1	3	0	0	−1	−1	1	5	−1	0
[741]	2	0	0	−2	−4	1	0	0	0	0	0	0	−2
[732]	1	−2	0	0	4	2	0	−1	−1	−1	5	0	0
$[731^2]$	0	2	2	1	−6	0	−1	0	−1	−4	−16	−1	1
$[72^2\,1]$	0	1	0	2	3	0	−1	−1	0	3	9	0	−1
$[721^3]$	0	2	1	1	6	0	0	0	0	0	0	−1	−1
$[71^5]$	1	−2	0	3	4	0	0	0	1	2	10	0	0
$[6^2]$	−1	0	−1	0	2	0	0	0	0	0	0	−1	1
[651]	−2	−1	0	1	1	0	0	1	1	−1	−5	0	1
[642]	0	0	0	−3	0	0	0	1	0	3	15	0	0
$[641^2]$	0	−2	0	1	−6	−1	0	0	3	4	0	0	−1
$[63^2]$	−1	−1	−1	0	5	0	0	0	1	−2	−20	−1	0
[6321]	0	0	−2	0	0	−2	0	0	0	0	0	1	2
$[631^3]$	−3	0	0	−2	0	0	1	0	−2	0	16	0	0
$[62^3]$	0	−1	0	0	−3	−1	0	1	0	−3	15	0	0
$[62^2\,1^2]$	0	0	−1	−2	0	0	1	0	0	0	−24	1	−1
$[5^2\,2]$	−1	2	2	1	2	0	0	0	−3	−4	0	1	1
$[5^2\,1^2]$	0	−1	1	4	3	0	0	−1	0	−3	−15	1	0
[543]	−2	2	1	−1	−2	0	0	0	0	0	0	−1	−1
[5421]	3	2	0	0	0	0	0	−1	−1	1	5	0	0
$[541^3]$	0	−2	−1	1	6	1	0	0	0	0	0	−1	−1
$[53^2\,1]$	0	−1	0	4	3	0	0	0	1	−2	−20	0	−1
$[532^2]$	0	0	2	−1	0	0	0	1	0	−3	15	−1	−1
$[4^3]$	−1	0	0	−1	−4	0	0	0	1	2	10	0	0
$[4^2\,31]$	0	−2	−1	−1	−6	0	0	0	1	2	10	1	1
*$[621^4]$	0	−2	0	0	0	0	0	0	0	−4	0	0	0
*$[5321^2]$	0	2	0	0	0	2	0	0	0	4	0	0	0
*$[4^2\,2^2]$	0	0	0	−4	0	0	0	0	0	0	0	0	0
$[1^{12}]$	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1

* Denotes a self-associated partition and character.

Table of characters of order 12 !—*continued.*

Class	$1^2 64$	$1^2 42$	$1^2 5^2$	$1^2 43^2$	$1^2 3^2 2^2$	156	1245	129	$12^4 3$	$12^3 5$	$12^2 7$	123^3	1236
Order	99 79200	37 42200	47 90016	33 26400	16 63200	159 66720	119 75040	266 11200	4 15800	19 95840	85 53600	14 78400	133 05600
[12]	1	1	1	1	1	1	1	1	1	1	1	1	1
[11, 1]	1	1	1	1	1	0	0	0	0	0	0	0	0
[10, 2]	−1	0	−1	−1	1	−1	0	0	3	2	1	0	0
[10, 1^2]	0	−1	0	0	−2	0	−1	−1	−4	−3	−2	−1	−1
[93]	−1	0	−1	1	3	0	0	0	1	0	0	3	1
[921]	0	0	0	−2	−2	1	1	1	0	1	1	−2	0
[91^3]	0	−1	0	2	0	0	0	0	1	0	0	3	1
[84]	1	1	0	3	1	0	0	−1	2	0	−1	−1	−1
[831]	0	−1	1	0	0	0	−1	0	−6	−3	−1	0	0
[82^2]	1	0	1	−1	−1	0	−1	−1	8	3	0	−1	−1
[821^2]	1	3	0	1	1	−1	1	0	−3	−1	0	0	0
[81^4]	−1	−2	0	1	3	0	−1	0	6	3	1	0	0
[75]	1	1	2	−1	1	1	1	0	3	1	0	0	0
[741]	0	0	−2	0	−4	−1	0	1	0	2	2	−2	0
[732]	−1	−3	0	−1	3	0	0	0	4	0	0	3	1
[731^2]	−1	0	1	−1	−1	1	1	0	0	1	0	0	0
[$72^2 1$]	0	1	−1	2	−2	0	0	0	−3	0	0	0	0
[721^3]	0	0	−2	−2	2	0	−1	0	0	−3	−1	0	0
[71^5]	−1	−2	2	−1	−3	1	1	0	−4	1	0	−3	−1
[6^2]	0	0	2	−2	2	1	0	0	−2	−2	−1	3	1
[651]	−1	−1	0	−1	1	−1	−1	0	−3	−1	0	0	0
[642]	0	3	−2	0	0	0	1	0	0	−3	−2	0	0
[641^2]	1	0	0	1	−1	1	−1	−1	0	1	0	2	0
[63^2]	1	0	0	1	5	0	0	0	−7	0	1	3	−1
[6321]	0	0	2	0	0	0	0	0	0	0	0	−6	0
[631^3]	0	0	1	0	2	−1	0	0	8	2	0	3	−1
[62^3]	0	−1	0	2	−2	0	0	1	3	0	0	4	0
[$62^2 1^2$]	0	0	−1	0	0	0	0	0	−6	0	1	0	0
[$5^2 2$]	−1	0	0	1	−1	0	1	0	8	3	0	−3	−1
[$5^2 1^2$]	0	−3	0	2	2	0	0	0	−3	0	1	0	0
[543]	0	0	2	2	−2	1	−1	0	0	1	1	0	0
[5421]	1	1	0	−3	−1	0	0	0	−2	0	0	3	1
[541^3]	0	0	0	2	2	0	1	1	0	−3	−1	−2	0
[$53^2 1$]	1	0	−2	−1	−1	−1	0	0	3	2	0	0	0
[532^2]	0	−1	0	0	0	0	−1	0	6	−3	0	0	0
[4^3]	−1	−2	2	1	−1	0	−1	0	−2	−3	0	−3	1
[$4^2 31$]	−1	−2	0	−1	1	−1	1	0	0	−1	−1	0	0
*[621^4]	2	0	0	0	−2	0	0	0	2	0	0	0	2
*[5321^2]	−2	0	0	0	−2	0	0	0	−6	0	0	0	0
*[$4^2 2^2$]	0	0	0	0	4	0	0	0	−8	0	2	0	−2
[1^{12}]	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

Table of characters of order 12 !—*continued.*

Class	$12^2\,34$	$13^2\,5$	147	134^2	138	1, 11	237	$2^2\,35$	$2^2\,8$	$2^2\,4^2$	246	2^6	25^2
Order	49 89600	53 22240	171 07200	49 89600	199 58400	435 45600	114 04800	39 91680	74 84400	18 71100	99 79200	10395	47 90016
[12]	1	1	1	1	1	1	1	1	1	1	1	1	1
[11, 1]	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1
[10, 2]	1	−1	−1	−1	−1	−1	1	2	2	2	1	6	1
$[10, 1^2]$	−2	0	0	0	0	0	0	−1	−1	−1	0	−5	0
[93]	1	2	0	1	1	0	0	−1	−2	−2	−1	−6	−1
[921]	0	−1	1	0	0	1	−1	−1	0	0	0	0	0
$[91^3]$	1	2	0	1	1	0	1	2	1	1	0	5	0
[84]	0	0	1	2	0	0	−1	0	1	3	1	15	0
[831]	−2	0	−1	−2	0	0	1	1	1	−1	0	−9	1
$[82^2]$	0	0	0	0	0	0	0	1	0	0	−1	24	−1
$[821^2]$	1	−1	0	1	−1	−1	0	−1	−1	1	1	−15	0
$[81^4]$	0	0	−1	−2	0	0	−1	−1	0	−2	−1	10	0
[75]	1	−1	0	−1	−1	0	1	2	−1	−3	−1	−15	2
[741]	0	−1	0	0	0	0	0	−2	0	0	0	0	−2
[732]	2	0	0	0	0	0	0	0	−1	1	1	−15	0
$[731^2]$	0	1	0	0	0	0	−1	1	0	0	−1	24	1
$[72^2\,1]$	−1	2	0	1	1	0	0	−1	1	−1	0	−9	1
$[721^3]$	0	−2	1	0	0	1	1	1	0	0	0	0	−2
$[71^5]$	−2	1	0	0	0	0	0	0	0	2	1	−10	2
$[6^2]$	0	1	−1	−2	0	0	−1	−3	0	4	2	20	−2
[651]	−1	1	0	1	1	0	0	1	1	−1	−1	−5	0
[642]	0	0	0	0	0	0	0	0	−1	1	0	45	2
$[641^2]$	0	1	0	0	0	0	0	1	0	0	1	−40	0
$[63^2]$	−1	0	1	1	−1	0	−1	0	2	−2	−1	−30	0
[6321]	0	−2	0	0	0	0	1	0	0	0	0	0	−2
$[631^3]$	0	1	0	0	0	0	0	−2	0	0	0	16	1
$[62^3]$	−3	0	0	−1	−1	0	0	0	−1	−3	0	45	0
$[62^2\,1^2]$	2	0	−1	−2	0	0	−1	1	0	4	0	−36	1
$[5^2\,2]$	0	0	0	0	0	0	−1	1	0	0	1	−40	0
$[5^2\,1^2]$	1	−2	1	1	−1	0	1	−2	−1	1	0	45	0
[543]	0	1	−1	0	0	0	1	−1	0	0	0	0	−2
[5421]	0	0	0	−2	0	0	0	0	1	−1	−1	−5	0
$[541^3]$	0	0	−1	0	0	0	−1	1	0	0	0	0	0
$[53^2\,1]$	−1	1	0	−1	1	0	0	1	−2	2	1	30	2
$[532^2]$	2	0	0	2	0	0	−1	−1	1	3	0	−45	0
$[4^3]$	2	−2	0	2	0	0	0	2	0	2	−1	30	2
$[4^2\,31]$	0	−1	1	0	0	0	−1	−1	0	−2	1	−30	0
$*[621^4]$	0	0	0	2	0	−1	0	0	0	−4	0	20	0
$*[5321^2]$	0	0	0	2	0	0	0	0	0	−4	0	20	0
$*[4^2\,2^2]$	0	2	0	0	0	0	0	2	0	0	0	80	0
$[1^{12}]$	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1

* Denotes a self-associated partition and character.

Table of characters of order 12 !—continued.

Class	2, 10	$2^3 3^2$	$2^3 6$	345	39	$3^2 6$	3^4	4^3	$23^2 4$	12	57	$2^4 4$	6^2	48
Order	239 50080	5 54400	16 63200	79 83360	177 40800	44 35200	2 46400	12 47400	33 26400	399 16800	136 85760	3 11850	66 52800	149 68800
[12]	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[11, 1]	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
[10, 2]	1	3	3	0	0	0	0	0	1	0	0	4	0	0
$[10, 1^2]$	0	−2	−2	1	1	1	1	1	0	1	1	−3	1	1
[93]	−1	−1	−3	1	1	2	4	0	1	0	0	−4	0	0
[921]	0	−2	0	−1	−1	−2	−4	0	−2	0	0	0	0	0
$[91^3]$	0	4	2	0	0	1	3	−1	2	−1	−1	3	−1	−1
[84]	0	1	3	0	−1	−2	−4	3	−1	0	0	7	0	1
[831]	1	0	0	−1	0	0	0	−3	0	0	0	−3	0	−1
$[82^2]$	−1	5	3	1	1	2	4	0	1	0	0	8	0	0
$[821^2]$	0	−3	−3	1	0	0	0	3	1	0	0	−5	0	1
$[81^4]$	0	−1	1	−1	0	−1	−3	−2	−3	1	1	2	1	0
[75]	0	3	−3	0	0	0	0	−3	1	0	1	−7	0	−1
[741]	0	−4	0	0	1	2	4	0	0	0	−1	0	0	0
[732]	0	1	−3	0	−1	−2	−4	3	1	0	0	−5	0	1
$[731^2]$	−1	3	3	1	0	0	0	0	−1	0	1	8	0	0
$[72^2 1]$	1	−6	0	−1	0	0	0	−3	−2	0	0	−3	0	−1
$[721^3]$	0	6	0	−1	0	0	0	0	2	0	−1	0	0	0
$[71^5]$	0	−5	−1	2	0	1	3	2	1	−1	0	−2	−1	0
$[6^2]$	0	−4	2	−1	0	2	6	0	0	0	−1	8	2	0
[651]	0	1	1	1	0	−2	−6	3	−1	0	0	−1	−2	1
[642]	0	0	0	0	0	0	0	−3	0	0	1	9	0	−1
$[641^2]$	0	3	−1	−1	−1	0	2	0	1	0	0	−8	2	0
$[63^2]$	0	−1	3	0	0	2	12	0	−1	0	0	−4	0	0
[6321]	0	0	0	0	1	0	−8	0	0	0	−1	0	0	0
$[631^3]$	1	−6	−2	0	0	0	6	0	0	0	0	0	−2	0
$[62^3]$	0	6	0	0	−1	0	8	3	2	0	0	3	0	1
$[62^2 1^2]$	−1	0	0	1	0	0	0	0	0	0	1	0	0	0
$[5^2 2]$	0	5	−1	−1	0	2	6	0	−1	0	−1	−8	2	0
$[5^2 1^2]$	0	−6	0	0	0	0	0	−3	2	0	1	9	0	−1
[543]	0	−2	0	1	0	−2	−12	0	2	0	0	0	0	0
[5421]	0	−3	1	0	0	0	6	3	−1	0	0	−1	−2	1
$[541^3]$	0	6	0	1	1	0	−8	0	−2	0	−1	0	0	0
$[53^2 1]$	0	3	−3	−1	0	0	0	0	−1	0	0	4	0	0
$[532^2]$	0	0	0	1	0	0	0	−3	0	0	1	−3	0	−1
$[4^3]$	0	−1	−3	−2	0	2	12	6	−3	0	0	2	0	0
$[4^2 31]$	0	3	3	1	0	0	0	−6	1	0	0	−2	0	0
$*[621^4]$	0	0	2	0	0	0	−6	0	−2	0	0	0	2	0
$*[5321]^2$	0	0	2	0	−1	0	2	0	2	0	0	0	2	0
$*[4^2 2^2]$	0	0	−4	0	0	0	−6	0	0	0	−1	0	2	0
$[1^{12}]$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13.

Class	1^{13}	$1^{11}2$	$1^{10}3$	$1^{9}4$	$1^{9}2^{2}$	$1^{8}23$	$1^{8}5$	$1^{7}6$	$1^{7}24$	$1^{7}2^{3}$	$1^{7}3^{2}$
Order	1	78	572	4290	2145	25740	30888	2 05920	1 54440	25740	68640
[13]	1	1	1	1	1	1	1	1	1	1	1
[12, 1]	12	10	9	8	8	7	7	6	6	6	6
[11, 2]	65	45	35	27	29	21	20	14	15	17	14
$[11, 1^{2}]$	66	44	36	28	26	20	21	15	14	12	15
[10, 3]	208	120	76	48	64	36	28	14	20	32	16
[10, 21]	429	231	159	105	105	63	64	35	35	35	33
$[10, 1^{3}]$	220	110	85	56	40	29	35	20	14	2	22
[94]	429	209	99	43	97	41	14	0	15	45	12
[931]	1365	595	315	161	217	91	70	21	35	63	21
$[92^{2}]$	936	384	216	120	120	48	56	21	20	24	9
$[921^{2}]$	1430	550	350	190	134	70	90	35	22	−10	35
$[91^{4}]$	495	165	135	69	15	21	35	15	−1	−27	27
[85]	572	242	77	8	104	35	−13	−14	6	46	14
[841]	2574	924	324	84	294	84	−1	−21	14	84	9
[832]	3432	1100	414	160	272	50	42	0	8	60	−30
$[831^{2}]$	4212	1242	567	216	216	63	57	0	6	−18	0
$[82^{2}1]$	3640	980	490	224	112	14	70	14	0	−28	−14
$[821^{3}]$	3003	693	483	189	−21	21	63	14	−21	−91	42
$[81^{5}]$	792	132	162	48	−48	6	22	6	−20	−36	36
[76]	429	165	33	−15	69	21	−21	−13	1	29	15
[751]	2860	880	160	−56	268	64	−70	−36	0	72	34
[742]	6006	1540	252	−28	350	28	−49	−21	−14	84	−21
$[741^{2}]$	6864	1584	384	0	240	48	−56	−34	0	−16	6
$[73^{2}]$	3575	825	125	15	155	−15	0	0	−15	45	−40
[7321]	12012	2310	609	168	168	−63	7	−14	−14	−14	−84
$[731^{3}]$	7800	1200	600	120	−120	0	0	−15	−20	−120	15
$[72^{3}]$	4004	616	224	112	−28	−56	14	0	0	0	−28
$[72^{2}1^{2}]$	7371	945	567	189	−189	−63	21	0	−21	−63	0
$[721^{4}]$	4290	330	510	78	−246	−6	20	1	−50	−62	69
$[6^{2}1]$	1287	363	27	−57	111	27	−43	−15	−1	27	27
[652]	5148	1122	−27	−168	264	27	−77	−6	−14	54	36
$[651^{2}]$	5720	1100	50	−160	176	50	−90	−20	8	−20	50
[643]	6435	1155	−135	−147	231	−21	−20	21	−35	63	−9
[6421]	17160	2420	−90	−224	208	−34	−70	6	0	−12	−6
$[641^{3}]$	10296	1056	216	−120	−120	48	−64	−15	20	−120	27
$[63^{2}1]$	11440	1320	−140	−48	64	−84	20	14	−20	32	−56
$[632^{2}]$	12012	1078	−63	56	−56	−119	7	0	14	42	−42
$[6321^{2}]$	20592	1320	324	48	−384	−84	−28	−6	20	−48	−36
$[631^{4}]$	9360	120	540	−48	−384	36	−20	−6	−20	−48	36
$[62^{3}1]$	9009	231	189	147	−231	−105	−21	0	21	63	0
$[5^{2}3]$	3432	528	−168	−120	120	0	−8	21	−20	24	21
$[5^{2}21]$	8580	990	−285	−216	120	21	−35	20	6	−22	48
$[5^{2}1^{3}]$	5005	385	−35	−119	−35	49	−35	0	21	−63	28
$[54^{2}]$	2574	330	−162	−78	78	−6	14	21	−22	18	9
[5431]	15015	1155	−735	−231	147	−21	70	49	−35	7	21
$[542^{2}]$	12870	660	−540	−84	30	−36	35	15	14	12	27
$[5421^{2}]$	21450	550	−450	−190	−190	70	0	15	50	−90	15
$[53^{2}2]$	11583	297	−567	27	27	−63	78	0	−15	45	0
$[4^{3}1]$	3432	132	−258	−48	48	6	42	14	−20	−4	12
*$[4^{2}32]$	8580	0	−600	0	84	0	90	0	−28	0	30
*$[53^{2}1^{2}]$	16016	0	−448	0	−112	0	56	0	0	0	−28
*$[71^{6}]$	924	0	168	0	−84	0	14	0	−28	0	42
$[1^{13}]$	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13—continued.

Class	$1^6 25$	$1^6 23^2$	$1^6 34$	$1^6 7$	$1^5 8$	$1^5 4^2$	$1^5 2^2 4$	$1^5 26$	$1^5 23^2$	$1^5 35$	$1^5 2^4$	$1^4 45$	$1^4 3^3$
Order	8 64864	3 60360	7 20720	12 35520	64 86480	16 21620	16 21620	43 24320	14 41440	34 59456	1 35135	129 72960	16 01600
[13]	1	1	1	1	1	1	1	1	1	1	1	1	1
[12, 1]	5	5	5	5	4	4	4	4	4	4	4	3	3
[11, 2]	10	11	9	9	5	5	7	6	6	5	9	2	2
$[11, 1^2]$	9	8	10	10	6	6	4	5	5	6	2	3	3
[10, 3]	10	16	6	5	0	0	8	4	6	1	16	−2	1
[10, 21]	16	15	15	16	5	5	5	5	3	4	5	0	−3
$[10, 1^3]$	5	1	11	10	4	4	−4	0	2	5	−12	1	4
[94]	4	19	1	−5	−5	−3	7	0	8	−1	21	−2	6
[931]	10	19	−1	0	−5	−7	5	1	1	−5	13	−4	−3
$[92^2]$	4	0	0	5	0	0	0	−1	−9	−4	8	0	−9
$[921^2]$	0	−10	10	9	0	2	−10	−5	−5	0	−26	0	−1
$[91^4]$	−5	−9	9	5	1	−1	−11	−5	3	5	−17	−1	9
[85]	−3	17	5	−9	−4	4	4	−4	8	2	20	3	5
[841]	−1	24	−6	−16	−6	−6	4	−1	9	−1	14	−1	9
[832]	0	2	−20	−5	0	−8	0	0	−10	−6	24	0	−6
$[831^2]$	−3	−9	−9	−9	−4	−4	−4	0	0	−3	−36	1	0
$[82^2 1]$	−10	−26	−4	0	0	8	0	−4	−16	−5	8	4	−5
$[821^3]$	−17	−21	9	0	−1	−1	−11	−6	6	3	−21	−1	6
$[81^5]$	−8	−6	6	1	0	−8	−8	−4	6	7	8	−2	9
[76]	−5	9	9	−5	−1	9	1	−5	3	3	13	5	−3
[751]	−10	16	16	−10	0	12	0	−6	4	5	12	4	−2
[742]	−5	8	−10	0	6	−6	−4	1	1	2	22	−3	3
$[741^2]$	4	0	0	−10	0	0	0	6	6	4	−48	0	6
$[73^2]$	0	5	−15	5	5	−5	−5	0	0	0	15	0	2
[7321]	5	−15	−15	0	4	4	4	4	−6	4	4	3	−3
$[731^3]$	0	0	0	−5	0	0	0	5	15	0	−40	0	−3
$[72^3]$	−4	−16	4	0	0	12	8	0	−8	−1	36	2	8
$[72^2 1^2]$	−15	−9	9	0	1	11	13	0	0	−3	27	−1	0
$[721^4]$	−10	6	6	−1	0	−10	−2	1	9	5	34	−2	−3
$[6^2 1]$	−7	3	15	−1	1	11	−1	−5	−3	2	7	3	−9
[652]	−13	−3	15	10	4	4	−4	−4	−6	−2	20	−3	−9
$[651^2]$	0	−10	20	1	0	8	0	0	−10	0	−24	0	−4
[643]	−10	9	−9	16	5	−5	−7	−1	9	−5	11	−2	9
[6421]	10	−14	4	10	0	−8	0	4	−4	5	−8	−4	−3
$[641^3]$	16	0	0	−1	0	0	0	5	3	−4	−40	0	9
$[63^2 1]$	10	16	−6	9	0	0	−8	−4	6	5	−16	2	1
$[632^2]$	13	−11	11	0	−4	4	−4	0	−2	7	36	1	6
$[6321^2]$	10	24	6	5	0	0	8	−4	6	−1	−16	−2	−9
$[631^4]$	10	24	−6	1	0	0	−8	4	6	−5	16	2	−9
$[62^3 1]$	−9	−3	15	0	−1	9	11	0	0	−6	33	−3	18
$[5^2 3]$	−12	0	0	9	0	0	0	−1	3	−8	8	0	3
$[5^2 21]$	−5	−21	9	5	−4	−4	4	0	−12	−5	12	−1	−6
$[5^2 1^3]$	5	−11	1	0	−1	1	1	0	−8	−5	−3	1	10
$[54^2]$	−10	6	−6	5	0	2	2	1	9	−7	−2	2	9
[5431]	−10	9	−9	0	−5	3	5	1	9	−5	−17	4	3
$[542^2]$	5	−24	6	−10	−6	−6	−4	5	−3	5	38	1	−9
$[5421^2]$	0	−10	−10	−5	0	−10	10	−5	−5	0	10	0	3
$[53^2 2]$	12	9	9	−9	−5	7	−17	0	0	3	−9	2	0
$[4^3 1]$	−8	6	−6	−5	0	8	8	4	6	−3	−8	2	3
*$[4^2 32]$	0	0	0	−16	0	12	0	10	0	0	4	0	−6
*$[53^2 1^2]$	0	32	0	0	0	0	0	−12	0	2	−48	0	−4
*$[71^6]$	0	0	0	0	0	−12	0	−2	0	8	28	0	6
$[1^{13}]$	−1	1	−1	1	−1	1	−1	1	−1	1	+1	−1	1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13—*continued.*

Class	$1^4\,36$	$1^4\,234$	$1^4\,27$	$1^4\,2^2\,5$	$1^4\,2^3\,3$	$1^4\,9$	$1^3\,10$	$1^3\,82$	$1^3\,62^2$	$1^3\,42^3$	$1^3\,2^5$	$1^3\,73$	$1^3\,532$
Order	144 14400	108 10800	185 32800	64 86480	18 01800	288 28800	1037 83680	648 64800	216 21600	54 05400	2 70270	494 20800	345 94560
$[13]$	1	1	1	1	1	1	1	1	1	1	1	1	1
$[12, 1]$	3	3	3	3	3	3	2	2	2	2	2	2	2
$[11, 2]$	2	3	3	4	5	2	0	1	2	3	5	0	1
$[11, 1^2]$	3	2	2	1	0	3	1	0	−1	−2	−4	1	0
$[10, 3]$	−1	2	1	4	8	−2	−2	0	2	4	8	−1	1
$[10, 21]$	−1	−1	0	0	−1	0	−1	−1	−1	−1	−1	−2	−2
$[10, 1^3]$	2	−1	−2	−5	−7	1	0	−2	−4	−6	−10	1	−1
$[94]$	0	3	−1	2	9	−3	−1	−1	0	3	9	1	1
$[931]$	−3	−1	0	2	3	−3	0	1	1	−1	−5	0	1
$[92^2]$	−3	−4	−1	0	0	0	1	0	1	4	16	−1	−2
$[921^2]$	−1	−2	−3	−6	−10	−1	0	0	−1	−2	−10	0	0
$[91^4]$	3	−1	−3	−5	−3	0	0	−1	−1	−1	5	2	1
$[85]$	1	3	−3	−1	7	−1	0	−2	−2	2	10	0	0
$[841]$	3	2	0	−1	0	0	1	0	−1	−2	−4	2	−1
$[832]$	0	−4	1	2	6	3	2	0	0	0	12	1	0
$[831^2]$	0	3	3	1	−9	0	0	2	0	−6	−30	0	3
$[82^2\,1]$	−1	0	0	2	2	1	0	0	2	8	20	0	−1
$[821^3]$	2	3	0	−1	5	0	0	1	2	3	5	0	1
$[81^5]$	3	−2	−1	2	6	0	0	0	2	4	20	1	1
$[76]$	−1	1	−3	−1	5	0	0	−1	−1	1	5	−2	1
$[751]$	0	0	−2	−2	0	1	0	0	0	0	0	−1	−1
$[742]$	3	−2	0	−5	0	3	0	0	−1	2	20	0	−2
$[741^2]$	2	0	2	0	−16	0	−1	0	2	0	−16	−1	−2
$[73^2]$	0	−3	−1	0	9	2	0	−1	0	−3	−15	−1	0
$[7321]$	1	1	0	3	1	0	−2	−2	−2	−2	−2	0	2
$[731^3]$	−3	4	3	0	0	0	0	0	−3	−4	0	−2	0
$[72^3]$	0	0	0	2	0	−1	−1	0	0	0	24	0	−1
$[72^2\,1^2]$	0	3	0	1	9	0	0	−1	0	3	−15	0	−3
$[721^4]$	1	−2	1	4	10	0	0	0	1	−2	10	−1	−1
$[6^2\,1]$	−3	−1	−1	1	3	0	0	1	1	−1	−5	−1	2
$[652]$	−3	1	2	−1	3	0	0	2	−2	−2	10	1	2
$[651^2]$	−2	−4	1	6	−2	−1	0	0	4	0	−20	1	0
$[643]$	−3	1	0	−4	3	0	0	1	1	1	−5	−2	−1
$[6421]$	3	0	−2	−2	−6	−3	0	0	2	8	20	+1	1
$[641^3]$	−3	−4	−1	0	0	0	1	0	1	4	16	−1	−2
$[63^2\,1]$	−1	−2	−3	4	8	−2	0	0	2	−4	−40	0	1
$[632^2]$	0	−1	0	−1	−3	−3	0	2	0	−6	30	0	1
$[6321^2]$	−3	2	−3	−4	0	0	2	0	−2	4	−8	2	1
$[631^4]$	−3	−2	1	−4	0	0	0	0	−2	−4	40	1	1
$[62^3\,1]$	0	−3	0	−1	−9	0	1	1	0	−3	−9	0	0
$[5^2\,3]$	−3	4	3	0	0	0	0	0	−3	−4	0	0	0
$[5^2\,21]$	2	3	3	5	5	0	0	−2	−4	−6	−10	2	1
$[5^2\,1^3]$	0	−3	0	5	9	1	0	−1	0	−3	−15	0	−1
$[54^2]$	−3	2	1	−2	−6	0	0	0	1	2	10	−1	−1
$[5431]$	1	1	0	2	−5	0	0	−1	1	1	−5	0	−1
$[542^2]$	3	2	2	−5	0	0	0	0	−1	−2	20	−1	−1
$[5421^2]$	3	2	−3	0	6	3	0	0	−1	2	−10	−2	0
$[53^2\,2]$	0	−3	3	2	9	0	0	1	0	−3	−15	0	−3
$[4^3\,1]$	−1	−2	−1	−2	−10	0	0	0	2	4	20	1	1
$*[4^2\,32]$	0	−4	0	−6	0	0	0	0	0	4	0	2	0
$*[53^2\,1^2]$	0	0	0	8	0	2	0	0	0	0	0	0	0
$*[71^6]$	0	−4	0	6	0	0	0	0	0	4	0	0	0
$[1^{13}]$	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13—continued.

Class	$1^3 64$	$1^3 4^2 2$	$1^3 5^2$	$1^3 43^2$	$1^3 3^2 2^2$	$1^2 56$	$1^2 245$	$1^2 29$	$1^2 2^4 3$	$1^2 2^3 5$	$1^2 2^2 7$	$1^2 23^3$	$1^2 236$
Order	432 43200	162 16200	207 56736	144 14400	72 07200	1037 83680	778 37760	1729 72800	27 02700	129 72960	555 98400	96 09600	864 86400
$[13]$	1	1	1	1	1	1	1	1	1	1	1	1	1
$[12, 1]$	2	2	2	2	2	1	1	1	1	1	1	1	1
$[11, 2]$	0	1	0	0	2	−1	0	0	3	2	1	0	0
$[11, 1^2]$	1	0	1	1	−1	0	−1	−1	−4	−3	−2	−1	−1
$[10, 3]$	−2	0	−2	0	4	−1	0	0	4	2	1	3	1
$[10, 21]$	−1	−1	−1	−3	−3	0	0	0	−1	0	0	−3	−1
$[10, 1^3]$	0	−2	0	2	−2	0	−1	−1	−3	−3	−2	2	0
$[94]$	0	1	−1	4	4	0	0	−1	3	0	−1	2	0
$[931]$	−1	−1	0	−1	1	1	0	1	−5	−2	0	1	1
$[92^2]$	1	0	1	−3	−3	1	0	0	8	4	1	−3	−1
$[921^2]$	1	2	0	1	−1	0	2	1	−2	0	1	1	1
$[91^4]$	−1	−3	0	3	3	0	−1	0	7	3	1	3	1
$[85]$	2	2	2	2	2	1	1	−1	5	1	−1	−1	−1
$[841]$	1	0	−1	3	−3	−1	−1	0	−4	−1	0	−3	−1
$[832]$	0	−4	2	−2	2	0	−2	−1	6	0	−1	2	0
$[831^2]$	0	2	2	0	0	0	1	0	−9	−3	−1	0	0
$[82^2 1]$	2	4	0	2	−2	−1	0	−1	2	2	0	−1	−1
$[821^3]$	0	1	−2	0	6	−1	−1	0	3	−1	0	0	0
$[81^5]$	−2	−4	2	0	0	1	0	0	2	4	1	−3	−1
$[76]$	1	1	4	−3	3	2	1	0	1	−1	−1	3	1
$[751]$	0	0	0	−2	−2	−1	0	1	0	2	2	−2	0
$[742]$	−1	0	−4	−1	−1	−1	1	1	4	−1	0	1	1
$[741^2]$	0	0	−1	0	−6	1	0	0	0	4	2	0	0
$[73^2]$	0	−3	0	0	8	0	0	0	−3	0	1	6	0
$[7321]$	−2	−2	2	0	0	1	1	0	1	1	0	−3	1
$[731^3]$	−1	0	0	−3	3	0	0	0	8	0	−1	3	−1
$[72^3]$	0	0	−1	4	−4	0	0	1	0	0	0	4	0
$[72^2 1^2]$	0	1	−4	0	0	0	−1	0	−9	−3	0	0	0
$[721^4]$	1	−2	0	−3	−3	1	0	0	−2	−2	−1	−3	1
$[6^2 1]$	−1	−1	2	−3	3	0	−1	0	−5	−3	−1	3	1
$[652]$	−2	2	−2	0	0	−1	1	0	5	−1	−2	−3	−1
$[651^2]$	0	−4	0	2	2	0	−2	−1	−6	0	1	2	0
$[643]$	1	3	0	3	3	1	0	0	−7	−2	0	3	−1
$[6421]$	2	4	0	−2	−2	1	0	−1	−2	−2	−2	−1	1
$[641^3]$	1	0	1	3	3	0	0	0	8	0	−1	3	−1
$[63^2 1]$	2	0	0	0	4	−1	0	0	−4	2	1	−3	−1
$[632^2]$	0	−2	2	2	−2	0	−1	1	9	−3	0	−2	0
$[6321^2]$	−2	0	2	0	0	−1	0	0	−4	2	1	−3	−1
$[631^4]$	2	0	0	0	0	−1	0	0	4	2	1	3	1
$[62^3 1]$	0	−1	−1	0	0	0	1	0	−3	3	0	6	0
$[5^2 3]$	−1	0	2	3	−3	1	0	0	8	4	1	−3	−1
$[5^2 21]$	0	−2	0	0	0	0	1	0	3	3	1	0	0
$[5^2 1^3]$	0	−3	0	4	4	0	1	1	−3	−3	0	−2	0
$[54^2]$	−1	−2	4	3	−3	1	−2	0	−2	−2	1	−3	1
$[5431]$	1	−1	0	−3	−3	−1	0	0	1	2	0	3	1
$[542^2]$	1	0	0	−3	3	0	−1	0	−4	−3	2	3	−1
$[5421^2]$	−1	2	0	−1	−1	0	0	1	−2	0	−1	1	1
$[53^2 2]$	0	1	−2	0	0	0	0	0	9	0	−1	0	0
$[4^3 1]$	−2	−4	2	0	0	−1	0	0	−2	−4	−1	−3	1
$*[4^2 32]$	−2	0	0	0	6	0	2	0	−8	0	0	0	−2
$*[53^2 1^2]$	0	0	−4	0	−4	0	0	0	0	0	0	0	0
$*[71^6]$	−2	0	4	0	−6	0	2	0	−8	0	0	0	−2
$[1^{13}]$	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13—continued.

Class	$1^2 2^2 34$	$1^2 3^2 5$	$1^2 47$	$1^2 34^2$	$1^2 38$	$1^2 11$	1237	$12^2 35$	$12^2 8$	$12^2 4^2$	1246	12^6	125^2
Order	324 32400	345 94560	1111 96800	324 32400	1297 29600	2830 46400	1482 62400	518 91840	972 97200	243 24300	1297 29600	1 35135	622 70208
$[13]$	1	1	1	1	1	1	1	1	1	1	1	1	1
$[12, 1]$	1	1	1	1	1	1	0	0	0	0	0	0	0
$[11, 2]$	1	−1	−1	−1	−1	−1	0	1	1	1	0	5	0
$[11, 1^2]$	−2	0	0	0	0	0	−1	−2	−2	−2	−1	−6	−1
$[10, 3]$	2	1	−1	0	0	−1	1	1	0	0	0	0	0
$[10, 21]$	−1	−2	0	−1	−1	0	0	0	1	1	1	1	1
$[10, 1^3]$	−1	2	0	1	1	0	1	1	0	0	0	0	0
$[94]$	1	2	1	3	1	0	−1	−1	−1	1	0	9	−1
$[931]$	−1	1	0	−1	1	1	0	−1	−1	−3	−1	−15	0
$[92^2]$	0	−1	1	0	0	1	−1	0	0	0	−1	24	−1
$[921^2]$	2	0	1	2	0	0	0	0	0	2	1	−10	0
$[91^4]$	1	2	−1	−1	1	0	0	1	1	−1	−1	15	0
$[85]$	1	−1	1	1	−1	0	0	2	0	0	0	0	2
$[841]$	−2	−1	0	0	0	0	0	−1	2	2	1	6	−1
$[832]$	0	0	−1	−2	0	0	1	2	0	0	0	0	0
$[831^2]$	−1	0	−1	−1	−1	−1	0	1	0	0	0	0	2
$[82^2 1]$	0	1	0	2	0	−1	0	−1	0	0	0	0	0
$[821^3]$	1	−3	0	−1	−1	0	0	−1	−1	−1	0	−5	−2
$[81^5]$	−2	1	−1	−2	0	0	−1	−1	0	0	0	0	2
$[76]$	1	0	−1	−3	−1	0	0	−1	−1	1	1	5	0
$[751]$	0	−1	0	0	0	0	1	1	0	−4	−2	−20	0
$[742]$	2	−1	0	0	0	0	0	−2	−2	+2	1	30	0
$[741^2]$	0	1	0	0	0	0	−1	0	0	0	0	−16	−1
$[73^2]$	1	0	1	1	−1	0	−1	0	1	−1	0	−45	0
$[7321]$	1	1	0	1	1	0	0	0	0	0	0	0	0
$[731^3]$	0	0	1	0	0	1	0	0	0	0	−1	40	0
$[72^3]$	−4	2	0	0	0	0	0	−1	0	−4	0	36	1
$[72^2 1^2]$	1	0	0	−1	1	1	0	1	1	3	0	−45	0
$[721^4]$	−2	−1	1	2	0	0	1	1	0	−2	1	10	0
$[6^2 1]$	−1	2	−1	−1	1	0	−1	−2	1	3	1	15	−2
$[652]$	−1	1	0	1	1	0	−1	2	0	0	0	0	2
$[651^2]$	0	0	1	2	0	0	1	0	0	0	0	0	0
$[643]$	−1	1	0	1	−1	0	0	−1	1	−1	−1	15	0
$[6421]$	0	−1	0	−2	0	0	1	1	0	0	0	0	0
$[641^3]$	0	2	−1	0	0	0	−1	0	0	0	1	−24	1
$[63^2 1]$	−2	−1	1	0	0	0	0	1	0	0	0	0	0
$[632^2]$	−1	−2	0	1	−1	0	0	−1	0	0	0	0	−2
$[6321^2]$	2	−1	−1	0	0	0	0	−1	0	0	0	0	0
$[631^4]$	−2	1	1	0	0	−1	1	−1	0	0	0	0	0
$[62^3 1]$	−1	0	0	−3	−1	0	0	2	−1	1	0	9	1
$[5^2 3]$	0	1	−1	0	0	0	0	0	0	0	1	−40	−2
$[5^2 21]$	1	−2	1	−1	−1	0	0	−1	0	0	0	0	0
$[5^2 1^3]$	1	−2	0	1	−1	0	0	−1	−1	1	0	45	0
$[54^2]$	2	−1	−1	2	0	0	1	1	0	2	−1	30	0
$[5431]$	−1	1	0	−3	1	0	0	−1	−1	−1	1	−5	0
$[542^2]$	2	2	0	0	0	0	−1	1	2	2	−1	30	0
$[5421^2]$	−2	0	−1	2	0	0	0	0	0	−2	−1	−30	0
$[53^2 2]$	1	0	−1	1	1	0	0	−1	−1	3	0	−45	2
$[4^3 1]$	2	−3	1	2	0	0	−1	1	0	0	0	0	2
*$[4^2 32]$	0	0	0	0	0	0	0	0	0	−4	0	20	0
*$[53^2 1^2]$	0	2	0	0	0	0	0	2	0	0	0	80	0
*$[71^6]$	0	2	0	0	0	0	0	0	0	4	0	−20	0
$[1^{13}]$	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13—continued.

Class	1, 2, 10	$12^3 3^2$	$12^3 6$	1345	139	$13^2 6$	13^4	14^3	$123^2 4$	1, 12	157	$12^4 4$	16^2
Order	3113 51040	72 07200	216 21600	1037 83680	2306 30400	576 57600	32 03200	162 16200	432 43200	5189 18400	1779 14880	40 54050	864 86400
[13]	1	1	1	1	1	1	1	1	1	1	1	1	1
[12, 1]	0	0	0	0	0	0	0	0	0	0	0	0	0
[11, 2]	0	2	2	−1	−1	−1	−1	−1	0	−1	−1	3	−1
[11, 1^2]	−1	−3	−3	0	0	0	0	0	−1	0	0	−4	0
[10, 3]	0	2	0	1	1	2	4	0	2	0	0	0	0
[10, 21]	1	−1	1	0	0	−1	−3	1	−1	1	1	1	1
[10, 1^3]	0	2	0	1	1	2	4	0	2	0	0	0	0
[94]	−1	0	0	1	0	0	0	3	0	0	0	3	0
[931]	0	−3	−3	−1	0	0	0	−3	−1	0	0	−7	0
[92^2]	−1	3	3	0	0	0	0	0	−1	0	0	8	0
[921^2]	0	−1	−1	0	−1	−1	−1	2	1	−1	−1	−2	−1
[91^4]	0	3	3	−1	0	0	0	−3	−1	0	0	5	0
[85]	0	4	0	0	−1	−2	−4	0	0	0	1	0	0
[841]	1	−3	3	−1	0	0	0	0	−1	0	−1	4	0
[832]	0	6	0	0	0	0	0	0	2	0	0	0	0
[831^2]	0	0	0	1	0	0	0	0	0	0	1	0	0
[$82^2 1$]	0	−4	0	1	1	2	4	0	0	0	0	0	0
[821^3]	0	2	−2	−1	0	−1	−3	1	0	1	0	−3	1
[81^5]	0	−6	0	1	0	0	0	0	−2	0	1	0	0
[76]	0	−1	−1	−1	0	2	6	−3	1	0	0	1	2
[751]	0	0	−2	1	1	0	−2	0	0	0	0	−8	−2
[742]	0	−3	−3	0	0	0	0	0	1	0	0	4	0
[741^2]	−1	2	2	0	0	2	6	0	0	0	0	0	2
[73^2]	0	0	0	0	−1	0	8	3	0	0	0	−9	0
[7321]	0	−2	0	0	0	−2	−12	0	−2	0	0	0	0
[731^3]	0	3	1	0	0	0	6	0	1	0	0	8	−2
[72^3]	1	0	0	−1	−1	0	8	0	0	0	0	0	0
[$72^2 1^2$]	0	0	0	−1	0	0	0	−3	0	0	0	−3	0
[721^4]	0	1	1	1	0	1	−3	2	1	−1	−1	−2	1
[$6^2 1$]	0	−3	3	0	0	0	0	3	−1	0	−1	7	0
[652]	0	6	0	0	0	0	0	0	−2	0	0	0	0
[651^2]	0	−2	0	0	−1	−2	−4	0	2	0	1	0	0
[643]	0	−3	3	1	0	0	0	−3	1	0	1	5	0
[6421]	0	0	0	−1	0	0	0	0	0	0	0	0	0
[641^3]	1	3	−3	0	0	0	0	0	−1	0	−1	−8	0
[$63^2 1$]	0	2	0	−1	1	2	4	0	−2	0	−1	0	0
[632^2]	0	6	0	1	0	0	0	0	2	0	0	0	0
[6321^2]	0	−6	0	1	0	0	0	0	2	0	0	0	0
[631^4]	0	−6	0	−1	0	0	0	0	−2	0	1	0	0
[$62^3 1$]	−1	0	0	0	0	0	0	3	0	0	0	3	0
[$5^2 3$]	0	3	−1	0	0	0	−6	0	1	0	−1	−8	2
[$5^2 21$]	0	−4	0	−1	0	2	12	0	0	0	0	0	0
[$5^2 1^3$]	0	0	0	1	1	0	−8	−3	0	0	0	9	0
[54^2]	0	−3	−3	−1	0	0	0	6	−1	0	0	2	0
[5431]	0	1	1	1	0	−2	−6	−3	1	0	0	1	−2
[542^2]	0	−3	−3	1	0	0	0	0	−1	0	0	−4	0
[5421^2]	0	3	3	0	0	0	0	6	−1	0	0	2	0
[$53^2 2$]	0	0	0	−1	0	0	0	3	0	0	1	3	0
[$4^3 1$]	0	2	0	−1	0	2	12	0	−2	0	0	0	0
*[$4^2 32$]	0	0	2	0	0	0	−6	0	2	0	−1	0	2
*[$53^2 1^2$]	0	0	−4	0	−1	0	2	0	0	0	0	0	2
*[71^6]	0	0	−2	0	0	0	6	0	2	0	0	0	−2
[1^{13}]	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13—*continued.*

Class	148	2, 11	$2^2 9$	$2^3 7$	$2^4 5$	$2^5 3$	$2^2 3^3$	$2^2 45$	238	$23^2 5$	247	234^2	256
Order	1945 94400	2830 46400	864 86400	185 32800	32 43240	5 40540	48 04800	389 18880	1297 29600	345 94560	1111 96800	324 32400	1037 83680
[13]	1	1	1	1	1	1	1	1	1	1	1	1	1
[12, 1]	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
[11, 2]	−1	1	2	3	4	5	2	2	1	1	1	1	1
[11, 1^2]	0	0	−1	−2	−3	−4	−1	−1	0	0	0	0	0
[10, 3]	0	−1	−2	−3	−4	−4	1	−2	0	1	−1	0	−1
[10, 21]	1	0	0	0	0	−1	−3	0	−1	−2	0	−1	0
[10, 1^3]	0	0	1	2	3	5	4	1	1	2	0	1	0
[94]	1	0	1	3	6	9	−2	2	−1	−2	1	1	0
[931]	−1	1	1	0	−2	−5	1	0	1	1	0	−1	1
[92^2]	0	−1	0	3	8	16	3	0	0	1	−1	0	−1
[921^2]	0	0	−1	−3	−6	−10	−1	0	0	0	1	2	0
[91^4]	−1	0	0	1	3	5	−3	−1	−1	−2	−1	−3	0
[85]	0	0	−1	−3	−5	−5	5	−1	1	3	−1	−1	1
[841]	0	0	0	0	−1	−4	−3	−1	0	−1	0	0	−1
[832]	0	0	−1	−3	−6	−6	2	0	0	0	1	2	0
[831^2]	0	−1	0	3	9	15	0	1	−1	0	−1	−1	0
[$82^2 1$]	0	1	1	0	−2	−10	−5	0	0	−1	0	−2	1
[821^3]	1	0	0	0	−1	5	6	−1	1	1	0	1	−1
[81^5]	0	0	0	−1	−2	−10	−3	2	0	1	1	2	1
[76]	−1	0	0	1	3	5	−3	1	−1	−2	1	1	0
[751]	0	0	1	2	2	0	−2	0	0	−1	0	0	−1
[742]	0	0	−1	0	7	20	−1	1	0	1	0	0	1
[741^2]	0	0	0	−2	−8	−16	6	0	0	1	0	0	1
[73^2]	1	0	2	3	0	−15	2	0	−1	0	−1	−3	0
[7321]	0	0	0	0	−1	1	−3	−1	1	−1	0	1	−1
[731^3]	0	1	0	−1	0	0	−3	0	0	0	1	0	0
[72^3]	0	0	−1	0	6	24	8	−2	0	2	0	0	0
[$72^2 1^2$]	−1	−1	0	0	−3	−15	0	3	−1	0	0	1	0
[721^4]	0	0	0	1	4	10	−3	−2	0	−1	−1	−2	1
[$6^2 1$]	1	0	0	−1	−3	−5	3	−1	1	2	−1	−1	0
[652]	0	0	0	−2	−5	−5	3	1	−1	−1	0	−1	1
[651^2]	0	0	−1	1	6	10	−4	0	0	0	1	2	0
[643]	−1	0	0	0	−4	−5	−3	−2	1	−1	0	3	−1
[6421]	0	0	1	2	2	−10	1	0	0	1	0	−2	−1
[641^3]	0	0	0	−1	0	16	−3	0	0	−2	−1	0	0
[$63^2 1$]	0	0	−2	−3	4	20	1	2	0	1	1	0	1
[632^2]	0	0	1	0	−9	−15	−2	1	−1	−2	0	1	0
[6321^2]	0	0	0	1	4	4	3	−2	0	1	−1	0	1
[631^4]	0	−1	0	1	−4	−20	3	2	0	1	1	0	−1
[$62^3 1$]	1	0	0	0	3	−9	−6	1	1	0	0	−1	0
[$5^2 3$]	0	0	0	3	8	0	3	0	0	3	1	0	−1
[$5^2 21$]	0	0	0	−1	−3	5	−6	−1	1	−2	−1	1	0
[$5^2 1^3$]	−1	0	1	0	−3	−15	10	1	−1	2	0	−3	0
[54^2]	0	0	0	−3	−2	10	−3	2	0	−1	−1	−2	1
[5431]	1	0	0	0	−2	−5	3	0	−1	−1	0	−1	1
[542^2]	0	0	0	−2	3	20	3	1	0	2	0	0	0
[5421^2]	0	0	−1	1	0	−10	−1	0	0	0	1	2	0
[$53^2 2$]	−1	0	0	3	6	−15	0	−2	1	0	−1	1	0
[$4^3 1$]	0	0	0	3	2	−10	3	−2	0	1	1	2	−1
*[$4^2 32$]	0	0	0	0	−6	0	−6	0	0	0	0	0	0
*[$53^2 1^2$]	0	0	2	0	−8	0	−4	0	0	0	0	0	−2
*[71^6]	0	0	0	0	−2	0	6	0	0	0	0	0	−2
[1^{13}]	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

The characters of the symmetric group of degree 13—continued.

Class	$2^2 36$	$2^3 34$	3, 10	$3^2 7$	$3^3 4$	346	49	35^2	58	$4^2 5$	67	13
Order	432 43200	108 10800	2075 67360	494 20800	96 09600	864 86400	1729 72800	415 13472	1556 75520	389 18880	1482 62400	4790 01600
[13]	1	1	1	1	1	1	1	1	1	1	1	1
[12, 1]	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
[11, 2]	2	3	0	0	0	0	0	0	0	0	0	0
[11, 1^2]	−1	−2	1	1	1	1	1	1	1	1	1	1
[10, 3]	−1	−2	1	2	3	1	0	1	0	0	0	0
[10, 21]	−1	−1	−1	−2	−3	−1	0	−1	0	0	0	0
[10, 1^3]	2	3	0	1	2	0	−1	0	−1	−1	−1	−1
[94]	0	3	−1	−2	−2	0	1	−1	0	2	0	0
[931]	1	−1	0	0	−1	−1	−1	0	0	−2	0	0
[92^2]	1	4	1	2	3	1	0	1	0	0	0	0
[921^2]	−1	−2	0	0	1	1	1	0	0	2	0	0
[91^4]	−1	−1	0	−1	−3	−1	0	0	1	−1	1	1
[85]	1	−1	0	0	−1	−1	−1	2	1	−1	0	0
[841]	−1	2	1	2	3	1	0	−1	−1	−1	0	0
[832]	0	0	−1	−2	−2	0	1	−1	0	2	0	0
[831^2]	0	3	0	0	0	0	0	2	1	1	0	0
[$82^2 1$]	−1	−4	0	0	−1	−1	−1	0	0	−2	0	0
[821^3]	2	3	0	0	0	0	0	−2	−1	−1	0	0
[81^5]	−1	−2	0	1	3	1	0	2	0	2	−1	−1
[76]	−1	1	0	1	3	1	0	−2	−1	−1	1	0
[751]	0	0	0	−1	−2	0	1	0	0	2	−1	0
[742]	−1	2	0	0	−1	−1	−1	2	1	−1	0	0
[741^2]	2	0	−1	−1	0	0	0	−1	0	0	1	0
[73^2]	0	−3	0	2	6	0	0	0	0	0	0	0
[7321]	1	1	1	0	−3	1	0	−1	−1	−1	0	0
[731^3]	−3	−4	0	1	3	−1	0	0	0	0	−1	0
[72^3]	0	0	−1	0	4	0	1	−1	0	2	0	0
[$72^2 1^2$]	0	3	0	0	0	0	0	2	1	1	0	0
[721^4]	1	−2	0	−1	−3	1	0	0	0	0	1	0
[$6^2 1$]	1	−1	0	−1	−3	−1	0	2	1	1	−1	0
[652]	1	1	0	1	3	1	0	−2	−1	−1	1	0
[651^2]	−2	0	0	1	2	0	−1	0	0	−2	1	0
[643]	1	1	0	−2	−3	1	0	0	0	0	0	0
[6421]	−1	−4	0	1	1	−1	1	0	0	2	−1	0
[641^3]	1	4	1	−1	−3	1	0	1	0	0	−1	0
[$63^2 1$]	−1	2	0	0	−3	−1	0	0	0	0	0	0
[632^2]	0	3	0	0	2	0	−1	2	1	−1	0	0
[6321^2]	1	−2	−1	−1	3	1	0	−1	0	0	1	0
[631^4]	1	2	0	1	−3	−1	0	0	0	0	1	0
[$62^3 1$]	0	−3	1	0	−6	0	0	−1	−1	−1	0	0
[$5^2 3$]	−3	−4	0	0	−3	−1	0	2	0	0	0	0
[$5^2 21$]	2	3	0	−1	0	0	0	0	1	1	−1	0
[$5^2 1^3$]	0	−3	0	0	−2	0	1	0	−1	1	0	0
[54^2]	1	2	0	2	3	−1	0	−2	0	2	0	0
[5431]	1	1	0	0	3	1	0	0	0	−2	0	0
[542^2]	−1	−2	0	−1	−3	1	0	0	−1	−1	1	0
[5421^2]	−1	2	0	1	−1	−1	−1	0	0	0	1	0
[$53^2 2$]	0	−3	0	0	0	0	0	−2	0	2	0	0
[$4^3 1$]	−1	−2	0	−2	−3	1	0	2	0	−2	0	0
*[$4^2 32$]	0	4	0	2	0	−2	0	0	0	2	0	0
*[$53^2 1^2$]	0	0	0	0	0	0	0	2	0	0	0	0
*[71^6]	0	4	0	0	0	−2	0	−2	0	−2	0	1
[1^{13}]	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1

* Denotes a self-associated partition and character.

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ON THE THEORY OF THE GEOMETRIC OBJECT

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(*Communicated by* G. Temple.)

[Received 2 July, 1936.—Read 12 November, 1936.]

1. *Introduction.*

Modern geometry is based on the idea of the geometric object. This idea is introduced by F. Klein in the following way. If we have different coordinate systems in a space, then there may be systems of quantities (*Grössen*), functions of these coordinates, with the property that they transform *into themselves* with a change of the coordinate system, viz. that the new quantities depend only on the old quantities and the transformation of the coordinates "without it being necessary to use the values of the coordinates themselves" (*ohne dass man die Koordinatenwerte selbst hinzuzunehmen braucht*). Such systems Klein calls *geometrische Gebilde**. Now this definition is fairly good as long as we consider only linear transformations of coordinates as Klein did, but, as we shall see, the restriction that the values of the coordinates have not to be used leads to difficulties, if the transformations are taken more generally.

In 1926 O. Veblen and J. M. Thomas† defined an "invariant" in the following way:

> "An invariant . . . is an entity with definite determining components in any coordinate system, such that the transformations of the components from one coordinate system to another form a group, isomorphic with the group of analytic transformations of the coordinates".

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* F. Klein, *Elementarmathematik vom höheren Standpunkte aus* (1909), 59.

† "Projective invariants of affine geometry of paths", *Ann. of Math.*, 27 (1926), 278–296.

In 1927 O. Veblen* gave the definition:

> "An object of any sort, which is not changed by transformations of coordinates, is called an invariant".

In 1932 O. Veblen and J. H. C. Whitehead†, adopting the expression "geometric object" proposed by J. A. Schouten and E. R. van Kampen‡ in 1930, gave the definition:

> "Anything which is unaltered by transformations of coordinates is called an invariant . . . Thus a point is an invariant and so is a curve or a system of curves. Also, strictly speaking, anything, such as a plant or an animal, which is unrelated to the space which we are talking about, is an invariant. For an invariant, which is related to the space, *i.e.* a property of the space . . ., we shall also use the term geometric object".

The last three definitions being rather vague, J. A. Schouten and D. v. Dantzig§ tried to give a more exact definition:

> "A geometric object is a system of functions of the coordinates, called components of the object, that transforms with a transformation of the coordinates 'in itself', viz. in such a way, that the new components depend only on the old components and the function of transformation".

This definition is nearly the same as that of F. Klein; the only difference is that the new components are expressly said to be functionals of the function of transformation, and that the restriction with regard to the non-using of the coordinates themselves is dropped, because no longer are only linear transformations of coordinates considered.

All these definitions have this in common, that they only try to give at the beginning of the deduction some central point of view of some geometry and not an axiom from which a theory of objects may be developed. The first writer who really tried to establish a theory of objects was A. Wundheiler. In a lecture delivered at the Congress in Moskau in 1934‖ he

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[*] "Invariants of quadratic differential forms", *Cambridge Tracts*, No. 24 (1927), 14.

[†] "The foundations of differential geometry", *Cambridge Tracts*, No. 29 (1932), 46.

[‡] "Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde", *Math. Annalen*, 103 (1930), 752–783.

[§] "Was ist Geometrie?" *Mitt. des Seminars für Vektor- und Tensoranalysis der Moskauer Universität*, 2–3 (1935), 15–48 (19).

[‖] "Objekte, Invarianten und Klassifikation der Geometrien", *Congress for tensorial differential geometry, Moskau*, 1934. (Not yet published.)

symbolized the old and the new coordinate system by B and B', the transformation that leads from B to B' by T, and the object with respect to B and B' by Ω and Ω' respectively. Then he defined a geometric object by an equation of the form*

$$(1) \qquad \Omega' = f(\Omega, T).$$

Now let us test the two latter definitions, using a very simple example. Let X_n be an n-dimensional space without any metric or pseudoparallelism with the coordinates ξ^κ $(\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau = 1, \ldots, n)$. Different coordinate systems are symbolized by $\xi^\kappa, \xi^{\kappa'}, \xi^{\kappa''}, \ldots$ $(\kappa', \ldots, \tau' = 1', \ldots, n'; \ldots)$. Let the transformation, that leads from $\xi^{\kappa'}$ to $\xi^{\kappa''}$, be written

$$(2) \qquad \xi^{\kappa''} = \overset{2,1}{t^\kappa}(\xi^{\lambda'});$$

then $\overset{2,1}{t^\kappa}$ stands for the functions of transformation. Let $v^{\kappa''}$ and $v^{\kappa'}$ be the components of a vectorfield with respect to the first and second system of coordinates respectively. Now there are two points of view. From the first point of view, which we call the *componental*, we have†

$$(3) \qquad \underbrace{v^{\kappa''}}_{\substack{\text{new}\\ \text{com-}\\ \text{ponents}}} = \underbrace{\delta^{\kappa''}_{\kappa} \overset{2,1}{t}_{\lambda}{}^{\kappa}}_{\substack{\text{transf.}\\ \text{function}}} (\underbrace{\xi^{\mu'}}_{\substack{\text{old coordinates}\\ \text{of point considered}}}) \delta^{\lambda}_{\lambda'} \underbrace{v^{\lambda'}}_{\substack{\text{old}\\ \text{com-}\\ \text{ponents}}}; \qquad \overset{2,1}{t}_{\lambda}{}^{\kappa} = \frac{\partial \overset{2,1}{t^\kappa}(\xi^\mu)}{\partial \xi^\lambda}.$$

Here the new *components* are *functionals* of the old *components*, the transformation *function* and the old *coordinates* of the point considered. Hence v^κ is a geometric object in the sense of Schouten and van Dantzig (the use of the coordinates not being forbidden) and it is a geometric object in the sense of Wundheiler only if we may look upon his Ω as a symbol for *components* and if T symbolizes the whole of $\overset{2,1}{t}_{\lambda}{}^{\kappa}(\xi^{\mu'})$, the use of the coordinates $\xi^{\mu'}$ being *included*.

From the second point of view, which we shall call the *functional* one, we may write all components as functions of the coordinates of the system to which they belong*:

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* Mr. Wundheiler's paper led to an extensive correspondence between him and the authors. The present paper has been greatly influenced by the correspondence and we wish to thank Mr. Wundheiler here for his kind remarks.

† $\delta^{\lambda'}_{\lambda}$ is the generalized Kronecker symbol and stands for 1, if λ and λ' take corresponding values (*e.g.*, 1, 1′; 2, 2′, …), and for 0 if they do not.

$$v^{\kappa'} = \delta^{\kappa'}_{\kappa}\overset{1}{\phi}{}^{\kappa}(\xi^{\mu'})\,; \quad v^{\kappa''} = \delta^{\kappa''}_{\kappa}\overset{2}{\phi}{}^{\kappa}(\xi^{\mu''}). \tag{4}$$

Then (3) can be written in the form

$$\underbrace{\overset{2}{\phi}{}^{\kappa}(\xi^{\lambda''})}_{\substack{\text{new}\\ \text{functions}}} = \underbrace{\underbrace{\overset{2,1}{t}{}_{\lambda}^{\kappa}}_{}\left(\underbrace{\overset{1,2}{t}{}^{\mu}(\xi^{\lambda''})}_{}\right)\underbrace{\overset{1}{\phi}{}^{\lambda}}_{\substack{\text{old}\\ \text{functions}}}\left(\underbrace{\overset{1,2}{t}{}^{\pi}(\xi^{\lambda''})}_{}\right)}_{\text{transformation functions}}, \tag{5}$$

holding for all values of $\xi^{\lambda''}$ in the domain considered. Here the new *functions* are *functionals* of the old *functions* and the transformation *function*, and the coordinates do not come in. Hence v^{κ} is a geometric object in the sense of Wundheiler, if we may look upon Ω as a symbol for *functions* and if T symbolizes only the transformation function, the use of the coordinates *not being included.*

From this example it follows that every theory of objects has to take into consideration these two different points of view. But there is something more in it. Take four coordinate systems $\xi^{\kappa'}$, $\xi^{\kappa''}$, $\xi^{\kappa'''}$, and $\xi^{\kappa^{\text{iv}}}$ in such a way that the transformation T leads from $\xi^{\kappa'}$ to $\xi^{\kappa''}$ as well as from $\xi^{\kappa'''}$ to $\xi^{\kappa^{\text{iv}}}$; in other words, that $\overset{2,1}{t}{}^{\kappa}$ and $\overset{4,3}{t}{}^{\kappa}$ are the same functions. Then

$$\begin{cases} v^{\kappa''} = \delta^{\kappa''}_{\kappa}\,\overset{2,1}{t}{}_{\lambda}^{\kappa}(\xi^{\mu'})\,\delta^{\lambda}_{\lambda'}\,v^{\lambda'}, \\ v^{\kappa^{\text{iv}}} = \delta^{\kappa^{\text{iv}}}_{\kappa}\,\overset{2,1}{t}{}_{\lambda}^{\kappa}(\xi^{\mu'''})\,\delta^{\lambda}_{\lambda'''}\,v^{\lambda'''}. \end{cases} \tag{6}$$

Take now a definite point $\underset{0}{\xi}$. At that point, generally,

$$\overset{2,1}{t}{}_{\lambda}^{\kappa}\left(\underset{0}{\xi}{}^{\mu'}\right) \neq \overset{2,1}{t}{}_{\lambda}^{\kappa}\left(\underset{0}{\xi}{}^{\mu'''}\right); \tag{7}$$

hence the parameters of the linear transformation of $v^{\kappa'}$ into $v^{\kappa''}$ at $\underset{0}{\xi}$ are not the same as those of the linear transformation of $v^{\kappa'''}$ into $v^{\kappa^{\text{iv}}}$ at the *same point*, though the transformation T *is the same in both cases.* From this

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[*] It is necessary to introduce the upper suffixes 1, 2, because a quantity p, which can be written as a function of x and also as a function of y, cannot be written $p = \phi(x) = \phi(y)$, but has to be written, *e.g.*, $p = \overset{1}{\phi}(x) = \overset{2}{\phi}(y)$, according to the rule, generally adopted in the theory of functions, that different functions have to be symbolized by different letters.

it follows, that, contrary to a widely spread opinion, the transformation of the components of a vector at a point, induced by a group $\mathfrak{G}$ of transformations of coordinates, is, in general, *not a representation* (*Darstellung*) of the group $\mathfrak{G}$ in the ordinary sense.

From the foregoing we see that there is some lack of rigour in all definitions hitherto published. If the geometric object is really at the base of all modern geometry, we need a definition which has the rigour usual in mathematical investigations.

2. *Coordinates and transformations.*

We shall investigate our subject only in an N-dimensional space and avoid all difficulties which may arise from the introduction of spaces with an infinite number of components.

For that purpose we start with N-dimensional *arithmetic space*, viz. the totality of all sets of N numbers $\Xi^1, \ldots, \Xi^N$, every number assuming all real values between $-\infty$ and $+\infty$*. Every set of numbers is called a *point of arithmetic space* or an *arithmetic point*. A set of arithmetic points given by inequalities of the form

$$(8) \qquad \left| \Xi^A - \underset{0}{\Xi}^A \right| < \delta \quad (\delta > 0;\ \ A = 1, \ldots, N)$$

is called a *box* and $\underset{0}{\Xi}$ its *centre*. A set of arithmetic points is called an *arithmetic region*, if and only if each of its points is the centre of a box whose points are all contained in the set.

Now we assume the existence of an N-dimensional topological space. We call this space *geometric space* and its points *geometric points*. A part of geometric space R_g, which can be brought into continuous (1, 1) point correspondence with a region R_a of arithmetic space, is called a *geometric region*. The correspondence itself is called a *coordinate system* in R_g, and the numbers Ξ, corresponding to a geometric point, are called *coordinates of the point*. If there is another correspondence between R_g and another arithmetic region R_a', then the transformation of R_a into R_a', which is continuous and reversible, is a transformation of *points* in arithmetic space and at the same time a transformation of *coordinates* in geometric space.

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* *Cf.* O. Veblen and J. H. C. Whitehead, *loc. cit.*

A transformation of arithmetic points is said to be regular and of class u, if, in a certain arithmetic region R_a,

1. It can be written in the form

$$\overset{2}{\Xi} = \overset{2,1}{t}\left(\overset{1}{\Xi}\right). \tag{9}$$

Here $\overset{1}{\Xi}$ stands for the totality of the $\Xi^{1'}, \ldots, \Xi^{N'}$, $\overset{2}{\Xi}$ for the totality of the transformed values $\Xi^{1''}, \ldots, \Xi^{N''}$, and $\overset{2,1}{t}$ symbolizes the functions of the transformation.

2. The functions $\overset{2,1}{t}$ are *of class* u, viz. they are continuous and have continuous derivatives up to the order u in the region R_a $\Big($called the *first region* of $\overset{2,1}{t}\Big)$.

3. It is (1, 1) and the Jacobian is different from zero at *all* arithmetic points in R_a; hence there exists an inverse transformation

$$\overset{1}{\Xi} = \overset{1,2}{t}\left(\overset{2}{\Xi}\right), \tag{10}$$

which is regular and of class u in an arithmetic region $\Big($called the *second region* of $\overset{2,1}{t}\Big)$, which results from the first region by $\overset{2,1}{t}$.

The regular transformations of class u, whose first and second regions coincide with the whole of arithmetic space, form a group $\mathfrak{G}_u'$.

The system of all regular transformations of class u (defined each for any region whatever) form a pseudogroup $\mathfrak{G}_u$, viz. two transformations give a resultant transformation, regular and of class u in a certain region, if and only if the second region of the first transformation and the first region of the second transformation have a region in common. Obviously $\mathfrak{G}_u'$ is a subgroup of $\mathfrak{G}_u$.

For the purpose of this paper we confine ourselves to coordinate systems that belong to the same geometric region R_g, viz. the region in which we wish to define our objects. In this region we start with one definitely chosen coordinate system and use only coordinate systems that can be got from this first one by a transformation of a definitely chosen group or pseudogroup $\mathfrak{G}$ which is a subgroup or subpseudogroup of $\mathfrak{G}_u$. These coordinate systems are called *adapted* (to R_g). It is clear that by this process we put a restriction on our geometry; the geometry which we study is not the geometry of the region R_g of topological space, but the geometry of the combination of R_g with the set of all adapted coordinate systems, or of R_g, one adapted coordinate system and $\mathfrak{G}$.

With regard to the transformations of $\mathfrak{G}$ and the adapted coordinate systems, we remark the following facts:

1. A transformation T of $\mathfrak{G}$ can be applied to a coordinate system B if, and only if, the first region of T contains the arithmetic region that corresponds to R_g with respect to B.

2. Every transformation of a coordinate system $\overset{1}{B}$ into another coordinate system $\overset{2}{B}$ is regular and of class u in the arithmetic region that corresponds to R_g with respect to $\overset{1}{B}$.

3. If S and T are two transformations of $\mathfrak{G}$ the transformation ST exists if, and only if, the second arithmetic region of T and the first arithmetic region of S have a region in common.

If $\mathfrak{G} = \mathfrak{G}_u$, we get the objects of an ordinary N-dimensional space X_N (of class u). But we may also suppose that the Ξ^A stand for $\xi^\kappa, d\xi^\kappa, \ldots, d^u\xi^\kappa$, ξ^κ being the coordinates of an X_n, and that $\mathfrak{G}$ is the u-th extension (in the sense of Lie) of $\mathfrak{G}_u$ in X_n. Then we get the objects of a Kawaguchi space. Also we may adjoin to every point of an X_n a flat space E_m with the coordinates y^h and suppose that the Ξ^A stand for ξ^κ, y^h and that $\mathfrak{G}$ is the product of a subpseudogroup of $\mathfrak{G}_u$ in X_n and the linear homogeneous group in y^h. Then we get the objects of a geometry of König. Or we can take Ξ^A to stand for ξ^κ, t^h, where t^h are parameters, and $\mathfrak{G}$ to be the product of a subpseudogroup of $\mathfrak{G}_u$ in X_n and some definite group in t^h. Then we get the objects of a geometry of Wundheiler, as generalized by Hlavatý and Hokari. There are, of course, many other possibilities. Apart from the examples, we do not specialize here in any way.

We use the following general notations. Points of geometric space are denoted by $\underset{0}{\Xi}, \underset{1}{\Xi}, \underset{2}{\Xi}, \ldots$, generally, $\underset{\Lambda}{\Xi}$ or Ξ; adapted coordinate systems are denoted by $\overset{0}{B}, \overset{1}{B}, \overset{2}{B}, \ldots$, generally, $\overset{K}{B}$ or B. For the coordinates of the point $\underset{\Lambda}{\Xi}$ with respect to $\overset{0}{B}, \overset{1}{B}, \ldots$ we write $\underset{\Lambda}{\Xi}^{A^0}, \underset{\Lambda}{\Xi}^{A'}, \ldots$; $A^0 = 1^0, \ldots, N^0$; $A' = 1', \ldots, N'$; and as a symbol for the whole set of coordinates belonging to $\underset{\Lambda}{\Xi}$ we take $\overset{0}{\underset{\Lambda}{\Xi}}, \overset{1}{\underset{\Lambda}{\Xi}}, \ldots$, generally $\overset{K}{\underset{\Lambda}{\Xi}}$.

3. *Objects.*

Definition of an object.

Corresponding to every coordinate system* $\overset{\kappa}{B}$ and every geometric point $\underset{\Lambda}{\Xi}$† in R_g, let a finite set of numbers $\Omega^{\kappa}_{B,\underset{\Lambda}{\Xi},1}$, $\Omega^{\kappa}_{B,\underset{\Lambda}{\Xi},2}$, ... be given. We symbolize these numbers by $\Omega^{\kappa}_{B,\underset{\Lambda}{\Xi}}$ and write the functional relations in the following way:

$$\Omega^{\kappa}_{B,\underset{\Lambda}{\Xi}} = \Omega^{\kappa}_{B}\left(\underset{\Lambda}{\overset{\kappa}{\Xi}}\right). \tag{11}$$

Hence Ω^{κ}_{B} stands for the functions which lead from the $\overset{\kappa}{\Xi}$ to the $\Omega^{\kappa}_{B,\Xi}$. (From now on a Λ will generally be omitted.)

The numbers $\Omega^{\kappa}_{B,\Xi}$ are called *components* of an *object.* If $\overset{0}{B}$ and $\overset{1}{B}$ are two arbitrarily chosen coordinate systems, the $\Omega^{\kappa}_{B,\Xi}$ can be written as functionals of the function of transformation $\overset{1,0}{t}$ and the coordinates $\overset{0}{\Xi}$ of the point Ξ with respect to the coordinate system $\overset{0}{B}$.

$$\Omega^{1}_{B,\Xi} = F^{(0)}\left(\overset{1,0}{t}, \overset{0}{\Xi}\right), \qquad \begin{pmatrix} \text{I}\,a \\ \text{Object} \end{pmatrix} \tag{Ia}$$

the form of the functional $F^{(0)}$ depending only on the choice of $\overset{0}{B}$, and not on the choice of $\overset{1}{B}$ or Ξ. By use of (11) we can write, instead of (Ia),

$$\Omega^{1}_{B} = \mathbf{F}^{(0)}\left(\overset{1,0}{t}\right), \qquad \begin{pmatrix} \text{I}\,b \\ \text{Object} \end{pmatrix} \tag{Ib}$$

in which equation the function Ω^{1}_{B} is expressed as a functional of the function $\overset{1,0}{t}$, and the form of the functional $\mathbf{F}^{(0)}$ depends only on the choice of $\overset{0}{B}$ and not on the choice of $\overset{1}{B}$. The dependence of the form of the functional $F^{(0)}$ on the choice of $\overset{0}{B}$ can be demonstrated by use of the equation

$$\Omega^{2}_{B,\Xi} = F^{(0)}\left(\overset{2,1}{t} \cdot \overset{1,0}{t}, \overset{0}{\Xi}\right) = G^{(0)}\left(\overset{2,1}{t}, \overset{1,0}{t}, \overset{1}{\Xi}\right) \tag{12}$$

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* In the following, by coordinate system we always mean adapted coordinate system.

† We only define objects here by components with respect to *one* coordinate system and *one* point, the generalization being as obvious as simple.

following from (I a). Now if we consider $\overset{0}{B}$ as fixed, the right side depends only on $\overset{2,1}{t}$, $\overset{1}{\Xi}$, and the choice of $\overset{1}{B}$, so that we can write

$$\Omega^{2}_{B,\Xi} = F^{(1)}\left(\overset{2,1}{t}, \overset{1}{\Xi}\right), \tag{13}$$

where the form of the functional $F^{(1)}$ depends only on the choice of $\overset{1}{B}$, and this equation has the same form as (I a), except that $F^{(1)}$ is a functional different from $\mathfrak{F}^{(0)}$.

Example 1. v^1 is a component of a contravariant vector field in a space with coordinates ξ^κ:

$$\underbrace{v^{1'}}_{\Omega^{1}_{B,\Xi}} = \underbrace{\overset{1,0}{t_\lambda{}^1}}_{\overset{1,0}{t}}(\underbrace{\xi^{\mu^0}}_{\overset{0}{\Xi}})\,\delta^\lambda_{\lambda^0}\,v^{\lambda^0}(\underbrace{\xi^{\mu^0}}_{\overset{0}{\Xi}}); \quad \overset{1,0}{t_\lambda{}^1} = \partial_\lambda \overset{1,0}{t^1}. \tag{14}$$

It is to be remarked that the form of the functional $F^{(0)}$ depends here on the form of the functions $v^{\lambda^0}(\xi^{\mu^0})$.

Definition of CLOSED *and* FUNCTIONALLY CLOSED *objects.*

To every coordinate system $\overset{\kappa}{B}$ belongs

1. A set of numbers $\Omega^{\kappa}_{B,\Xi}$ (components of Ω) at every point of R_g.

2. A set of functions $\Omega^{\kappa}_{B}\left(\overset{\kappa}{\Xi}\right)$ in R_g.

If $\Omega^{1}_{B,\Xi} = \Omega^{2}_{B,\Xi}$ at all points in R_g, we say that $\overset{1}{B}$ and $\overset{2}{B}$ belong *to the same class* with respect to the object Ω.

Example 2. Contravariant vector field v^κ in an X_n, $\overset{1}{B}$ and $\overset{2}{B}$ belong for instance to the same class if $\overset{2,1}{t}$ is a translation.

Example 3. The coordinates ξ^κ of the points in an X_n are the components of an object. $\overset{1}{B}$ and $\overset{2}{B}$ belong to the same class only if $\overset{1}{B} = \overset{2}{B}$.

If $\Omega^{1}_{B} = \Omega^{2}_{B}$, we say that $\overset{1}{B}$ and $\overset{2}{B}$ belong to the same *functional class* with respect to the object Ω.

Example 4. The object ξ^κ. There is only one functional class and each B belongs to it.

Example 5. To every coordinate system in an X_n we adjoin the same functions of the coordinates, thus $\phi(\xi^\kappa)$ to the system with coordinates ξ^κ, $\phi(\xi^{\kappa'})$ to the system with coordinates $\xi^{\kappa'}$, and so on. There is only one functional class and every B belongs to it.

An object is said to be *closed* or *functionally closed** if the system of *classes* or *functional classes* respectively is invariant with $\mathfrak{G}$.

Hence necessary and sufficient conditions for a closed object are

$$\Omega_{TU\overset{1}{B},\,\Xi} = \Omega_{T\overset{1}{B},\,\Xi} \quad \text{always if} \quad \Omega_{U\overset{1}{B},\,\Xi} = \Omega_{\overset{1}{B},\,\Xi}, \tag{15}$$

and for a functionally closed object

$$\Omega_{TU\overset{1}{B}} = \Omega_{T\overset{1}{B}} \quad \text{always if} \quad \Omega_{U\overset{1}{B}} = \Omega_{\overset{1}{B}}, \tag{16}$$

T and U being transformations belonging to $\mathfrak{G}$ (TB is the coordinate system obtained from B by the transformation T).

An object is said to be *complete* or *functionally complete* if all classes or functional classes respectively contain only one B.

Obviously every complete or functionally complete object is closed or functionally closed respectively.

Definition of a macrogeometric object.

An object is called a *macrogeometric object* if from

$$\begin{cases} \Omega_{\overset{1}{B}} = \mathbf{F}^{(0)}\left(\overset{1,0}{t}\right), \\[2ex] \Omega_{\overset{2}{B}} = \mathbf{F}^{(0)}\left(\overset{2,0}{t}\right) \end{cases} \tag{17}$$

we can deduce an equation of the form

$$\Omega_{\overset{2}{B}} = \overset{\Omega}{\mathbf{T}}\left(\Omega_{\overset{1}{B}},\ \overset{2,1}{t}\right) \qquad \left(\begin{matrix} \text{II}\,a \\ \text{Macrogeom. object} \end{matrix}\right), \tag{IIa}$$

the form of the functional $\overset{\Omega}{\mathbf{T}}$ not depending on the choice of $\overset{1}{B}$ and $\overset{2}{B}$. We can write (IIa) also in the form

$$\Omega_{\overset{2}{B},\,\Xi} = \overset{\Omega}{T}\left(\Omega_{\overset{1}{B}},\ \overset{2,1}{t},\ \overset{1}{\Xi}\right) \qquad \left(\begin{matrix} \text{II}\,b \\ \text{Macrogeom. object} \end{matrix}\right), \tag{IIb}$$

the form of the functional $\overset{\Omega}{T}$ not depending on the choice of $\overset{1}{B}$, $\overset{2}{B}$, and Ξ.

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* The notion "closed" is m.m. the same notion as "operator-homomorph" in modern algebra. *Cf.* B. L. van der Waerden, *Moderne algebra*, 1 (1930), 134.

We prove the theorem:

An object is functionally closed if and only if it is macrogeometric.

From (II a) follows

$$\Omega_{TU\overset{1}{B}} = \overset{\Omega}{\mathbf{T}}(\Omega_{U\overset{1}{B}}, T) = \overset{\Omega}{\mathbf{T}}(\Omega_{\overset{1}{B}}, T) = \Omega_{T\overset{1}{B}}; \tag{18}$$

hence every macrogeometric object is functionally closed. If the object is functionally closed, $\Omega_{\overset{1}{B}}$ and T determine the class of $T\overset{1}{B}$ and therefore $\Omega_{T\overset{1}{B}}$; hence there exists an equation of the form

$$\Omega_{T\overset{1}{B}} = \overset{\Omega}{\mathbf{T}}(\Omega_{\overset{1}{B}}, T). \tag{19}$$

Example 6. The object ξ^κ is complete but not functionally complete. It is closed and functionally closed.

Example 7. The vectorfield v^κ in an X_n is not closed. Because if U is a translation, the transformation that leads from $T\overset{1}{B}$ to $TU\overset{1}{B}$ is, in general, a transformation which changes the values of the components. But from (5) it follows that v^κ is macrogeometric, and thus functionally closed.

Example 8. The object (ξ^κ, v^1), v^1 being one component of a vector v^κ in X_n, is complete and thus closed. Now take a system $\overset{0}{B}$, so that $v^{1^0} = \xi^{1^0}$, $v^{2^0} = 0$, $v^{\alpha^0} =$ arbitrary, $\alpha = 3, \ldots, n$, and take for U the transformation $\xi^{1'} = \xi^{1^0}$, $\xi^{2'} = \xi^{1^0} + \xi^{2^0}$, $\xi^{\alpha'} = \xi^{\alpha^0}$. Then we have $v^{1'} = \xi^{1'}$; hence $\Omega_{U\overset{0}{B}} = \Omega_{\overset{0}{B}}$ (but $\Omega_{U\overset{0}{B}, \Xi} \neq \Omega_{\overset{0}{B}, \Xi}$). Now take for T the transformation that leads to

$$\xi^{1''} = \xi^{2'}, \quad \xi^{2''} = \xi^{1'}, \quad \xi^{\alpha''} = \xi^{\alpha'} \tag{20}$$

operating on $\overset{1}{B}$, and thus to

$$\xi^{1'''} = \xi^{2^0}, \quad \xi^{2'''} = \xi^{1^0}, \quad \xi^{\alpha''} = \xi^{\alpha^0} \tag{21}$$

operating on $\overset{0}{B}$. Then TU is the transformation that leads to

$$\xi^{1''} = \xi^{1^0} + \xi^{2^0}, \quad \xi^{2''} = \xi^{1^0}, \quad \xi^{\alpha''} = \xi^{\alpha^0} \tag{22}$$

operating on $\overset{0}{B}$. Substituting, we find for $\Omega_{TU\overset{0}{B}, \Xi}$,

$$v^{1''} = \xi^{1^0} = \xi^{2''}, \tag{23}$$

and for $\Omega_{T\overset{0}{B}, \Xi}$,

$$v^{1'''} = 0, \tag{24}$$

from which equations we see that $\Omega_{TU\overset{0}{B}} \neq \Omega_{T\overset{0}{B}}$ (and $\Omega_{TU\overset{0}{B},\Xi} \neq \Omega_{T\overset{0}{B},\Xi}$). Hence (ξ^κ, v^1) is not functionally closed and so is no macrogeometric object.

From these examples we see that an object can be closed but not functionally closed and also functionally closed but not closed.

Definition of objects with local transformation.

The new components $\Omega_{\overset{2}{B},\Xi}$ of a macrogeometric object at a point are, according to (II b), functionals of $\Omega_{\overset{1}{B}}$, that is of the whole function, not only of the components $\Omega_{\overset{1}{B},\Xi}$ at the *same* point.

Now it is easy to deduce from the equation of definition of any object (I a) an equation that contains on the left side the $\Omega_{\overset{2}{B},\Xi}$ and on the right side the $\Omega_{\overset{1}{B},\Xi}$ belonging to the same point Ξ. From (I a) we deduce

$$\Omega_{\overset{2}{B},\Xi} = F^{(0)}\left(\overset{2,0}{t}, \overset{0}{\Xi}\right), \tag{25}$$

and a functional of $\overset{2,0}{t}$ and $\overset{0}{\Xi}$ is also a functional of $\Omega_{\overset{1}{B},\Xi}$, $\overset{2,1}{t}$, $\overset{1,0}{t}$, and $\overset{1}{\Xi}$, so that we can write, instead of (25),

$$\Omega_{\overset{2}{B},\Xi} = T^{(0)}\left(\Omega_{\overset{1}{B},\Xi}, \overset{2,1}{t}, \overset{1,0}{t}, \overset{1}{\Xi}\right) \qquad \begin{pmatrix}\text{I}c\\ \text{Object}\end{pmatrix}, \tag{Ic}$$

the functional $T^{(0)}$ depending only on the choice of $\overset{0}{B}$ and not on the choice of $\overset{1}{B}$, $\overset{2}{B}$, and Ξ; but this is only a formal result.

Example 9. v^1 is one component of a contravariant vectorfield in an X_n with the coordinates ξ^κ.

$$\underbrace{v^{1''}}_{\Omega_{\overset{2}{B},\Xi}} = \underbrace{\overset{2,1}{t}{}_1^{\,1}}_{\overset{2,1}{t}}\underbrace{(\xi^{\rho'})}_{\overset{1}{\Xi}}\underbrace{v^{1'}}_{\Omega_{\overset{1}{B},\Xi}} + \underbrace{\overset{2,1}{t}{}_\alpha^{\,1}}_{\overset{2,1}{t}}\underbrace{(\xi^{\rho'})}_{\overset{1}{\Xi}}\ \underbrace{\overset{1,0}{t}{}_\lambda^{\,\alpha}}_{\overset{1,0}{t}}\underbrace{(\xi^{\rho^0})}_{\overset{0,1}{t}\left(\overset{1}{\Xi}\right)}\delta^\lambda_{\lambda^0}v^{\lambda^0}\underbrace{(\xi^{\rho^0})}_{\overset{0,1}{t}\left(\overset{1}{\Xi}\right)}. \tag{26}$$

It is to be remarked that the form of the functionals $T^{(0)}$ depends here on the form of the functions $v^{\lambda^0}(\xi^{\lambda^0})$. On the other hand, every system $\Omega_{B,\Xi}$ for which (I c) holds is an object as defined by (I a). Indeed, from (I c) it follows that

$$\Omega_{\overset{1}{B},\Xi} = T^{(0)}\left(\Omega_{\overset{0}{B},\Xi}, \overset{1,0}{t}, I, \overset{0}{\Xi}\right) = S^{(0)}\left(\Omega_{\overset{0}{B},\Xi}, \overset{1,0}{t}, \overset{0}{\Xi}\right), \tag{27}$$

and the $\Omega_{\overset{0}{B},\Xi}$ may be considered as functions of the $\overset{0}{\Xi}$;

$$\Omega_{\overset{0}{B},\Xi} = \Omega_{\overset{0}{B}}\left(\overset{0}{\Xi}\right). \tag{28}$$

Substituting this in (27), we get an equation of the form (I a). Hence (I c) may, equally with (I a), be used as the definition of an object. This result is not merely formal.

So far no special condition is imposed on the object. But, starting with (I c) as equation of definition, we can now introduce the condition that the right side of (I c) can be written in a form which does not contain either $\overset{1}{\Xi}$ or $\overset{1,0}{t}$ or both and that, in the latter cases, the form of the functional $T^{(0)}$ no longer depends on the choice of $\overset{0}{B}$. The objects thus obtained are called objects with *local transformation*. We have three cases:

1. The *localised object*:

$$\Omega_{\overset{2}{B},\Xi}=\overset{\Omega}{T}{}^{(0)}\left(\Omega_{\overset{1}{B},\Xi},\ \overset{2,1}{t},\ \overset{1,0}{t}\right). \tag{29}$$

Example 10. The object (ξ^{κ}, v^{1}), because here the $\overset{1}{\Xi}$ belong to the components of the object.

2. The *microgeometric* or *geometric object*:

$$\Omega_{\overset{2}{B},\Xi}=\overset{\Omega}{T}\left(\Omega_{\overset{1}{B},\Xi},\ \overset{2,1}{t},\ \overset{1}{\Xi}\right), \qquad \left(\begin{matrix}\text{III}\\ \text{Geom. object}\end{matrix}\right), \tag{III}$$

the form of the functionals $\overset{\Omega}{T}$ not depending on the choice of $\overset{1}{B}$, $\overset{2}{B}$, or Ξ.

Example 11. The vectorfield v^{κ}.

Because $\Omega_{\overset{1}{B},\Xi}=\Omega_{\overset{1}{B}}\left(\overset{1}{\Xi}\right)$, *every geometric object is a macrogeometric object, but not conversely.*

3. The *localized geometric* object:

$$\Omega_{\overset{2}{B},\Xi}=\overset{\Omega}{T}\left(\Omega_{\overset{1}{B},\Xi},\ \overset{2,1}{t}\right), \tag{30}$$

the form of the functionals $\overset{\Omega}{T}$ not depending on the choice of $\overset{1}{B}$, $\overset{2}{B}$, or Ξ.

Example 12. The object (ξ^{κ}, v^{κ}), because the $\overset{1}{\Xi}$ belong here to the components of the object.

As we have seen, from a general point of view the macrogeometric object presents itself first. But it has never until now occurred in geometrical investigations. Always objects with local transformations have been dealt with. We therefore give here an example of a macrogeometric object which is not microgeometric.

Example 13. We take a one-dimensional space and take $\mathfrak{G}$ to be the group of all affine transformations:

$$\overset{2}{\Xi} = \overset{2,1}{a}\,\overset{1}{\Xi} + \overset{2,1}{b}, \tag{31}$$

the a and b being constants. The object Ω has only one component and is defined by the equation

$$\Omega_{\overset{1}{B}}\Big(\overset{1}{\Xi}\Big) = \Omega_{\overset{1}{B},\,\Xi} = \overset{1,0}{a}\,\phi\Big(\overset{1}{\Xi}\Big) + \overset{1,0}{b}\,\psi\Big(\overset{1}{\Xi}\Big), \tag{32}$$

ϕ and ψ being arbitrary given functions, defined for every value of the argument, which are not zero for every argument and are not such that ϕ/ψ is a constant. From (32) it follows that $\phi\Big(\overset{0}{\Xi}\Big) = \Omega_{\overset{0}{B}}\Big(\overset{0}{\Xi}\Big)$; hence

$$\Omega_{\overset{1}{B},\,\Xi} = \Omega_{\overset{1}{B}}\Big(\overset{1}{\Xi}\Big) = \overset{1,0}{a}\,\Omega_{\overset{0}{B}}\Big(\overset{1}{\Xi}\Big) + \overset{1,0}{b}\,\psi\Big(\overset{1}{\Xi}\Big), \tag{33}$$

which holds for all values of $\overset{1}{\Xi}$. Because

$$\begin{cases} \overset{2,0}{a} = \overset{2,1}{a}\;\overset{1,0}{a}, \\ \overset{2,0}{b} = \overset{2,1}{a}\;\overset{1,0}{b} + \overset{2,1}{b}, \end{cases} \tag{34}$$

it follows from (32) that

$$\Omega_{\overset{2}{B},\,\Xi} = \overset{2,0}{a}\,\phi\Big(\overset{2}{\Xi}\Big) + \overset{2,0}{b}\,\psi\Big(\overset{2}{\Xi}\Big) = \overset{2,1}{a}\;\overset{1,0}{a}\,\phi\Big(\overset{2}{\Xi}\Big) + \Big(\overset{2,1}{a}\;\overset{1,0}{b} + \overset{2,1}{b}\Big)\,\psi\Big(\overset{2}{\Xi}\Big). \tag{35}$$

Now it is impossible to eliminate *both* $\overset{1,0}{a}$ and $\overset{1,0}{b}$ from (32) and (35). By eliminating $\overset{1,0}{b}$, we get

$$\Omega_{\overset{2}{B},\,\Xi} = \overset{2,1}{a}\,\frac{\psi\Big(\overset{2}{\Xi}\Big)}{\psi\Big(\overset{1}{\Xi}\Big)}\,\Omega_{\overset{1}{B},\,\Xi} + \overset{2,1}{a}\;\overset{1,0}{a}\left(\phi\Big(\overset{2}{\Xi}\Big) - \frac{\psi\Big(\overset{2}{\Xi}\Big)}{\psi\Big(\overset{1}{\Xi}\Big)}\,\phi\Big(\overset{1}{\Xi}\Big)\right) + \overset{2,1}{b}\,\psi\Big(\overset{2}{\Xi}\Big), \tag{36}$$

the right side depending not only on $\Omega_{\overset{1}{B},\,\Xi}$, $\overset{2,1}{a}$, $\overset{2,1}{b}$, and $\overset{1}{\Xi}$, but (for a general choice of $\overset{1}{B}$, $\overset{2}{B}$, and Ξ) *also* on $\overset{1,0}{a}$; hence Ω *cannot be a microgeometric object.*

On the other hand, from (33) it follows that

$$\Omega_{\overset{1}{B}}(Z) = \overset{1,0}{a}\,\Omega_{\overset{0}{B}}(Z) + \overset{1,0}{b}\,\psi(Z), \tag{37}$$

which holds for every value of Z. From (37) and the same equation for $\Omega_{\overset{2}{B}}(Z)$, namely

$$\Omega_{\overset{2}{B}}(Z) = \overset{2,0}{a}\,\Omega_{\overset{0}{B}}(Z) + \overset{2,0}{b}\,\psi(Z), \tag{38}$$

it follows that

$$\Omega_{\overset{2}{B}}(Z) = \overset{2,1}{a}\,\Omega_{\overset{1}{B}}(Z) + \overset{2,1}{b}\,\psi(Z), \tag{39}$$

the right-hand side being a functional of $\overset{2,1}{t}$ $\left(\text{viz. } \overset{2,1}{a} \text{ and } \overset{2,1}{b}\right)$ and $\Omega_{\overset{1}{B}}$. Hence Ω *is a macrogeometric object*. From (38) and (39) it follows that Ω is a functionally complete object.

4. *The special object*.*

An object is called a *special object* if the general equation of definition (I a),

$$\Omega_{\overset{1}{B},\,\Xi} = F^{(0)}\left(\overset{1,0}{t}\,;\ \overset{0}{\Xi}\right), \tag{40}$$

can be written in the form

$$\Omega_{\overset{1}{B},\,\Xi} = G^{(0)}\left(\tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi},\ \overset{0}{\Xi},\ \overset{1}{\Xi}\right) \qquad \left(\begin{matrix}\text{IV}\\ \text{Special object}\end{matrix}\right), \tag{IV}$$

where the $\tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$ symbolize a finite number of parameters, satisfying the following conditions:

(a) The $\tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$ are functionals of $\overset{1,0}{t}$ and functions of the $\overset{0}{\Xi}$, the form of which is independent of the choice of $\overset{1}{B}$ or $\overset{0}{B}$.

(b) They satisfy the equations

$$\tau_{\overset{2}{B},\,\overset{0}{B},\,\Xi} = \phi\left(\tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi}\right) \tag{41}$$

for every choice of $\overset{0}{B}$, $\overset{1}{B}$, and $\overset{2}{B}$, the form of the function ϕ not depending on the choice of the coordinate systems involved. The $\tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$ can be obtained by solving from these equations.

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* The idea of the special object, apart from condition (c), is due to Mr. Wundheiler.

(*c*) For every choice of $\overset{1}{B}$ and $\overset{2}{B}$ and for every point Ξ_{Λ} it is possible to change $\overset{1}{B}$ infinitesimally in such a way that the $\overset{1}{\Xi}_{\Lambda}$ and the $\tau_{\overset{2}{B},\overset{1}{B},\Xi}$ (or the $\tau_{\overset{1}{B},\overset{2}{B},\Xi}$) are varied arbitrarily.

From (IV) it follows that

$$\Omega_{\overset{2}{B},\Xi} = G^{(0)}\left(\tau_{\overset{2}{B},\overset{0}{B},\Xi},\ \overset{0}{\Xi},\ \overset{2}{\Xi}\right). \tag{42}$$

Hence, by (41),

$$\Omega_{\overset{2}{B},\Xi} = H^{(0)}\left(\tau_{\overset{2}{B},\overset{1}{B},\Xi},\ \tau_{\overset{1}{B},\overset{0}{B},\Xi},\ \overset{0}{\Xi},\ \overset{2}{\Xi}\right). \tag{43}$$

If it is possible to eliminate the $\tau_{\overset{1}{B},\overset{0}{B},\Xi}$ and the $\overset{0}{\Xi}$ from (IV) and (43), we get an equation of the form

$$\Omega_{\overset{2}{B},\Xi} = \overset{\Omega}{T}\left(\Omega_{\overset{1}{B},\Xi},\ \tau_{\overset{2}{B},\overset{1}{B},\Xi},\ \overset{2}{\Xi},\ \overset{1}{\Xi}\right) \quad \left(\begin{matrix}\text{V}\\ \text{Special geom. obj.}\end{matrix}\right), \tag{V}$$

holding for every choice of $\overset{2}{B}$, $\overset{1}{B}$, and Ξ. In this case Ω is a geometric object and we call it a *special geometric object*.

5. *Objects of class $v \leqslant u$.*

In the general equation of definition of an object (I*a*),

$$\Omega_{\overset{1}{B},\Xi} = F^{(0)}\left(\overset{1,0}{t},\ \overset{0}{\Xi}\right), \tag{44}$$

$F^{(0)}$ is a functional of the function $\overset{1,0}{t}$. Now it may happen that we can write the $\Omega_{\overset{1}{B},\Xi}$ as functions (no longer functionals) of $\overset{0}{\Xi}$, $\overset{1}{\Xi}$, and the derivatives of the $\overset{1}{\Xi}$ with regard to $\overset{0}{\Xi}$ up to the order $v \leqslant u$ at the point Ξ. The totality of these derivatives we symbolize by $\omega_{\overset{1}{B},\overset{0}{B},\Xi}$. An object of this kind is called an object of class v:

$$\Omega_{\overset{1}{B},\Xi} = G^{(0)}\left(\omega_{\overset{1}{B},\overset{0}{B},\Xi},\ \overset{0}{\Xi},\ \overset{1}{\Xi}\right) \quad \left(\begin{matrix}\text{VI}\\ \text{Object of class } v\end{matrix}\right). \tag{VI}$$

The $\omega_{\overset{1}{B},\overset{0}{B},\Xi}$, satisfying conditions (*a*)–(*c*) of § 4, constitute a special case of the $\tau_{\overset{1}{B},\overset{0}{B},\Xi}$; hence *every object of class v is a special object.* According to (V), a *geometric* object of class v is thus characterized by the existence of an equation of the form

$$\text{(VII)}\qquad \Omega_{\overset{2}{B},\,\Xi} = \overset{\Omega}{T}\left(\Omega_{\overset{1}{B},\,\Xi},\ \omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \overset{2}{\Xi},\ \overset{1}{\Xi}\right)\quad \left(\begin{matrix}\text{VII}\\ \text{Geom. object of class } v\end{matrix}\right),$$

holding for every choice of $\overset{2}{B}$, $\overset{1}{B}$, and Ξ.

Thus we have the following classification of objects:*

	General.	Special.	Class v.
General	I	(I), IV	(I, IV), VI
Macrogeom.	(I), II	(I), II, IV	(I), II, (IV), VI
Microgeom.	(I, II), III	(I, II, III, IV), V	(I, II, III, IV, V, VI), VII

with the logical relations between the conditions

$$(45)\qquad \begin{tikzcd} & \mathrm{I} \arrow[r, phantom, "\subset"] \arrow[dl, phantom, "\supset"] & \mathrm{IV} \arrow[r, phantom, "\subset"] \arrow[dd, phantom, "\cap"] & \mathrm{VI} \arrow[dd, phantom, "\cap"] \\ \mathrm{II} \arrow[dr, phantom, "\subset"] & & & \\ & \mathrm{III} \arrow[r, phantom, "\subset"] & \mathrm{V} \arrow[r, phantom, "\subset"] & \mathrm{VII} \end{tikzcd}$$

6. *The fundamental theorem.*

In the general case elimination of all the $\tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$ and the $\overset{0}{\Xi}$ from (IV) and (43) is not possible. Let the number of these parameters be m and let us call them $\phi_1, \ldots, \phi_m$. Then, if m_1 of the equations (IV) are independent with respect to $\phi_1, \ldots, \phi_m$, these equations are equivalent to the set of equations

$$(46)\qquad \begin{cases} \phi_1 = f_1^{(0)}\left(\phi_{m_1+1}, \ldots, \phi_m,\ \Omega_{\overset{1}{B},\,\Xi},\ \overset{1}{\Xi}\right),\\ \vdots \\ \phi_{m_1} = f_{m_1}^{(0)}\left(\phi_{m_1+1}, \ldots, \phi_m,\ \Omega_{\overset{1}{B},\,\Xi},\ \overset{1}{\Xi}\right). \end{cases}$$

Substituting in (43), we get

$$(47)\qquad \Omega_{\overset{2}{B},\,\Xi} = K^{(0)}\left(\Omega_{\overset{1}{B},\,\Xi},\ \tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \phi_{m_1+1}, \ldots, \phi_m,\ \overset{1}{\Xi},\ \overset{2}{\Xi}\right).$$

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* Conditions which depend on the other conditions are put in parentheses.

For a definite point $\underset{\Lambda}{\Xi}$ and a definite choice of $\overset{0}{B}$ and $\overset{1}{B}$ the $\phi_{m_1+1}, \dots, \phi_m$ are constant parameters. Now, in general, these parameters are not "essential" (*wesentlich*) in the sense of Lie. Lie called parameters essential if it is not possible to replace them by a smaller number of parameters; we use here two theorems from his theory:

1. The parameters $\phi_{m_1+1}, \dots, \phi_m$ are essential if, and only if, it is impossible to find $m-m_1$ functions χ of the ϕ *alone*, so that

$$\chi_{m_1+1}\frac{\partial K^{(0)}}{\partial\phi_{m_1+1}}+\dots+\chi_m\frac{\partial K^{(0)}}{\partial\phi_m}=0 \tag{48}$$

for every choice of $\Omega_{\overset{1}{B},\,\Xi}$, $\tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi}$, $\phi_{m_1+1}, \dots, \phi_m$, $\overset{1}{\Xi}$, and $\overset{2}{\Xi}$.

2. If the parameters $\phi_{m_1+1}, \dots, \phi_m$ are essential, every variation $\delta\phi_{m_1+1}, \dots, \delta\phi_m$, with constant $\Omega_{\overset{1}{B},\,\Xi}$, $\tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi}$, $\overset{1}{\Xi}$, and $\overset{2}{\Xi}$, leads to variations of the $\Omega_{\overset{2}{B},\,\Xi}$, which cannot *all* vanish.

In the general case we can introduce essential parameters $\psi_1, \dots, \psi_r$ $(r \leqslant m-m_1)$, and *completing* parameters $\theta_1, \dots, \theta_s$ $(s=m-m_1-r)$, so that $\phi_{m_1+1}, \dots, \phi_m$ are functions of the ψ and θ and *vice versa*, and $\phi_{m_1+1}, \dots, \phi_m$ can be replaced in (47) by functions of the ψ alone. The number r we call the characteristic number of the special object. The totality of all ψ we symbolize by $\pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$. Then from (47) it follows that

$$\Omega_{\overset{2}{B},\,\Xi}=L^{(0)}\left(\Omega_{\overset{1}{B},\,\Xi},\ \tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi},\ \overset{1}{\Xi},\ \overset{2}{\Xi}\right), \tag{49}$$

which holds for every choice of $\overset{0}{B}$, $\overset{1}{B}$, $\overset{2}{B}$, and Ξ. We give some examples of special objects which are all objects of class v.

Example 14. (*Cf.* Example 9) $v=1$.

$$\underbrace{v^{1''}}_{\Omega_{\overset{2}{B},\,\Xi}}=\underbrace{A^{1''}_{1'}\underbrace{v^{1'}}_{\Omega_{\overset{1}{B},\,\Xi}}+A^{1''}_{\alpha'}}_{\omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi}}\underbrace{v^{\alpha'}}_{\pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi}};\quad A^{\kappa''}_{\lambda'}=\delta^{\kappa''}_{\kappa}\,t_{\lambda}{}^{\kappa}(\xi^{\rho'})\,\delta^{\lambda}_{\lambda'}. \tag{50}$$

Example 15. $\Gamma^{1}_{\mu\lambda}$ is a system of n^2 of the n^3 parameters of a linear connection in X_n; $v=2$.

$$\underbrace{\Gamma^{1''}_{\mu''\lambda''}}_{\Omega_{\overset{2}{B},\,\Xi}}=\underbrace{A^{1''\mu'\lambda'}_{1'\mu''\lambda''}}_{\omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi}}\underbrace{\Gamma^{1'}_{\mu'\lambda'}}_{\Omega_{\overset{1}{B},\,\Xi}}+\underbrace{A^{1''\mu'\lambda'}_{\alpha'\mu''\lambda''}}_{\omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi}}\underbrace{\Gamma^{\alpha'}_{\mu'\lambda'}}_{\pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi}}+\underbrace{A^{1''}_{\lambda'}\,\partial_{\mu''}A^{\lambda'}_{\lambda''}}_{\omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi}}. \tag{51}$$

Example 16. (*Cf*. Example 13) $v=1$.

$$(52)\qquad \Omega_{\overset{2}{B},\,\Xi}=\underbrace{\overset{2,\,1}{a}}_{\omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi}}\underbrace{\frac{\psi\left(\overset{2}{\Xi}\right)}{\psi\left(\overset{2}{\Xi}\right)}}_{\overset{1}{\Xi},\ \overset{2}{\Xi}}\Omega_{\overset{1}{B},\,\Xi}+\underbrace{\overset{2,\,1}{a}\left\{\phi\left(\overset{2}{\Xi}\right)-\frac{\psi\left(\overset{2}{\Xi}\right)}{\psi\left(\overset{1}{\Xi}\right)}\,\phi\left(\overset{1}{\Xi}\right)\right\}}_{\omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \overset{1}{\Xi},\ \overset{2}{\Xi}}\underbrace{\overset{1,\,0}{a}}_{\pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi}}+\underbrace{\overset{2,\,1}{b}\,\psi\left(\overset{2}{\Xi}\right)}_{\omega_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \overset{2}{\Xi}}.$$

From (49) it follows, on the one hand, that

$$(53)\qquad \Omega_{\overset{3}{B},\,\Xi}=L^{(0)}\left(\Omega_{\overset{1}{B}.\,\Xi},\ \tau_{\overset{3}{B},\,\overset{1}{B},\,\Xi},\ \pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi}\ \overset{1}{\Xi},\ \overset{3}{\Xi}\right),$$

and, on the other hand, that

$$(54)\qquad \Omega_{\overset{3}{B},\,\Xi}=L^{(0)}\left(\Omega_{\overset{2}{B},\,\Xi},\ \tau_{\overset{3}{B},\,\overset{2}{B},\,\Xi},\ \pi_{\overset{2}{B},\,\overset{0}{B},\,\Xi},\ \overset{1}{\Xi},\ \overset{3}{\Xi}\right).$$

Now the $\pi_{\overset{2}{B},\,\overset{0}{B},\,\Xi}$ are, according to their definition, functions of the $\tau_{\overset{2}{B},\,\overset{0}{B},\,\Xi}$ and the $\overset{0}{\Xi}$. But the $\tau_{\overset{2}{B},\,\overset{0}{B},\,\Xi}$ can, according to (41), be expressed as functions of the $\tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi}$ and the $\tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$, and the $\tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$ and $\overset{0}{\Xi}$ can, according to (46), be expressed as functions of the $\pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi}$, $\theta_1, \ldots, \theta_s$, the $\Omega_{\overset{1}{B},\,\Xi}$, and the $\overset{1}{\Xi}$. Hence there exist equations of the form

$$(55)\qquad \pi_{\overset{2}{B},\,\overset{0}{B},\,\Xi}=F\left(\tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \tau_{\overset{1}{B},\,\overset{0}{B},\,\Xi},\ \overset{0}{\Xi}\right),$$

or, from (46),

$$(56)\qquad \pi_{\overset{2}{B},\,\overset{0}{B},\,\Xi}=G^{(0)}\left(\tau_{\overset{2}{B},\,\overset{1}{B},\,\Xi},\ \Omega_{\overset{1}{B},\,\Xi},\ \pi_{\overset{1}{B},\,\overset{0}{B},\,\Xi},\ \theta_1,\ \ldots,\ \theta_s,\ \overset{1}{\Xi}\right),$$

holding for every choice of $\overset{0}{B}$, $\overset{1}{B}$, $\overset{2}{B}$, and Ξ.

We shall prove that the right-hand side of (56) does not really depend on $\theta_1, \ldots, \theta_s$.

For that purpose, we begin with a definite chosen point $\underset{\Lambda}{\Xi}$ and give this point a variation in such a way that the $\underset{\Lambda}{\overset{0}{\Xi}}$ have arbitrary variations, chosen at will. $\left(\overset{0}{B}\text{ remains constant.}\right)$ By this process, the $\underset{\Lambda}{\overset{1}{\Xi}}$, $\underset{\Lambda}{\overset{2}{\Xi}}$, and $\underset{\Lambda}{\overset{3}{\Xi}}$ also vary. Now we use condition (*c*) on page 371 and give $\overset{1}{B}$ a variation in such a way that the variation of $\underset{\Lambda}{\overset{1}{\Xi}}$, just obtained, is annihilated and the

$\tau_{\overset{1}{B},\overset{0}{B},\Xi}$ have arbitrary variations, chosen at will. By this process, the variations of the $\underset{\Lambda}{\overset{2}{\Xi}}$ and the $\underset{\Lambda}{\overset{3}{\Xi}}$ do not change, but the $\tau_{\overset{3}{B},\overset{1}{B},\Xi}$ and the $\tau_{\overset{2}{B},\overset{1}{B},\Xi}$ vary. Finally, we give $\overset{2}{B}$ and $\overset{3}{B}$ variations in such a way that these variations of the $\overset{2}{\Xi}$, $\tau_{\overset{2}{B},\overset{1}{B},\Xi}$ and the $\underset{\Lambda}{\overset{3}{\Xi}}$, $\tau_{\overset{3}{B},\overset{1}{B},\Xi}$ are annihilated, this being possible in virtue of condition (*c*) on page 371. The final result is that the $\underset{\Lambda}{\overset{0}{\Xi}}$ and the $\tau_{\overset{1}{B},\overset{0}{B},\Xi}$ have arbitrary variations, chosen at will, and that the $\underset{\Lambda}{\overset{1}{\Xi}}$, $\underset{\Lambda}{\overset{2}{\Xi}}$, $\tau_{\overset{2}{B},\overset{1}{B},\Xi}$ and $\tau_{\overset{3}{B},\overset{1}{B},\Xi}$ remain constant. Now the $\underset{\Lambda}{\overset{0}{\Xi}}$ together with the $\tau_{\overset{1}{B},\overset{0}{B},\Xi}$ are equivalent to $\phi_1, \ldots, \phi_m$ and this set is equivalent to the $\pi_{\overset{1}{B},\overset{0}{B},\Xi}$, $\theta_1, \ldots, \theta_s$. Hence it is possible to choose the variations in such a way that

1. The $\pi_{\overset{1}{B},\overset{0}{B},\Xi}$ remain constant,

2. $\theta_1, \ldots, \theta_s$ have arbitrary, *non-vanishing* variations,

3. $\phi_1, \ldots, \phi_{m_1}$ vary in such a way that they satisfy the equation (46) with constant $\Omega_{\overset{1}{B},\Xi}$, $\overset{1}{\Xi}$, and $\pi_{\overset{1}{B},\overset{0}{B},\Xi}$ and changed $\theta_1, \ldots, \theta_s$.

If these variations are substituted in (IV) we naturally find that the variations of the $\Omega_{\overset{1}{B},\Xi}$ are zero, and so we have shown that the $\Omega_{\overset{1}{B},\Xi}$, $\overset{1}{\Xi}$, $\overset{2}{\Xi}$, $\overset{3}{\Xi}$, $\pi_{\overset{1}{B},B,\Xi}$, $\tau_{\overset{2}{B},\overset{1}{B},\Xi}$, and $\tau_{\overset{3}{B},\overset{1}{B},\Xi}$ remain constant, while the $\theta_1, \ldots, \theta_s$ have arbitrary, *non-vanishing* variations. From (49) and (53), it follows also that the $\Omega_{\overset{2}{B},\Xi}$ and $\Omega_{\overset{3}{B},\Xi}$ remain constant. Now, if the right-hand side of (56) depended on $\theta_1, \ldots, \theta_s$, it would be possible to choose the variations of the $\theta_1, \ldots, \theta_s$ in such a way that at least one of the variations of the $\pi_{\overset{2}{B},\overset{0}{B},\Xi}$ would not be zero. But, the $\pi_{\overset{2}{B},\overset{0}{B},\Xi}$ being essential parameters in (54), it follows from the second theorem of Lie, mentioned above, that, in that case, at least one of the variations of the $\Omega_{\overset{3}{B},\Xi}$ would not vanish. This being in contradiction with the result that the $\Omega_{\overset{3}{B},\Xi}$ remain constant, we have proved that the right-hand side of (56) does not really depend on $\theta_1, \ldots, \theta_s$ and this equation is, therefore, equivalent to an equation of the form

$$(57) \qquad \pi_{\overset{2}{B},\overset{0}{B},\Xi} = \overset{\pi}{T}\left(\pi_{\overset{1}{B},\overset{0}{B},\Xi},\ \Omega_{\overset{1}{B},\Xi},\ \tau_{\overset{2}{B},\overset{1}{B},\Xi},\ \overset{1}{\Xi}\right).$$

From this equation and (49) we deduce the *fundamental theorem,*

If Ω *is a special object, which is not geometric, the* $\Omega_{B,\Xi}$ *together with the* $\pi_{B,\overset{0}{B},\Xi}$ *constitute a special geometric object.*

Or, in another form,

Every special object Ω *which is not geometric with* M *components and a characteristic number* r *can be completed to a special geometric object with* $M+r$ *components.*

The $\pi_{B, \overset{0}{B}, \Xi}$ being essential parameters in (47), it is clear that the geometric object at which we have arrived has the smallest possible number of components. Geometric objects with more components that contain Ω can easily be constructed. For example, the $\Omega_{B, \Xi}$, $\tau_{B, \overset{0}{B}, \Xi}$, and $\overset{0}{\Xi}$ constitute a geometric object, as follows from the equations (41) and (43).

We give some examples of the completion of an object:

Example 17. ξ^1; completion ξ^α; $\alpha = 2, \ldots, n$.

Example 18. (*Cf.* Example 14.)

$$v^1; \text{ completion } v^\alpha.$$

Example 19. (*Cf.* Example 15.)

$$\Gamma^1_{\mu\lambda}; \text{ completion } \Gamma^\alpha_{\mu\lambda}.$$

It follows, from the fundamental theorem, that every macrogeometric object, which is special, but not geometric, can be completed to a special geometric object with the smallest possible number of components.

Example 20. (*Cf.* Examples 13 and 16.)

Because
$$\overset{2,0}{a} = \underbrace{\overset{2,1}{a}}_{\omega^{}_{\overset{2}{B}, \overset{1}{B}, \Xi}} . \overset{1,0}{a},$$
the object $\left(\Omega_{\overset{1}{B}, \Xi}, \overset{1,0}{a}\right)$ is a geometric object of class one.

One of the reasons why macrogeometric objects have never occurred till now is that in differential geometry we nearly always have to deal with objects of a distinct class, and these objects, if they are macrogeometric, can be considered as parts of microgeometric objects, which are much easier to handle.

Delft, Holland.

SINGULAR MODULI (5)

By G. N. WATSON.

[Read June 18, 1936.—Received July 2, 1936.]

The degree of the equation satisfied by singular moduli of elliptic functions with periods connected by the relation $K' = K\sqrt{n}$ is intimately connected with the class-number of the negative determinant $-n$. In consequence of this, modern tables of singular moduli are usually concerned with sets of values of n having the same class-number. Thus, Weber[†] has constructed the values, in the form of surds, of the sixty-five singular moduli corresponding to the sixty-five negative determinants of quadratic forms which (according to the classification of Gauss) have one class in each genus, the number of genera being I, II, IV, VIII, or XVI; it is believed that there are no other quadratic forms with one class only in each genus. Later Berwick (**1**) constructed the values of the fifty-four Dedekind invariants $j(\tau)$ which are algebraic integers of the second and third degrees. Ramanujan's notebooks, moreover, contain evidence that he was making advances on the lines initiated by Weber, since he seems to have started on the task of calculating systematically the singular moduli corresponding to negative determinants of quadratic forms which (according to the classification of Gauss) have two classes in each genus. So far as is known, there are, in all, 231 such negative determinants, the numbers of determinants with II, IV, VIII, XVI, XXXII genera being 20, 69, 97, 40, 5 respectively. More recently I have myself, in my paper (**10**), gone in a direction different from that of Weber and Ramanujan, by considering negative determinants with I genus and 1, 3, 5, 7, 9, 11, 13, and 15 classes in that genus; such a piece of work cannot be regarded as complete, since the number of negative determinants of genus I is infinite, while the work of constructing the equation satisfied by a singular modulus becomes prohibitively heavy when the class-number gets large.

[†] See Weber (**12**), 720–726. He had published the results previously, *Math. Annalen*, 33 (1889), 390–410.

Pending the completion of Ramanujan's work just described, I have carried out the much lighter task of investigating the negative determinants with II genera and either 3 or 4 classes in each genus; there are 48 of the former and 31 of the latter. In the present paper I deal with the 48 negative determinants with II genera and 3 classes in each genus. It is convenient to divide the 48 values of $-n$ into types, according to the residues of n modulo 8. I adopt Greenhill's mode of division, namely

Class A: $n \equiv 3 \pmod 8$. Class B: $n \equiv 7 \pmod 8$.

Class C: $n \equiv 1 \pmod 4$. Class D: $n \equiv 2 \pmod 4$.

To these I add Class E: $n \equiv 0 \pmod 4$, which Greenhill does not need in his work. The 48 values of n with their classifications are given in the adjoining table.

26	D	75	A	115	A	172	E	268	E
29	C	76	E	118	D	175	B	277	C
35	A	81	C	121	C	187	A	298	D
38	D	87	B	123	A	202	D	358	D
44	E	91	A	124	E	207	B	397	C
50	D	92	E	135	B	214	D	403	A
51	A	99	A	147	A	235	A	427	A
53	C	106	D	157	C	247	B	652	E
54	D	108	E	162	D	262	D		
61	C	109	C	169	C	267	A		

This classification into types is adequate for the purposes of the present paper, but it is not complete, since the classes C and D should be subdivided; the complete classification is given by Weber (**12**, 473).

It is convenient to use Weber's notation for the class-invariants which will be investigated, with a few additions, some of which are due to Russell and Ramanujan.

Let
$$q = e^{i\pi\tau} \quad [I(\tau) > 0,\ |q| < 1)],$$

$$f(\tau) = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1+q^{2m-1}), \quad f_1(\tau) = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1-q^{2m-1}),$$

$$f(\sqrt{-n}) = f_n, \quad 2^{-\frac{1}{4}} f(\sqrt{-n}) = G_n, \quad 2^{-\frac{1}{4}} f_1(\sqrt{-n}) = g_n, \quad 2^{-\frac{1}{2}} f(\sqrt{-n}) = F_n.$$

For odd values of n the class-invariant to be discussed is either f_n or F_n, or a power of one of these; for even values of n it is g_n or a power of g_n. In each case the appropriate class-invariant satisfies a sextic equation with rational coefficients. This sextic equation[†] is resoluble into a cubic equation whose coefficients are quadratic irrationalities; for the smaller values of n this reduction has been effected by previous writers. It does not, however, seem to have been fully realised that the sextic is also resoluble into a quadratic equation whose coefficients are rational in a cubic field, so that the solution of the sextic equation is ultimately expressible as a rational function of a quadratic surd and a cubic irrationality[‡]. The main object of the present paper is to exhibit the forty-eight class-invariants under discussion in such a form. Usually, but not invariably, it has been convenient to employ the cubic irrationalities constructed by Berwick for Mathews (**4**). To each cubic equation[§] introduced I append the value of its discriminant Δ.

For the twenty-seven quadratfrei values of n in classes A, C, and D, the equations were constructed empirically; for example, when the value of G_{397}^4 has been computed to be $16974 \cdot 78887\,15892$, it is an easy matter (with the help of a calculating machine) to construct a cubic equation which it satisfies (at any rate approximately), the coefficients in the equation having the properties that (i) they are integers in the quadratic field defined by $\sqrt{397}$, (ii) their moduli are of the same kind of order of magnitude as G_{397}^4; say not larger than 10^6, (iii) their norms are comparatively small. In none of the cases under discussion did more than one cubic seem to require serious consideration. When the cubic had been obtained, the fact that its solution could be expressed as a rational function of a quadratic surd and a cubic irrationality seemed to provide an adequate guarantee that the correct cubic equation had been discovered. The class-invariants in class B were obtained by the method described in my previous papers; the remainder were obtained by means of the modular equations of Schläfli's type cited by Weber (**12**, 115, 262, 263).

[†] It was remarked by the Referee that the sextic equations of this type belong to the group of order 12. For the classification of sextic equations and the properties of this group, with full references to the literature of the subject, see W. E. H. Berwick, *Proc. London Math. Soc.* (2), 29 (1929), 1–28 (27).

[‡] This was brought home to me when investigating the class-invariants of Class B which are common to this paper and to (**11**). Since, for instance,

$$\frac{F_{247}}{F_{19/13}} + \frac{F_{19/13}}{F_{247}} \quad \text{and} \quad F_{247} F_{19/13}$$

are cubic irrationalities, it is clear that F_{247} is the root of a quadratic equation whose coefficients are cubic irrationalities, and so the solution of the sextic can be effected in the form stated. It was then natural to examine the other classes of class-invariants by empirical methods, and, in each case, the solution of the sextic was effected in the form required.

[§] In these equations, and elsewhere, the Weierstrassian * indicates an absent term.

With these preliminary explanations it is now practicable to give the catalogue of results; the results are arranged by classes, the values of n in each class being arranged in order of magnitude.

Class A. $n \equiv 3 \pmod 8$.

There are thirteen values of n in class A, namely 35, 51, 75, 91, 99, 115, 123, 147, 187, 235, 267, 403, 427. The class-invariant to be evaluated is $f(\sqrt{-n})$ when n is prime to 3, and $\frac{1}{2}f^3(\sqrt{-n})$ when n is a multiple of 3.

The results which I have obtained are as follows:

n = 35. The sextic equation satisfied by $f(\sqrt{-35})$ is

$$f^6-2f^5-2f^4+4f^3-*-4f+4=0;$$

by adjunction of $\sqrt{5}$, this reduces to the cubic given by Weber (**12**)

$$f^3-(1+\sqrt{5})f^2+(1+\sqrt{5})f-2=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3+*+2\theta-2=0 \quad (\Delta=-140)$$

we have
$$2f(\sqrt{-35})=\theta^2+\theta+2+\frac{-\theta^2+\theta+2}{\sqrt{5}}.$$

n = 51. The sextic equation satisfied by $f^3(\sqrt{-51})$ is

$$f^{18}-16f^{15}-12f^{12}+48f^9+144f^6+64f^3+64=0;$$

by adjunction of $\sqrt{17}$, this reduces to the cubic given by Weber (**12**)

$$f^9-(8+2\sqrt{17})f^6-4f^3-8=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-\theta^2+\theta-3=0 \quad (\Delta=-204)$$

we have
$$f^3(\sqrt{-51})=2\theta^2+\theta+3+\frac{4\theta^2+5\theta+11}{\sqrt{17}};$$

also
$$f(\sqrt{-51})f(\tfrac{1}{3}\sqrt{-51})=\theta^2+1.$$

n = 75. The sextic equation satisfied by $f(\sqrt{-75})\,.\sqrt[3]{2}$ is

$$x^6-4x^5+*+*+*+16x+16=0,$$

which, by adjunction of $\sqrt{5}$, reduces to the cubic

$$x^3-2x^2-(2+2\sqrt{5})x-4=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-4\theta^2+2\theta-2=0 \quad (\Delta=-300)$$

we have
$$f(\sqrt{-75}) . \sqrt[3]{16}=\theta^2-2\theta+\frac{-\theta^2+6\theta-4}{\sqrt{5}}$$
$$=\theta^2-2\theta+\frac{(\theta^2-3\theta)^2}{\sqrt{5}}$$
$$=\theta^2-2\theta+\sqrt[6]{80};$$

also
$$f(\sqrt{-75})\, f(\tfrac{1}{3}\sqrt{-75})=\theta\sqrt[3]{2}; \quad (\theta^2-3\theta)^3=10.$$

Some other results† about $f(\sqrt{-75})$ have been given by Russell (**7**).

n = 91. The sextic equation satisfied by $f(\sqrt{-91})$ is

$$f^6-4f^5+2f^4-*-4f^2+4f+4=0,$$

which, by adjunction of $\sqrt{13}$, reduces to

$$f^3-2f^2-(1+\sqrt{13})f-2=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3+*+4\theta-2=0 \quad (\Delta=-364)$$

we have
$$2f(\sqrt{-91})=\theta^2+\theta+4+\frac{3\theta^2-\theta+8}{\sqrt{13}}.$$

n = 99. The sextic equation satisfied by $f^3(\sqrt{-99})$ is

$$f^{18}-52f^{15}+116f^{12}-240f^9+144f^6+256f^3+64=0,$$

which, by adjunction of $\sqrt{33}$, reduces to

$$f^9-(26+4\sqrt{33})f^6-(16+4\sqrt{33})f^3-8=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-2\theta^2+2\theta-2=0 \quad (\Delta=-44),$$

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† [*Added November* 6, 1936. The Referee has reminded me that the value of $j\{\frac{1}{2}(5+\sqrt{-75})\}$, as well as values of j associated with other members of the set of thirteen class-invariants in Class A, are to be found in a paper, whose existence I had temporarily forgotten, by W. E. H. Berwick, *Quart. J. of Math.*, 50 (1927), 1–32. Berwick's paper also contains values of j associated with negative determinants whose classification is other than II.3.

which gives $f(\sqrt{-11})=\theta$, we have

$$f^3(\sqrt{-99})=9\theta^2-5\theta+12+\frac{45\theta^2-15\theta+54}{\sqrt{33}};$$

also
$$f(\sqrt{-99})\,f(\tfrac{1}{11}\sqrt{-99})=\theta^2+2.$$

n = 115. The sextic equation satisfied by $f(\sqrt{-115})$ is

$$f^6-6f^5+10f^4-12f^3+16f^2-12f+4=0;$$

by adjunction of $\sqrt{5}$, this reduces to the cubic given by Greenhill (**2**)

$$f^3-(3+\sqrt{5})\,f^2+(3+\sqrt{5})\,f-2=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-\theta^2+5\theta-3=0 \quad (\Delta=-460)$$

we have
$$2f(\sqrt{-115})=\theta^2+5+\frac{\theta^2-2\theta+7}{\sqrt{5}}.$$

n = 123. The sextic equation satisfied by $f^3(\sqrt{-123})$ is

$$f^{18}-80f^{15}+156f^{12}+672f^9+240f^6-256f^3+64=0,$$

which, by adjunction of $\sqrt{41}$, reduces to

$$f^9-(40+6\sqrt{41})f^6+(16+4\sqrt{41})f^3-8=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-2\theta^2+4\theta-6=0 \quad (\Delta=-492)$$

we have

$$f^3(\sqrt{-123})=7\theta^2-2\theta+24+\frac{39\theta^2-12\theta+142}{\sqrt{41}};$$

also
$$f(\sqrt{-123})\,f(\tfrac{1}{3}\sqrt{-123})=\theta^2+4.$$

n = 147. The sextic equation satisfied by $f(\sqrt{-147})/\sqrt[3]{2}$ is

$$x^6-5x^5+4x^4+x^3+2x^2-3x+1=0,$$

which, by adjunction of $\sqrt{21}$, reduces to

$$x^3-\frac{5+\sqrt{21}}{2}\,x^2+\frac{3+\sqrt{21}}{2}\,x-1=0.$$

In terms of the cubic irrationality†

$$\theta^3-5\theta^2-\theta-1=0 \quad (\Delta=-588)$$

we have
$$f(\sqrt{-147})\,.\,\sqrt[3]{4}=\theta+\frac{2\theta^2-9\theta+4}{\sqrt{21}};$$

also
$$f(\sqrt{-147})\,f(\tfrac{1}{3}\sqrt{-147})=\theta\,.\,\sqrt[3]{4}.$$

A different form of this result has been given by Ramanujan (**6**).

n = 187. The sextic equation satisfied by $f(\sqrt{-187})$ is

$$f^6-6f^5-2f^4+12f^3+4f^2-12f+4=0,$$

which, by adjunction of $\sqrt{17}$, reduces to

$$f^3-(3+\sqrt{17})f^2+(3+\sqrt{17})f-2=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-2\theta^2+2\theta-6=0 \quad (\Delta=-748)$$

we have
$$2f(\sqrt{-187})=\theta^2+2+\frac{\theta^2+2\theta+10}{\sqrt{17}}.$$

n = 235. The sextic equation satisfied by $f(\sqrt{-235})$ is

$$f^6-10f^5+22f^4-24f^3+16f^2-4f+4=0,$$

which, by adjunction of $\sqrt{5}$, reduces to

$$f^3-(5+\sqrt{5})f^2+(1-\sqrt{5})f-2=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-3\theta^2+\theta-5=0 \quad (\Delta=-940)$$

we have
$$2f(\sqrt{-235})=\theta^2-\theta+2+\frac{\theta^2+\theta}{\sqrt{5}}.$$

n = 267. The sextic equation satisfied by $f^3(\sqrt{-267})$ is

$$f^{18}-620f^{15}+4884f^{12}+2000f^9+5136f^6+640f^3+64=0,$$

which, by adjunction of $\sqrt{89}$, reduces to

$$f^9-(310+32\sqrt{89})f^6-(40+4\sqrt{89})f^3-8=0.$$

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† This is connected with Berwick's cubic irrationality

$$\phi^3-2\phi^2+6\phi-6=0 \quad (\Delta=-588)$$

by the relation
$$\phi=1+\theta^{-1}.$$

In terms of the cubic irrationality

$$\theta^3-2\theta^2-*-12=0 \quad (\Delta=-4272)$$

we have

$$2f^3(\sqrt{-267})=35\theta^2+42\theta+132+\frac{323\theta^2+378\theta+1216}{\sqrt{89}};$$

also $$f(\sqrt{-267})\,f(\tfrac{1}{3}\sqrt{-267})=\theta^2+\theta+4.$$

n = 403. The sextic equation satisfied by $f(\sqrt{-403})$ is

$$f^6-12f^5-26f^4+4f^3+36f^2+20f+4=0,$$

which, by adjunction of $\sqrt{13}$, reduces to

$$f^3-(6+2\sqrt{13})f^2-(5+\sqrt{13})f-2=0.$$

In terms of the cubic irrationality

$$\theta^3-\theta^2-3\theta-7=0 \quad (\Delta=-1612)$$

we have $$2f(\sqrt{-403})=\theta^2+2\theta+1+\frac{3\theta^2+4\theta+9}{\sqrt{13}}.$$

n = 427. The sextic equation satisfied by $f(\sqrt{-427})$ is

$$f^6-14f^5-16f^4+24f^3+32f^2+8f+4=0;$$

by adjunction of $\sqrt{61}$, this reduces to the cubic given by Russell (7)

$$f^3-(7+\sqrt{61})f^2-2f-2=0.$$

In terms of the cubic irrationality

$$\theta^3-2\theta^2-4\theta-6=0 \quad (\Delta=-1708)$$

we have $$2f(\sqrt{-427})=\theta^2+\theta+\frac{5\theta^2+7\theta+16}{\sqrt{61}}.$$

Seven of the thirteen values of n just considered are of the form ab, where a and b are primes other than 3; in these seven cases, the values of $f(\sqrt{-a/b})$ are obtainable from the values given for $f(\sqrt{-ab})$ by merely changing the sign of the square root which occurs in each expression.

Class B. $n\equiv 7 \pmod 8$.

There are five values of n in class B, namely 87, 135, 175, 207, 247. The class-invariant to be evaluated is $F_n=f(\sqrt{-n})/\sqrt{2}$ when n is prime to 3, and $F_n{}^3$ when n is a multiple of 3.

A full investigation of these class-invariants has already been given by Watson (**11**). To maintain a balance between completeness in this paper and unnecessary repetition of the previous paper, I give merely the values of the five class-invariants in terms of cubic and quadratic irrationalities; the computation of the sextic and cubic equations given in the previous paper is then a matter of straightforward arithmetic for those to whom that paper is inaccessible.

The results are as follows.

n = 87. With
$$\theta^3-2\theta^2+3\theta-3=0 \quad (\Delta=-87)$$
we have
$$2F_{87}^3=4\theta^2-3\theta+9+\frac{24\theta^2-15\theta+55}{\sqrt{29}},$$
$$F_{87}F_{29/3}=\theta^2-\theta+2.$$

n = 135. With
$$\theta^3-3\theta^2-*-1=0 \quad (\Delta=-135)$$
we have
$$2F_{135}^3=\theta^2+\theta+1+\frac{7\theta^2+3\theta+1}{\sqrt{5}},$$
$$F_{135}F_{27/5}=\theta.$$

n = 175. With
$$\theta^3-\theta^2+2\theta-3=0 \quad (\Delta=-175)$$
we have
$$2F_{175}=\theta^2+\theta+2+\frac{\theta^2+\theta+4}{\sqrt{5}}.$$

n = 207. With $F_{23}=\theta$,
$$\theta^3-*-\theta-1=0 \quad (\Delta=-23)$$
we have
$$2F_{207}^3=24\theta^2+31\theta+18+\frac{198\theta^2+255\theta+144}{\sqrt{69}},$$
$$F_{207}F_{23/9}=\theta^2+\theta+1.$$

n = 247. With
$$\theta^3-3\theta^2+4\theta-5=0 \quad (\Delta=-247)$$
we have
$$2F_{247}=\theta^2-\theta+2+\frac{5\theta^2-7\theta+14}{\sqrt{13}}.$$

Class C. $n \equiv 1 \pmod 4$.

There are ten values of n in class C, namely 29, 53, 61, 81, 109, 121, 157, 169, 277, 397. For the seven quadratfrei values of n the class-invariant to be evaluated is $G_n{}^4 \equiv \frac{1}{2} f^4(\sqrt{-n})$; for these values of n the sextic equation satisfied by $G_n{}^4$ is skew-reciprocal, and I accordingly state it as a cubic equation in $T_n \equiv G_n{}^4 - G_n^{-4}$.

The results which I have obtained are as follows.

n = 29. The cubic equation satisfied by T_{29}, due to Russell (**7**), is

$$T^3 - 9T^2 + 8T - 16 = 0 \quad (\Delta = -29 \times 32^2);$$

by adjunction of $\sqrt{29}$, this reduces to an equation given by Weber (**12**), namely

$$G^{12} - \frac{9+\sqrt{29}}{2} G^8 - (4+\sqrt{29})\, G^4 - \frac{5+\sqrt{29}}{2} = 0.$$

In terms of Berwick's cubic irrationality

$$\theta^3 - \theta^2 - * - 2 = 0 \quad (\Delta = -116)$$

we have
$$2G_{29}^{\pm 4} = \frac{11\theta^2 + 6\theta + 4}{\sqrt{29}} \pm (\theta^2 + 2\theta + 2).$$

n = 53. The cubic equation satisfied by T_{53}, due to Mathews (**4**) and Russell (**7**), is

$$T^3 - 23T^2 + 15T - 121 = 0 \quad (\Delta = -53 \times 320^2);$$

by adjunction of $\sqrt{53}$, this reduces to

$$G^{12} - \frac{23+3\sqrt{53}}{2} G^8 - \frac{1+\sqrt{53}}{2} G^4 - \frac{7+\sqrt{53}}{2} = 0.$$

In terms of Berwick's cubic irrationality

$$\theta^3 - \theta^2 + 4\theta - 2 = 0 \quad (\Delta = -212)$$

we have
$$2G_{53}^{\pm 4} = \frac{44\theta^2 - 16\theta + 161}{\sqrt{53}} \pm (6\theta^2 - 4\theta + 23).$$

n = 61. The cubic equation satisfied by T_{61}, due to Mathews (**4**) and Russell (**7**), is

$$T^3 - 30T^2 + 9T - 108 = 0 \quad (\Delta = -61 \times 432^2);$$

by adjunction of $\sqrt{61}$, this reduces to

$$G^{12} - (15+2\sqrt{61})\, G^8 + \frac{25+3\sqrt{61}}{2} G^4 - \frac{39+5\sqrt{61}}{2} = 0.$$

In terms of the cubic irrationality†

$$\theta^3-2\theta^2-3\theta-2=0 \quad (\Delta=-244)$$

we have
$$2G_{61}^{\pm 4}=\frac{17\theta^2+16\theta+14}{\sqrt{61}}\pm 3\theta^2.$$

n = 81. The cubic equation satisfied by $x\equiv G_{81}^3+G_{81}^{-3}$ is

$$x^3-18x^2-48x-32=0,$$

which, by adjunction of $\sqrt{3}$, reduces to

$$G^9-(9+6\sqrt{3})\,G^6-(9+6\sqrt{3})\,G^3-(7+4\sqrt{3})=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-3\theta^2-*-2=0 \quad (\Delta=-324)$$

we have
$$G_{81}^{\pm 3}=\theta^2\pm\frac{(\theta+1)^2}{\sqrt{3}}.$$

Another representation of G_{81} is given by Ramanujan (**6**).

n = 109. The cubic equation satisfied by T_{109} is

$$T^3-117T^2-153T-243=0 \quad (\Delta=-109\times 3456^2),$$

which, by adjunction of $\sqrt{109}$, reduces to

$$G^{12}-\frac{117+11\sqrt{109}}{2}G^8-\frac{281+27\sqrt{109}}{2}G^4-\frac{261+25\sqrt{109}}{2}=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-3\theta^2+4\theta-6=0 \quad (\Delta=-436)$$

we have

$$2G_{109}^{\pm 4}=\frac{188\theta^2-120\theta+457}{\sqrt{109}}\pm(18\theta^2-12\theta+45).$$

n = 121. The cubic equation satisfied by $x\equiv(G_{121}-G_{121}^{-1})\sqrt{2}$, which, according to Greenhill (**3**), is due to Russell, is

$$x^3-4x^2-2x-4=0 \quad (\Delta=-1936);$$

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† This is connected with Berwick's cubic irrationality

$$2\phi^3-2\phi^2+3\phi-2=0 \quad (4\Delta=-61)$$

by the relation
$$\phi=\theta/(\theta+1).$$

by adjunction of $\sqrt{11}$, this reduces to

$$G^3-\sqrt{2}\,.\,G^2-(3+\sqrt{11})\,G-\frac{3\sqrt{2}+\sqrt{22}}{2}=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-2\theta^2+5\theta-6=0 \quad (\Delta=-484)$$

we have
$$2G_{121}^{\pm 1}=\frac{5\theta^2-3\theta+12}{\sqrt{22}}\pm\frac{\theta^2-\theta+4}{\sqrt{2}}.$$

Another representation of G_{121} is given by Ramanujan (**6**) and Watson (**9**).

n = **157.** The cubic equation satisfied by T_{157} is

$$T^3-354T^2+225T-972=0 \quad (\Delta=-157\times 32400^2),$$

which, by adjunction of $\sqrt{157}$, reduces to

$$G^{12}-(177+14\sqrt{157})\,G^8-\frac{335+27\sqrt{157}}{2}\,G^4-\frac{213+17\sqrt{157}}{2}=0.$$

In terms of the cubic irrationality†

$$\theta^3-12\theta^2+13\theta-6=0 \quad (\Delta=-10048)$$

we have
$$2G_{157}^{\pm 4}=\frac{43\theta^2-60\theta+14}{\sqrt{157}}\pm 3\theta^2.$$

n = **169.** The cubic equation satisfied by $x\equiv G_{169}-G_{169}^{-1}$, which, according to Greenhill (**3**), is due to Russell, is

$$x^3-4x^2+x-12=0 \quad (\Delta=-6084);$$

by adjunction of $\sqrt{13}$, this reduces to

$$G^3-(2+\sqrt{13})\,G^2+\frac{7+\sqrt{13}}{2}\,G-\frac{3+\sqrt{13}}{2}=0.$$

In terms of Berwick's cubic irrationality

$$2\theta^3-2\theta^2+5\theta-2=0 \quad (4\Delta=-169)$$

we have
$$G_{169}^{\pm 1}=\frac{3\theta^2-2\theta+9}{\sqrt{13}}\pm(\theta^2+2).$$

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† This is connected with Berwick's cubic irrationality

$$2\phi^3-5\phi^2+6\phi-5=0 \quad (4\Delta=-157)$$

by the relation
$$2\phi=(3\theta-1)/(\theta-1).$$

Another representation of G_{169} is given by Ramanujan (6) and Watson (9).

n = 277. The cubic equation satisfied by T_{277} is

$$T^3-3045T^2-153T-11907=0$$

$$(\Delta=-277\times 22\,03200^2),$$

which, by adjunction of $\sqrt{277}$, reduces to

$$G^{12}-\frac{3045+183\sqrt{277}}{2}G^8+\frac{951+57\sqrt{277}}{2}G^4-\frac{2613+157\sqrt{277}}{2}=0.$$

In terms of the cubic irrationality

$$\theta^3-5\theta^2-*-2=0 \quad (\Delta=-1108)$$

we have

$$2G_{277}^{\pm 4}=\frac{1908\,\theta^2+144\theta+757}{\sqrt{277}}\pm(114\theta^2+12\theta+45).$$

n = 397. The cubic equation satisfied by T_{397} is

$$T^3-16974T^2+13383T-1\,17612=0$$

$$(\Delta=-397\times 752\,76000^2),$$

which, by adjunction of $\sqrt{397}$, reduces to

$$G^{12}-(8487+426\sqrt{397})\,G^8+\frac{3417+171\sqrt{397}}{2}G^4-\frac{3447+173\sqrt{397}}{2}=0.$$

In terms of the cubic irrationality

$$\theta^3-6\theta^2+\theta-2=0 \quad (\Delta=-1588)$$

we have

$$2G_{397}^{\pm 4}=\frac{9837\theta^2-1044\theta+3350}{\sqrt{397}}\pm(495\theta^2-60\theta+168).$$

Class D. $n\equiv 2 \pmod 4$.

There are twelve values of n in class D, namely 26, 38, 50, 54, 106, 118, 162, 202, 214, 262, 298, 358. The class-invariant to be evaluated is $g_n^2\equiv 2^{-\frac{1}{2}}f_1^2(\sqrt{-n})$ when n is quadratfrei; when n is not quadratfrei, the class-invariant is g_n or g_n^3 according as n is prime to 3 or is a multiple of 3.

For quadratfrei values of n, the sextic equation satisfied by g_n^2 is skew-reciprocal, and I accordingly state it as a cubic in $t_n\equiv g_n^2-g_n^{-2}$.

The results which I have obtained are as follows.

n = 26. The cubic equation satisfied by t_{26} is

$$t^3-2t^2+t-4=0 \quad (\Delta=-416);$$

by adjunction of $\sqrt{13}$, this reduces to an equation given by Weber (**12**), namely

$$g^6-g^4-\frac{3+\sqrt{13}}{2}g^2-\frac{3+\sqrt{13}}{2}=0.$$

In terms of the cubic irrationality†

$$\theta^3-*-\theta-2=0 \quad (\Delta=-104)$$

we have

$$g_{26}^{\pm 2}=\frac{3\theta^2+4\theta-2}{\sqrt{13}}\pm\theta^2.$$

n = 38. The cubic equation satisfied by t_{38} is

$$t^3-4t^2+3t-2=0 \quad (\Delta=-152);$$

by adjunction of $\sqrt{2}$, this reduces to an equation given by Weber (**12**), namely

$$g^6-2g^4-(2+2\sqrt{2})g^2-(1+\sqrt{2})=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-4\theta^2+3\theta-2=0 \quad (\Delta=-152)$$

(the expression on the left was misprinted as $2\theta^3-4\theta^2+3\theta-2$ in the paper by Mathews) we have

$$2g_{38}^{\pm 2}=\frac{\theta^2-\theta-2}{\sqrt{2}}\pm\theta.$$

n = 50. The cubic equation satisfied by $x\equiv g_{50}-g_{50}^{-1}$ is

$$x^3-2x^2+3x-4=0 \quad (\Delta=-200);$$

by adjunction of $\sqrt{5}$, this reduces to an equation given by Weber (**12**), namely

$$g^3-g^2-\frac{1+\sqrt{5}}{2}g-\frac{1+\sqrt{5}}{2}=0.$$

---

† This is connected with Berwick's cubic irrationality

$$2\phi^3-2\phi^2+3\phi-1=0 \quad (4\Delta=-26)$$

by the relation

$$\phi=1/(\theta+1).$$

In terms of Berwick's cubic irrationality

$$\theta^3-2\theta^2+3\theta-4=0 \quad (\Delta=-200)$$

we have
$$2g_{50}^{\pm 1}=\frac{2\theta^2-\theta+2}{\sqrt{5}}\pm\theta.$$

n = 54. The cubic equation satisfied by $x \equiv g_{54}^6-g_{54}^{-6}$ is

$$x^3-114x^2+60x-184=0 \quad (\Delta=-54\times 4352^2),$$

which, by adjunction of $\sqrt{2}$, reduces to

$$g^{18}-(57+39\sqrt{2})g^{12}-(75+54\sqrt{2})g^6-(41+29\sqrt{2})=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3+*+3\theta-2=0 \quad (\Delta=-864)$$

we have

$$g_{54}^{\pm 6}=(10\theta^2+6\theta+33)\sqrt{2}\pm(14\theta^2+8\theta+47),$$

$$2\left\{\frac{g_{54}^3}{\sqrt{(1+\sqrt{2})}}\right\}^{\pm 1}=(\theta^2+\theta+4)\sqrt{2}\pm(2\theta^2+6).$$

n = 106. The equation satisfied by t_{106} is

$$t^3-11t^2+8t-16=0 \quad (\Delta=-106\times 24^2),$$

which, by adjunction of $\sqrt{53}$, reduces to

$$g^6-\frac{11+\sqrt{53}}{2}g^4-(6+\sqrt{53})g^2-\frac{7+\sqrt{53}}{2}=0.$$

In terms of the cubic irrationality[†]

$$\theta^2-2\theta^2+\theta-8=0 \quad (\Delta=-1696)$$

we have
$$2g_{106}^{\pm 2}=\frac{7\theta^2+6\theta+9}{\sqrt{53}}\pm(\theta^2+3).$$

n = 118. The equation satisfied by t_{118} is

$$t^3-12t^2+3t-46=0 \quad (\Delta=-118\times 54^2),$$

which, by adjunction of $\sqrt{2}$, reduces to

$$g^6-(6+4\sqrt{2})g^4-(2+2\sqrt{2})g^2-(7+5\sqrt{2})=0.$$

[†] This is connected with Berwick's cubic irrationality

$$3\phi^3-4\phi^2+5\phi-2=0 \quad (81\Delta=-424)$$

by the relation
$$\phi=2/(\theta+1).$$

In terms of the cubic irrationality†

$$\theta^3-3\theta^2-2\theta-2=0 \quad (\Delta=-472)$$

we have
$$2g_{118}^{\pm 2}=\frac{\theta^2+\theta}{\sqrt{2}}\pm(1+3\theta).$$

n = 162. The cubic equation satisfied by $x\equiv g_{162}^{3}+g_{162}^{-3}$ is

$$x^3-90x^2+168x-80=0 \quad (\Delta=-2592\times 28^2),$$

which, by adjunction of $\sqrt{6}$, reduces to

$$g^9-(45+18\sqrt{6})g^6+(45+18\sqrt{6})g^3-(49+20\sqrt{6})=0.$$

In terms of the cubic irrationality (connected with Berwick's cubic irrationality by the relation $\theta=2\phi$)

$$\theta^3+*+6\theta-8=0 \quad (\Delta=-2592)$$

we have
$$2g_{162}^{\pm 3}=9\theta^2+10\theta+66\pm\frac{22\theta^2+26\theta+160}{\sqrt{6}}.$$

n = 202. The cubic equation satisfied by t_{202} is

$$t^3-29t^2-t-115=0 \quad (\Delta=-202\times 240^2),$$

which, by adjunction of $\sqrt{101}$, reduces to

$$g^6-\frac{29+3\sqrt{101}}{2}g^4+\frac{13+\sqrt{101}}{2}g^2-(10+\sqrt{101})=0.$$

In terms of Berwick's cubic irrationality

$$\theta^3-\theta^2+2\theta-6=0 \quad (\Delta=-808)$$

we have
$$2g_{202}^{\pm 2}=\frac{36\theta^2+30\theta+127}{\sqrt{101}}\pm(4\theta^2+2\theta+13).$$

n = 214. The cubic equation satisfied by t_{214} is

$$t^3-32t^2-13t-142=0 \quad (\Delta=-214\times 306^2),$$

which, by adjunction of $\sqrt{2}$, reduces to

$$g^6-(16+12\sqrt{2})g^4+(8+4\sqrt{2})g^2-(7+5\sqrt{2})=0.$$

---

† This is connected with Berwick's cubic irrationality

$$2\phi^3-4\phi^2+5\phi-4=0 \quad (4\Delta=-118)$$

by the relation
$$\phi=(\theta+1)/\theta.$$

In terms of the cubic irrationality (connected with Berwick's cubic irrationality by the relation $\theta=2\phi$)

$$\theta^3-2\theta^2+2\theta-12=0 \quad (\Delta=-3424)$$

we have
$$2g_{214}^{\pm 2}=\frac{3\theta^2+3\theta+14}{\sqrt{2}}\pm\frac{5\theta^2+2\theta+20}{2}.$$

n = 262. The cubic equation satisfied by t_{262} is

$$t^3-48t^2-45t-50=0 \quad (\Delta=-262\times 270^2),$$

which, by adjunction of $\sqrt{2}$, reduces to

$$g^6-(24+16\sqrt{2})g^4-(56+40\sqrt{2})g^2-(41+29\sqrt{2})=0.$$

In terms of the cubic irrationality

$$\theta^3-\theta^2+8\theta-12=0 \quad (\Delta=-4192)$$

we have
$$2g_{262}^{\pm 2}=\frac{13\theta^2+5\theta+106}{2\sqrt{2}}\pm\frac{9\theta^2+3\theta+76}{2}.$$

n = 298. The cubic equation satisfied by t_{298} is

$$t^3-65t^2+8t-16=0 \quad (\Delta=-298\times 240^2),$$

which, by adjunction of $\sqrt{149}$, reduces to

$$g^6-\frac{65+5\sqrt{149}}{2}g^4-(60+5\sqrt{149})g^2-\frac{61+5\sqrt{149}}{2}=0.$$

In terms of the cubic irrationality

$$\theta^3-7\theta^2-8\theta-4=0 \quad (\Delta=-4768)$$

we have
$$2g_{298}^{\pm 2}=\frac{11\theta^2+12\theta-18}{\sqrt{149}}\pm\theta^2.$$

n = 358. The cubic equation satisfied by t_{358} is

$$t^3-100t^2-13t-338=0 \quad (\Delta=-358\times 1950^2),$$

which, by adjunction of $\sqrt{2}$, reduces to

$$g^6-(50+36\sqrt{2})g^4+(38+26\sqrt{2})g^2-(41+29\sqrt{2})=0.$$

In terms of the cubic irrationality

$$\theta^3-2\theta^2-7\theta-12=0 \quad (\Delta=-5728)$$

we have
$$2g_{358}^{\pm 2}=\frac{9\theta^2+21\theta+28}{2\sqrt{2}}\pm\frac{7\theta^2+13\theta+16}{2}.$$

Class E. $n \equiv 0 \pmod 4$.

There are eight values of n in class E, namely 44, 76, 92, 108, 124, 172, 268, 652. The class-invariant to be evaluated is

$$f_1^4(\sqrt{-n})/\sqrt{8} \equiv g_n^4/\sqrt{2} \equiv U_n,$$

except in the case of 108 when it is U_n^3. Instead of using Berwick's cubic irrationalities in these results, I have avoided a little calculation by using the cubic irrationalities $f(\sqrt{-11})$, $f(\sqrt{-19})$, ..., $f(\sqrt{-163})$, whether they are the same as Berwick's irrationalities or not, since these occur naturally in the determination of the class-invariants now under consideration.

The results which I have obtained are as follows.

n = 44. The sextic equation satisfied by U_{44} is

$$U^6-12U^5+8U^4-14U^3+12U^2-4U-2=0;$$

by adjunction of $\sqrt{11}$, this reduces to an equation given by Weber (**12**), namely

$$U^3-(6+2\sqrt{11})\,U^2+(8+2\sqrt{11})\,U-(3+\sqrt{11})=0.$$

If $f(\sqrt{-11})=\theta$, so that

$$\theta^3-2\theta^2+2\theta-2=0 \quad (\Delta=-44),$$

we have
$$2\,\frac{f_1^4(\sqrt{-44})}{\sqrt{8}}=\frac{13\theta^2-8\theta+20}{\sqrt{11}}+3\theta^2+4.$$

n = 76. The sextic equation satisfied by U_{76} is

$$U^6-32U^5-64U^4-74U^3-52U^2-20U-2=0,$$

which, by adjunction of $\sqrt{19}$, reduces to

$$U^3-(16+4\sqrt{19})\,U^2-(8+2\sqrt{19})\,U^2-(13+3\sqrt{19})=0.$$

If $f(\sqrt{-19})=\theta$, so that

$$\theta^3-*-2\theta-2=0 \quad (\Delta=-76),$$

we have
$$2\,\frac{f_1^4(\sqrt{-76})}{\sqrt{8}}=\frac{23\theta^2+32\theta+20}{\sqrt{19}}+5\theta^2+8\theta+4.$$

n = 92. The sextic equation satisfied by U_{92} is

$$U^6-52U^5-88U^4-28U^3+164U^2-88U+8=0,$$

which, by adjunction of $\sqrt{23}$, reduces to

$$U^3-(26+6\sqrt{23})\,U^2+(32+6\sqrt{23})\,U-(10+2\sqrt{23})=0.$$

If $F_{23}=\theta$, so that

$$\theta^3-*-\theta-1=0 \quad (\Delta=-23),$$

we have
$$\frac{f_1^4(\sqrt{-92})}{\sqrt{8}}=7\theta^2+8\theta+4+\frac{27\theta^2+40\theta+28}{\sqrt{23}}.$$

n = 108. The sextic equation satisfied by U_{108}^3 is

$$U^{18}-5\,43090U^{15}-4\,82106U^{12}-2\,15840U^9-55788U^6-9960U^3-8=0,$$

which, by adjunction of $\sqrt{3}$, reduces to

$$U^9-(2\,71545+1\,56777\sqrt{3})\,U^6-(42972+24810\sqrt{3})\,U^3$$
$$-(27550+15906\sqrt{3})=0.$$

If $\frac{1}{2}f^3(\sqrt{-27})=\theta$, so that

$$\theta^3-3\theta^2-3\theta-1=0 \quad (\Delta=-108),$$

we have

$$2\,\frac{f_1^4(\sqrt{-108})}{f_1^2(\sqrt{-27})\sqrt{8}}=\frac{2\theta^2+2\theta-1}{\sqrt{3}}+(2\theta^2-2\theta-1).$$

n = 124. The sextic equation satisfied by U_{124} is

$$U^6-122U^5+194U^4-180U^3+112U^2-112U+8=0,$$

which, by adjunction of $\sqrt{31}$, reduces to

$$U^3-(61+11\sqrt{31})\,U^2+(112+20\sqrt{31})U-(78+14\sqrt{31})=0.$$

If $F_{31}=\theta$, so that

$$\theta^3-\theta^2-*-1=0 \quad (\Delta=-31),$$

we have
$$\frac{f_1^4(\sqrt{-124})}{\sqrt{8}}=17\theta^2+8\theta+12+\frac{97\theta^2+40\theta+68}{\sqrt{31}}.$$

n = 172. The sextic equation satisfied by U_{172} is

$$U^6-340U^5+200U^4-70U^3+80U^2-40U-2=0,$$

which, by adjunction of $\sqrt{43}$, reduces to

$$U^3-(170+26\sqrt{43})\,U^2+(184+28\sqrt{43})\,U-(59+9\sqrt{43})=0.$$

If $f(\sqrt{-43})=\theta$, so that

$$\theta^3-2\theta^2-*-2=0 \quad (\Delta=-172),$$

we have

$$2\,\frac{f_1^4(\sqrt{-172})}{\sqrt{8}}=\frac{305\theta^2+112\theta+264}{\sqrt{43}}+47\theta^2+16\theta+40.$$

n = 268. The sextic equation satisfied by U_{268} is

$$U^6-1868U^5+2312U^4-1442U^3+524U^2-92U-2=0,$$

which, by adjunction of $\sqrt{67}$, reduces to

$$U^3-(934+114\sqrt{67})\,U^2+(344+42\sqrt{67})-(221+27\sqrt{67})=0.$$

If $f(\sqrt{-67})=\theta$, so that

$$\theta^3-2\theta^2-2\theta-2=0 \quad (\Delta=-268),$$

we have

$$2\,\frac{f_1^4(\sqrt{-268})}{\sqrt{8}}=\frac{1287\theta^2+1176\theta+876}{\sqrt{67}}+157\theta^2+144\theta+108.$$

n = 652. The sextic equation satisfied by U_{652} is

$$U^6-2\,26388U^5+1\,56872U^4-54362U^3+11024U^2-1112U-2=0,$$

which, by adjunction of $\sqrt{163}$, reduces to

$$U^3-(1\,13194+8866\sqrt{163})\,U^2+(23032+1804\sqrt{163})\,U$$
$$-(8005+627\sqrt{163})=0.$$

If $f(\sqrt{-163})=\theta$, so that

$$\theta^3-6\theta^2+4\theta-2=0 \quad (\Delta=-652),$$

we have

$$2\,\frac{f_1^4(\sqrt{-652})}{\sqrt{8}}=\frac{1\,15427\theta^2-78640\theta+43400}{\sqrt{163}}+9041\theta^2-6160\theta+3400.$$

Bibliography.

1. W. E. H. Berwick, " Modular invariants expressible in terms of quadratic and cubic irrationalities ", *Proc. London Math. Soc.* (2), 28 (1928), 53–69.
2. A. G. Greenhill, " Complex multiplication moduli of elliptic functions ", *Proc. London Math. Soc.* (1), 19 (1889), 301–364.
3. A. G. Greenhill, " Tables of complex multiplication moduli ", *Proc. London Math. Soc.* (1), 21 (1891), 403–422.
4. G. B. Mathews, " Complex multiplication moduli of elliptic functions for the determinants -53 and -61 ", *Proc. London Math. Soc.* (1), 21 (1891), 215–217.
5. G. B. Mathews, " The reduction and classification of binary cubics ", *Proc. London Math. Soc.* (2), 10 (1912), 128–138.
6. S. Ramanujan, " Modular equations and approximations to π ", *Quart. J. of Math.*, 45 (1914), 350–372.
7. R. Russell, " On modular equations ", *Proc. London Math. Soc.* (1), 21 (1891), 351–395.
8. G. N. Watson, " Some singular moduli (1) ", *Quart. J. of Math.*, 3 (1932), 81–98.
9. G. N. Watson, " Some singular moduli (2) ", *Quart. J. of Math.*, 3 (1932), 189–212.
10. G. N. Watson, " Singular moduli (3) ", *Proc. London Math. Soc.* (2), 40 (1936), 83–142.
11. G. N. Watson, " Singular moduli (4) ", *Acta Arithmetica*, 1 (1936), 284–323.
12. H. Weber, *Lehrbuch der Algebra*, 3 (Braunschweig, 1908).

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SINGULAR MODULI (6)

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[Read June 18, 1936.—Received July 2, 1936.]

This paper is a companion to its immediate predecessor in the series. There are thirty-one values of n for which the negative determinant $-n$ has II genera and four classes in each genus, and my object in this paper is to investigate the corresponding thirty-one class-invariants. These class-invariants satisfy octavic equations which I either solve completely or reduce to quadratic equations whose coefficients are quartic[*] irrationalities. The notations used in this paper are the same as in the preceding paper, and, so far as standard notations are concerned, I do not waste any space in explaining them anew.

The thirty-one values of n with their classifications into types according to their residues modulo 8 are given in the adjoining table, class E consisting of multiples of 4.

41	C	128	E	289	C	466	D
62	D	137	C	292	E	478	D
68	E	158	D	295	B	562	D
94	D	178	D	313	C	577	C
95	B	183	B	337	C	583	B
98	D	196	E	382	D	772	E
111	B	226	D	388	E	862	D
113	C	256	E	457	C		

It is to be observed that there are no values of n in class A.

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[*] Each of the corresponding quartic equations, like all the quartic equations which occur in this paper, is such that its reducing cubic has a root which is rational in the field of the coefficients of the quartic. Such quartic equations are sometimes appropriately described as biquadratic equations; and, now that the term biquadratic as the description of the general quartic has become archaic, the use of the term biquadratic for these special quartics might well be more generally adopted.

The results stated for values of n in class B were obtained by the method described in my paper (**4**). For all other quadratfrei values of n, the results were obtained empirically. For $n = 289$ and the values of n in class E, the results were obtained by the use of modular equations.

Class B. $n \equiv 7 \pmod 8$.

There are five values of n in class B, namely 95, 111, 183, 295, 583. The class-invariant to be evaluated is $F_n \equiv 2^{-\frac{1}{4}} f(\sqrt{-n})$ when n is prime to 3, and $F_n{}^3$ when n is a multiple of 3. The values of these class-invariants have already been given in full by Watson (**4**). It will now be sufficient to give the quartic equations (whose coefficients are quadratic irrationalities) satisfied by these class-invariants; these equations will be so arranged that they can be solved by inspection.

n = 95. The quartic equation satisfied by F_{95} is

$$\left[F^2 - \frac{1+\sqrt{5}}{2} F - \frac{1+\sqrt{5}}{4}\right]^2 = \left[\frac{1+\sqrt{5}}{4}\right]^2 (2\sqrt{5}-1).$$

n = 111. The quartic equation satisfied by F_{111}^3 is

$$\left[F^6 - \frac{21+3\sqrt{37}}{4} F^3 - \frac{31+5\sqrt{37}}{4}\right]^2$$
$$= \left[\frac{29+5\sqrt{37}}{2} F^3 + \frac{43+7\sqrt{37}}{2}\right]^2 \left(\frac{-5+\sqrt{37}}{8}\right).$$

n = 183. The quartic equation satisfied by F_{183}^3 is

$$\left[F^6 - \frac{71+9\sqrt{61}}{4} F^3 - \frac{149+19\sqrt{61}}{4}\right]^2$$
$$= \left[\frac{27+3\sqrt{61}}{4} F^3 + \frac{55+7\sqrt{61}}{4}\right]^2 \left(\frac{7+\sqrt{61}}{2}\right).$$

n = 295. The quartic equation satisfied by F_{295} is

$$\left[F^2 - \frac{4+\sqrt{5}}{2} F - \frac{7+3\sqrt{5}}{4}\right]^2 = \left[F + \frac{1+\sqrt{5}}{2}\right]^2 \left(\frac{11+6\sqrt{5}}{4}\right).$$

n = 583. The quartic equation satisfied by F_{583} is

$$\left[F^2 - \frac{8+\sqrt{53}}{2}(F+1)\right]^2 = (F+1)^2 \frac{(3+\sqrt{53})(7+\sqrt{53})^2}{32}.$$

For other investigations of F_{95} and F_{111} see Greenhill (**1**) and Hanna (**2**).

Class C. $n \equiv 1 \pmod 4$.

There are eight values of n in class C, namely 41, 113, 137, 289, 313, 337, 457, 577. The simplest class-invariant in each case (except 289) is $G_n{}^2 \equiv 2^{-\frac{1}{2}} f^2(\sqrt{-n})$; but, since the octavic equation which this class-invariant satisfies is a reciprocal equation, I give the octavic equation in the form of a quartic equation in $X_n \equiv G_n{}^2 + G_n^{-2}$. I also give expressions for $G_n + G_n^{-1}$ and $G_n - G_n^{-1}$ in terms of radicals. Apart from the equation for X_{41} given by Weber (**6**) and the value of G_{289} given by Ramanujan (**3**), no results about these eight class-invariants have previously been published.

The results which I have obtained are as follows.

n = 41. The quartic equation satisfied by X_{41} is

$$X^4 - 5X^3 + 3X^2 + 3X + 2 = 0,$$

whence Weber's equation

$$X^2 - \tfrac{1}{2}(5 + \sqrt{41})X + \tfrac{1}{2}(7 + \sqrt{41}) = 0$$

follows immediately. Hence we have

$$G_{41} + G_{41}^{-1} = \sqrt{\frac{19 + 3\sqrt{41}}{8}} + \sqrt{\frac{7 - \sqrt{41}}{8}},$$

$$G_{41} - G_{41}^{-1} = \left[\sqrt{\frac{-3 + \sqrt{41}}{4}} + \sqrt{\frac{7 - \sqrt{41}}{4}}\right] \sqrt[4]{\frac{5 + \sqrt{41}}{8}}.$$

n = 113. The quartic equation satisfied by X_{113} is

$$X^4 - 9X^3 - 26X^2 - 32X - 32 = 0,$$

whence we have

$$G_{113} + G_{113}^{-1} = \sqrt{\frac{25 + \sqrt{113}}{8}} + \sqrt{\frac{9 + \sqrt{113}}{8}},$$

$$G_{113} - G_{113}^{-1} = \left[\sqrt{\frac{57 + \sqrt{113}}{8}} + \sqrt{\frac{-7 + \sqrt{113}}{8}}\right] \sqrt[4]{\frac{7 + \sqrt{113}}{32}}.$$

n = 137. The quartic equation satisfied by X_{137} is

$$X^4 - 21X^3 + 99X^2 - 173X + 98 = 0,$$

whence we have

$$G_{137}+G_{137}^{-1}=\sqrt{\frac{35+3\sqrt{137}}{8}}+\sqrt{\frac{23-\sqrt{137}}{8}},$$

$$G_{137}-G_{137}^{-1}=\left[\sqrt{\frac{13+\sqrt{137}}{8}}+\sqrt{\frac{-3+\sqrt{137}}{8}}\right]\sqrt[4]{\frac{11+\sqrt{137}}{8}}.$$

n = 289. The quartic equation satisfied by $G_{289}+G_{289}^{-1}$ is

$$(G+G^{-1})^4-9(G+G^{-1})^3+7(G+G^{-1})^2+15(G+G^{-1})-18=0;$$

this equation is easily obtained by applying the modular equation of order 17 to $G_1=1$. From the quartic equation it is possible to derive Ramanujan's result

$$G_{289}=\left[\sqrt{\frac{17+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}}+\sqrt{\frac{1+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}}\right]^2.$$

It is also to be observed that

$$G_{289}+G_{289}^{-1}=\frac{9+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{4},$$

$$G_{289}-G_{289}^{-1}=\sqrt{\frac{51+15\sqrt{17}+(31+7\sqrt{17})\sqrt[4]{17}}{4}}.$$

n = 313. The quartic equation satisfied by X_{313} is

$$X^4-76X^3+249X^2-288X+108=0,$$

whence we have

$$G_{313}+G_{313}^{-1}=\sqrt{\frac{53+3\sqrt{313}}{4}}+\sqrt{\frac{31+\sqrt{313}}{4}},$$

$$G_{313}-G_{313}^{-1}=\left[\sqrt{\frac{21+\sqrt{313}}{8}}+\sqrt{\frac{5+\sqrt{313}}{8}}\right]\sqrt[4]{\frac{87+5\sqrt{313}}{8}}.$$

n = 337. The quartic equation satisfied by X_{337} is

$$X^4-85X^3-129X^2+315X+486=0,$$

whence we have

$$G_{337}+G_{337}^{-1}=\sqrt{\frac{131+7\sqrt{337}}{8}}+\sqrt{\frac{55+3\sqrt{337}}{8}},$$

$$G_{337}-G_{337}^{-1}=\left[\sqrt{\frac{41+3\sqrt{337}}{8}}+\sqrt{\frac{50+2\sqrt{337}}{8}}\right]\sqrt[4]{\frac{9+\sqrt{337}}{8}}.$$

n = 457. The quartic equation satisfied by X_{457} is

$$X^4-189X^3-301X^2-117X+18=0,$$

whence we have

$$G_{457}+G_{457}^{-1}=\sqrt{\frac{235+11\sqrt{457}}{8}}+\sqrt{\frac{159+7\sqrt{457}}{8}},$$

$$G_{457}-G_{457}^{-1}=\left[\sqrt{\frac{35+3\sqrt{457}}{4}}+\sqrt{\frac{45+\sqrt{457}}{4}}\right]\sqrt[4]{\frac{21+\sqrt{457}}{8}}.$$

n = 577. The quartic equation satisfied by X_{577} is

$$X^4-376X^3-1791X^2-2628X-972=0,$$

whence we have

$$G_{577}+G_{577}^{-1}=\sqrt{\frac{215+9\sqrt{577}}{4}}+\sqrt{\frac{169+7\sqrt{577}}{4}},$$

$$G_{577}-G_{577}^{-1}=\left[4+\sqrt{\frac{-1+\sqrt{577}}{2}}\right]\sqrt[4]{24+\sqrt{577}}.$$

Class D. $n\equiv 2 \pmod 4$.

There are eleven values of n in class D, namely 62, 94, 98, 158, 178, 226, 382, 466, 478, 562, 862. The simplest class-invariant in each case is $g_n{}^2\equiv 2^{-\frac{1}{2}}f_1{}^2(\sqrt{-n})$; but, since the octavic equation which this class-invariant satisfies is a reciprocal equation, I give the octavic equation in the form of a quartic equation in $x_n\equiv g_n{}^2+g_n^{-2}$. I also give expressions for $g_n+g_n^{-1}$ and $g_n-g_n^{-1}$ in terms of radicals. For $n=62$, 94, 98, 158 the values of $g_n+g_n^{-1}$ have been given by Ramanujan (**3**), while Greenhill (**1**) has given some formulae (not free from numerical errors) which can be derived from Ramanujan's first two results.

The results which I have obtained are as follows.

n = 62. The quartic equation satisfied by x_{62} is

$$x^4-2x^3-17x^2-24x-8=0,$$

whence we have Ramanujan's result

$$g_{62}+g_{62}^{-1}=\tfrac{1}{2}(\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}})$$

and

$$g_{62}-g_{62}^{-1}=\tfrac{1}{2}[\sqrt[4]{11+46\sqrt{2}+(8+4\sqrt{2})\sqrt{31}}+\sqrt[4]{11+46\sqrt{2}-(8+4\sqrt{2})\sqrt{31}}].$$

n = 94. The quartic equation satisfied by x_{94} is

$$x^4-6x^3-25x^2-24x-8=0,$$

whence we have Ramanujan's result

$$g_{94}+g_{94}^{-1}=\tfrac{1}{2}(\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}})$$

and

$$g_{94}-g_{94}^{-1}=\tfrac{1}{2}[\sqrt[4]{99+90\sqrt{2}+(16+12\sqrt{2})\sqrt{47}} \\ +\sqrt[4]{99+90\sqrt{2}-(16+12\sqrt{2})\sqrt{47}}].$$

n = 98. The quartic equation satisfied by x_{98} is

$$x^4-8x^3-18x^2+24x+65=0.$$

By applying the modular equation of order 7 to g_2, we can easily prove that

$$(g_{98}+g_{98}^{-1})^4-4(g_{98}+g_{98}^{-1})^2-5=\{2(g_{98}+g_{98}^{-1})^3+6(g_{98}+g_{98}^{-1})\}\sqrt{2},$$

whence the previous equation is easily derived. From the latter equation we have Ramanujan's result

$$g_{98}+g_{98}^{-1}=\tfrac{1}{2}(\sqrt{2}+\sqrt{14+4\sqrt{14}})$$

and $$g_{98}-g_{98}^{-1}=\tfrac{1}{2}(\sqrt[4]{16\sqrt{14}+8\sqrt{7}}+\sqrt[4]{16\sqrt{14}-8\sqrt{7}}).$$

n = 158. The quartic equation satisfied by x_{158} is

$$x^4-18x^3-17x^2-56x-8=0,$$

whence we have Ramanujan's result

$$g_{158}+g_{158}^{-1}=\tfrac{1}{2}(\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}})$$

and

$$g_{158}-g_{158}^{-1}=\tfrac{1}{2}[\sqrt[4]{235+198\sqrt{2}+(24+20\sqrt{2})\sqrt{79}} \\ +\sqrt[4]{235+198\sqrt{2}-(24+20\sqrt{2})\sqrt{79}}].$$

n = 178. The quartic equation satisfied by x_{178} is

$$x^4-21x^3-55x^2+33x+106=0,$$

whence we have

$$g_{178}+g_{178}^{-1}=\sqrt{\frac{47+5\sqrt{89}}{8}}+\sqrt{\frac{11+\sqrt{89}}{8}},$$

$$g_{178}-g_{178}^{-1}=\tfrac{1}{2}\left[\sqrt[4]{\frac{477+63\sqrt{89}}{2}+(116+12\sqrt{89})\sqrt{2}}\right.$$
$$\left.+\sqrt[4]{\frac{477+63\sqrt{89}}{2}-(116+12\sqrt{89})\sqrt{2}}\right].$$

n = **226.** The quartic equation satisfied by x_{226} is

$$x^4-39x^3+101x^2-21x-98=0,$$

whence we have

$$g_{226}+g_{226}^{-1}=\sqrt{\frac{53+5\sqrt{113}}{8}}+\sqrt{\frac{41+\sqrt{113}}{8}},$$

$$g_{226}-g_{226}^{-1}=\tfrac{1}{2}\left[\sqrt[4]{\frac{1749+165\sqrt{113}}{2}+(512+48\sqrt{113})\sqrt{2}}\right.$$
$$\left.+\sqrt[4]{\frac{1749+165\sqrt{113}}{2}-(512+48\sqrt{113})\sqrt{2}}\right].$$

n = **382.** The quartic equation satisfied by x_{382} is

$$x^4-120x^3+242x^2-120x+25=0,$$

whence we have

$$g_{382}+g_{382}^{-1}=\tfrac{1}{2}(\sqrt{82+58\sqrt{2}}+\sqrt{46+26\sqrt{2}}),$$

$$g_{382}-g_{382}^{-1}=\tfrac{1}{2}[\sqrt[4]{6912+4896\sqrt{2}+(136+96\sqrt{2})\sqrt{191}}$$
$$+\sqrt[4]{6912+4896\sqrt{2}-(136+96\sqrt{2})\sqrt{191}}].$$

n = **466.** The quartic equation satisfied by x_{466} is

$$x^4-207x^3+1154x^2-2340x+1576=0,$$

whence we have

$$g_{466}+g_{466}^{-1}=\sqrt{\frac{259+17\sqrt{233}}{8}}+\sqrt{\frac{171+9\sqrt{233}}{8}},$$

$$g_{466}-g_{466}^{-1}=\tfrac{1}{2}\left[\sqrt[4]{\frac{40449+2651\sqrt{233}}{2}+(3108+204\sqrt{233})\sqrt{2}}\right.$$
$$\left.+\sqrt[4]{\frac{40449+2651\sqrt{233}}{2}-(3108+204\sqrt{233})\sqrt{2}}\right].$$

n = 478. The quartic equation satisfied by x_{478} is

$$x^4-216x^3-94x^2+648x+553=0,$$

whence we have

$$g_{478}+g_{478}^{-1}=\tfrac{1}{2}(\sqrt{82+58\sqrt{2}}+\sqrt{142-98\sqrt{2}}),$$

$$g_{478}-g_{478}^{-1}=\tfrac{1}{2}[\sqrt[4]{23040+16320\sqrt{2}+(136+96\sqrt{2})\sqrt{239}}$$
$$+\sqrt[4]{23040+16320\sqrt{2}-(136+96\sqrt{2})\sqrt{239}}].$$

n = 562. The quartic equation satisfied by x_{562} is

$$x^4-351x^3+122x^2+108x+40=0,$$

whence we have

$$g_{562}+g_{562}^{-1}=\sqrt{\frac{419+25\sqrt{281}}{8}}+\sqrt{\frac{299+17\sqrt{281}}{8}},$$

$$g_{562}-g_{562}^{-1}=\tfrac{1}{2}\left[\sqrt[4]{\frac{123921+7395\sqrt{281}}{2}+(9860+588\sqrt{281})\sqrt{2}}\right.$$
$$\left.+\sqrt[4]{\frac{123921+7395\sqrt{281}}{2}-(9860+588\sqrt{281})\sqrt{2}}\right].$$

n = 862. The quartic equation satisfied by x_{862} is

$$x^4-1542x^3+2423x^2+3336x-5768=0,$$

whence we have

$$g_{862}+g_{862}^{-1}=\tfrac{1}{2}(\sqrt{799+565\sqrt{2}}+\sqrt{751+521\sqrt{2}}),$$

$$g_{862}-g_{862}^{-1}=\tfrac{1}{2}[\sqrt[4]{1199571+848226\sqrt{2}+(10912+7716\sqrt{2})\sqrt{431}}$$
$$+\sqrt[4]{1199571+848226\sqrt{2}-(10912+7716\sqrt{2})\sqrt{431}}].$$

Class E. $n\equiv 0 \pmod{4}$.

There are seven values of n in class E, namely 68, 128, 196, 256, 292, 388, 772. The class-invariant to be evaluated is $f_1{}^4(\sqrt{-n})/\sqrt{8}\equiv g_n{}^4/\sqrt{2}\equiv U_n$, except in the cases of 128, 196, and 256. Quadratic equations (whose coefficients are quartic irrationalities) for the class-invariants are obtainable by writing the quadratic transformation

$$U_{4m}^4-2U_{4m}^2\,G_m^{16}+G_m{}^8=0$$

in the form

$$U_{4m}^2 - U_{4m} G_m{}^2 \sqrt{(2G_m^{12}+2)} + G_m{}^4 = 0$$

and then expressing $\sqrt{(2G_m^{12}+2)}$ as a rational function of G_m for $m = 17$, 49, 73, 97, 193. For $n = 196$, however, simpler results are obtained by applying the modular equation of order seven to $g_4 = \sqrt[8]{2}$. To deal with the remaining two cases (128 and 256) we take the quadratic transformation

$$g_{4m}^{16} - 4g_{4m}^8 g_m^{16} - 4g_m{}^8 = 0,$$

and, by iteration, we have successively

$$g_{16m}^{32} - g_{16m}^{24}(64g_m^{32} + 32g_m{}^8) + 240g_{16m}^{16} g_m^{16} - 256g_{16m}^8 g_m^{24} - 64g_m{}^8 = 0,$$

$$g_{64m}^{64} - g_{64m}^{56}(16384g_m^{64} + 16384g_m^{40} + 2176g_m^{16})$$

$$-g_{64m}^{48}(5\,24288g_m^{56} - 6\,59200g_m^{32} + 2176g_m{}^8) - g_{64m}^{40}(39\,48544g_m^{48} + 18\,55488g_m^{24})$$

$$-g_{64m}^{32}(47\,18592g_m^{40} - 6\,59200g_m^{16}) - g_{64m}^{24}(10\,48576g_m^{56} + 47\,18592g_m^{32} + 16384g_m{}^8)$$

$$-39\,48544g_{64m}^{16} g_m^{24} - g_{64m}^8(10\,48576g_m^{40} + 5\,24288g_m^{16}) - 16384g_m{}^8 = 0.$$

By taking $m = 2$ and $m = 4$ in turn in these equations and using the results $g_2{}^8 = 1$ and $g_4{}^8 = 2$, we can calculate g_8, g_{32}, g_{128}, and g_{16}, g_{64}, g_{256} in succession.

The results which I have obtained for the seven values of n under consideration are as follows.

n = 68. The quadratic equation (whose coefficients are quartic irrationalities) for U_{68} is

$$U^2 - 2UG_{17}^4(G_{17}^2 + 1) + G_{17}^4 = 0,$$

with

$$G_{17}^2 + G_{17}^{-2} = \tfrac{1}{2}(1 + \sqrt{17}).$$

Hence we have in succession

$$U^4 - (15 + 3\sqrt{17})\,U^3 + \tfrac{1}{2}(25 + 5\sqrt{17})\,U^2 - (5 + \sqrt{17})\,U + 1 = 0,$$

$$U^8 - 30U^7 + 97U^6 - 130U^5 + 100U^4 - 70U^3 + 33U^2 - 10U + 1 = 0.$$

n = 128. We first obtain the results given by Weber (**6**)

$$g_8^{16} - 4g_8{}^8 - 4 = 0, \quad g_8{}^8 = 2 + 2\sqrt{2},$$

$$g_{32}^{16} - (48 + 32\sqrt{2})\,g_{32}^8 - (8 + 8\sqrt{2}) = 0,$$

and then we have

$$g_{32}^{8} = 24+16\sqrt{2}+(10+8\sqrt{2})\sqrt{2+2\sqrt{2}}$$

$$= (2+\sqrt{2}+\sqrt{4+4\sqrt{2}})^2\sqrt{2+2\sqrt{2}},$$

$$g_{128}^{16}-4g_{128}^{8}g_{32}^{16}-4g_{32}^{8}=0,$$

$$g_{128}^{32}-(17472+12352\sqrt{2})g_{128}^{24}+(2880+1920\sqrt{2})g_{128}^{24}$$
$$-(14336+10240\sqrt{2})g_{128}^{8}-(128+128\sqrt{2})=0,$$

$$g_{128}^{64}-34944\,g_{128}^{56}+1\,32736g_{128}^{48}-58\,04032g_{128}^{40}-40\,59392g_{128}^{32}-57\,83552g_{128}^{24}$$
$$-39\,48544g_{128}^{16}-15\,72864g_{128}^{8}-16384=0.$$

n = 196. The quadratic equation (whose coefficients are quartic irrationalities) for U_{196} is

$$U^2-2U(G_{49}^{7}+G_{49}^{3})\sqrt{2}+G_{49}^{4}=0,$$

with $$G_{49}^{2}+G_{49}^{-2}=2+\sqrt{7}.$$

Hence we have in succession

$$U^4-(270+102\sqrt{7})U^3+(97+36\sqrt{7})U^2-(18+6\sqrt{7})U+1=0,$$

$$U^8-540U^7+266U^6-1008U^5+1491U^4-1008U^3+266U^2-36U+1=0.$$

The modular equation of order seven, however, gives

$$g_{196}^{8}-g_{196}^{7}g_4^{\,7}\sqrt{8}-7g_{196}^{4}g_4^{\,4}-g_{196}g_4\sqrt{8}+g_4^{\,8}=0,$$

whence we can derive the simpler results

$$g_{196}^{4}-(2+\sqrt{7})g_{196}^{3}g_4^{\,3}+\frac{3+\sqrt{7}}{\sqrt{2}}g_{196}^{2}g_4^{\,2}-(1+\sqrt{7})g_{196}g_4+g_4^{\,4}=0,$$

$$g_{196}^{2}-G_{49}^{2}g_{196}g_4^{\,3}+G_{49}g_4^{\,2}=0,$$

with $$G_{49}^{2}+G_{49}^{-2}=2+\sqrt{7},\quad g_4^{\,8}=2.$$

n = 256. We first obtain the result given by Weber (6)

$$g_{16}^{8}=8+6\sqrt{2}=2(1+\sqrt{2})^2\sqrt{2},$$

and then we find that

$$g_{64}^{16}-(544+384\sqrt{2})g_{64}^{8}-(32+24\sqrt{2})=0,$$

$$g_{64}^{8}=272+192\sqrt{2}+(228+162\sqrt{2})\sqrt[4]{2}$$

$$=(1+\sqrt{2})^3(1+\sqrt[4]{2})^4.\sqrt[4]{128},$$

whence we have

$$g_{256}^{16}-4g_{256}^{8}\,g_{64}^{16}-4g_{64}^{8}=0,$$

$$g_{256}^{32}-(23\,63648+16\,71360\sqrt{2})\,g_{256}^{24}+(32640+23040\sqrt{2})\,g_{256}^{16}$$
$$-(5\,73440+4\,05504\sqrt{2})\,g_{256}^{8}-(512+384\sqrt{2})=0,$$

$$g_{256}^{64}-47\,27296g_{256}^{56}-565\,66016g_{256}^{48}-2675\,50720g_{256}^{40}-1483\,58144g_{256}^{32}$$
$$-2097\,47968g_{256}^{24}-315\,88352g_{256}^{16}-356\,51584g_{256}^{8}-32768=0.$$

n = 292. The quadratic equation (whose coefficients are quartic irrationalities) for U_{292} is

$$U^2-\tfrac{4}{3}U\,G_{73}^4(G_{73}^2-1)(8+\sqrt{73})+G_{73}^4=0,$$

with $$G_{73}^2+G_{73}^{-2}=\tfrac{1}{2}(5+\sqrt{73}).$$

Hence we have in succession

$$U^4-(1359+159\sqrt{73})\,U^3-\tfrac{1}{2}(535+63\sqrt{73})\,U^2-(27+3\sqrt{73})\,U+1=0,$$

$$U^8-2718U^7+833U^6-4230U^5+2868U^4-2070U^3-463U^2-54U+1=0.$$

n = 388. The quadratic equation (whose coefficients are quartic irrationalities) for U_{388} is

$$U^2-2U(G_{97}^2+1)(2G_{97}^6-G_{97}^4+2G_{97}^2)+G_{97}^4=0,$$

with $$G_{97}^2+G_{97}^{-2}=\tfrac{1}{2}(9+\sqrt{97}).$$

Hence we have in succession

$$U^4-(5329+541\sqrt{97})\,U^3+\tfrac{1}{2}(1705+173\sqrt{97})\,U^2-(67+7\sqrt{97})\,U+1=0,$$

$$U^8-10658U^7+9889U^6-7558U^5-19612U^4-7426U^3$$
$$+1441U^2-134U+1=0.$$

n = 772. The quadratic equation (whose coefficients are quartic irrationalities) for U_{772} is

$$U^2-\tfrac{1}{4}U\,G_{193}^4(G_{193}^2+1)(77+5\sqrt{193})+G_{193}^4=0,$$

with $$G_{193}^2+G_{193}^{-2}=13+\sqrt{193}.$$

Hence we have in succession

$$U^4-(3\,67790+26474\sqrt{193})\,U^3+(19730+1420\sqrt{193})\,U^2 \\ -(530+38\sqrt{193})\,U+1=0,$$

$$U^8-7\,35580\,U^7+10\,97092U^6-20\,65580U^5+16\,44470U^4 \\ -8\,20820U^3+41668U^2-1060U+1=0.$$

Bibliography.

1. A. G. Greenhill, " Tables of complex multiplication moduli ", *Proc. London Math. Soc.* (1), 21 (1891), 403–422.
2. M. Hanna, " The modular equations ", *Proc. London Math. Soc.* (2), 28 (1928), 46–52.
3. S. Ramanujan, " Modular equations and approximations to π ", *Quart. J. of Math.*, 45 (1914), 350–372.
4. G. N. Watson, " Singular moduli (4) ", *Acta Arithmetica*, 1 (1936), 284–323.
5. G. N. Watson, " Singular moduli (5) ", *Proc. London Math. Soc.* (2), 42 (1937), 377–397.
6. H. Weber, *Lehrbuch der Algebra*, 3 (Braunschweig, 1908).

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WELL-POISED HYPERGEOMETRIC SERIES AND COGNATE TRIGONOMETRIC SERIES

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[Received 9 March, 1936.—Read 19 March, 1936.—Revised 28 December, 1936.]

1. In a recent study of relations between well-poised series of the type $_7F_6$ equations were found each of which involved three of these series. The proofs of these equations depended on formulae previously established by Dr. W. N. Bailey. A more direct method of establishing the equations having been found, it was noticed that the same method could be used to obtain not only new relations involving well-poised series of any order but also relations involving trigonometric series. The method is merely an application of contour integration with integrands of the type introduced by Barnes in 1908, but it seems to have been overlooked hitherto that the integrand can be so chosen that the contributions to the integral by one set of poles are proportional to the terms of a well-poised series, and the contributions by another set are proportional to the terms of a series which is identical with the other except perhaps for a different argument. If the integrand has $2n$ sets of poles and the integral round a circular contour with a large radius tends to zero as the radius tends to infinity, then an equation involving n well-poised series or n trigonometric series can be obtained.

In the paper most of the equations involve three series, but the extensions to more parameters are obvious. Relations between well-poised series are regarded as special cases of those between trigonometric series. Eight formulae derived in this way can be deduced from the last of them by taking special values of the parameters, but it has been thought worth while to write out all for facility of reference.

2. The subject may be approached by considering an integral related to the generalized hypergeometric series ${}_4F_3\left[\begin{matrix}a,\ b,\ c,\ d\\ e,\ f,\ g\end{matrix}\right]$, viz. the integral

$$\frac{1}{2i\pi}\int \frac{\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c+z)\,\Gamma(-z)\,\Gamma(1-e-z)\,\Gamma(1-f-z)}{\Gamma(1-d-z)\,\Gamma(g+z)}\,dz$$

taken round a large circle. It can be shown* that, if the parameters $a, b, \ldots, g$ are real, and if $s \equiv e+f+g-a-b-c-d > 0$, the integral tends to zero as the radius of the circle tends to infinity. The sum of the residues at the poles of the integrand likewise tends to zero, and it follows that

$$\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(1-e)\,\Gamma(1-f)}{\Gamma(1-d)\,\Gamma(g)}\,{}_4F_3\left[\begin{matrix}a,\ b,\ c,\ d\\ e,\ f,\ g\end{matrix}\right]$$

$$+\frac{\Gamma(1+a-e)\,\Gamma(1+b-e)\,\Gamma(1+c-e)\,\Gamma(e-1)\,\Gamma(e-f)}{\Gamma(e-d)\,\Gamma(1+g-e)}$$

$$\times\,{}_4F_3\left[\begin{matrix}1+a-e,\ 1+b-e,\ 1+c-e,\ 1+d-e\\ e,\ 1+f-e,\ 1+g-e\end{matrix}\right]$$

$$+\text{idem}\ (e \rightleftarrows f)$$

$$-\frac{\Gamma(b-a)\,\Gamma(c-a)\,\Gamma(a)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a-d)\,\Gamma(g-a)}$$

$$\times\,{}_4F_3\left[\begin{matrix}a,\ 1+a-e,\ 1+a-f,\ 1+a-g\\ 1+a-b,\ 1+a-c,\ 1+a-d\end{matrix}\right]$$

$$-\text{idem}\ (a \rightleftarrows b)-\text{idem}\ (a \rightleftarrows c)=0, \tag{2.1}$$

where "idem $(e \rightleftarrows f)$" stands for "the same expression with e and f interchanged". In the particular case in which $e=1+a-b$, $f=1+a-c$, and $g=1+a-d$, the series are all well-poised and they are equal in pairs. Unfortunately the multipliers of the series are also equal in pairs, so that the identity (2.1) is, in this case, trivial. The difficulty is avoided, however, by considering a modified integral, viz.

$$\frac{1}{2i\pi}\int \frac{\Gamma(b+z)\,\Gamma(c+z)\,\Gamma(-z)\,\Gamma(1-e-z)\,\Gamma(1-f-z)}{\Gamma(1-a-z)\,\Gamma(1-d-z)\,\Gamma(g+z)}\exp(i\pi z)\,dz.$$

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* *Cf.* below, § 3.

The real part of the sum of the residues can be equated to zero, and it follows that

$$\frac{\Gamma(b)\Gamma(c)\Gamma(1-e)\Gamma(1-f)}{\Gamma(1-a)\Gamma(1-d)\Gamma(g)}\,{}_4F_3\left[\begin{matrix}a,\ b,\ c,\ d\\ e,\ f,\ g\end{matrix}\right]$$

$$=\cos\pi e\,\frac{\Gamma(1+b-e)\Gamma(1+c-e)\Gamma(e-1)\Gamma(e-f)}{\Gamma(e-a)\Gamma(e-d)\Gamma(1+g-e)}$$

$$\times\,{}_4F_3\left[\begin{matrix}1+a-e,\ 1+b-e,\ 1+c-e,\ 1+d-e\\ 2-e,\ 1+f-e,\ 1+g-e\end{matrix}\right]$$

$$+\text{idem }(e\rightleftarrows f)$$

$$+\cos\pi b\,\frac{\Gamma(c-b)\Gamma(b)\Gamma(1+b-e)\Gamma(1+b-f)}{\Gamma(1+b-a)\Gamma(1+b-d)\Gamma(g-b)}$$

$$\times\,{}_4F_3\left[\begin{matrix}b,\ 1+b-e,\ 1+b-f,\ 1+b-g\\ 1+b-a,\ 1+b-c,\ 1+b-d\end{matrix}\right]$$

$$+\text{idem }(b\rightleftarrows c). \qquad (2.2)$$

As before, the series can be made well-poised by writing $e=1+a-b$, $f=1+a-c$, and $g=1+a-d$ and the series on the right are identical in pairs. The sum of the coefficients of the series which are shown explicitly is

$$\frac{\Gamma(2b-a)\Gamma(b+c-a)\Gamma(a-b)\Gamma(c-b)\Gamma(b)}{\Gamma(1+a-b-d)\Gamma(1+b-d)}$$

$$\times\left[\frac{-\cos\pi(a-b)\sin\pi b}{\pi}+\frac{\cos\pi b\sin\pi(a-b)}{\pi}\right].$$

The product of $\Gamma(2b-a)$ and the expression in square brackets is

$$-\frac{\Gamma(2b-a)\sin\pi(2b-a)}{\pi}\quad\text{which is equal to}\quad-\frac{1}{\Gamma(1+a-2b)}.$$

It follows that

$$\frac{\Gamma(b)\Gamma(c)\Gamma(b-a)\Gamma(c-a)}{\Gamma(1-a)\Gamma(1-d)\Gamma(1+a-d)}\,{}_4F_3\left[\begin{matrix}a,\ b,\ c,\ d\\ 1+a-b,\ 1+a-c,\ 1+a-d\end{matrix}\right]$$

$$+\frac{\Gamma(b)\Gamma(b+c-a)\Gamma(a-b)\Gamma(c-b)}{\Gamma(1+a-2b)\Gamma(1+a-b-d)\Gamma(1+b-d)}\,{}_4F_3\left[\begin{matrix}2b-a,\ b,\ b+c-a,\ b+d-a\\ 1+b-a,\ 1+b-c,\ 1+b-d\end{matrix}\right]$$

$$+\text{idem }(b\rightleftarrows c)=0. \qquad (2.3)$$

Other relations of like character between well-poised series could be obtained in the same way. It is more interesting, however, to regard such relations as cases of the relations between trigonometric series. Further consideration of the algebraic series is therefore postponed to § 5.

3. The trigonometric series which are to be considered in the first instance may be defined as follows. If u_n is the typical term of the series

$$ {}_rF_{r-1}\left[\begin{matrix} a,\ b,\ c,\ \dots \\ 1+a-b,\ 1+a-c,\ \dots \end{matrix}\right], $$

then the series with typical term $(-1)^{rn}\,u_n \sin(a+2n)\theta$ is denoted by

$$ {}_r\Phi_{r-1}\left[\begin{matrix} a,\ b,\ c,\ \dots;\ \theta \\ 1+a-b,\ 1+a-c,\ \dots \end{matrix}\right]. $$

Now consider the integral

$$ \frac{1}{2i\pi}\int \frac{\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c+z)\,\Gamma(-z)\,\Gamma(b-a-z)\,\Gamma(c-a-z)}{\Gamma(1-d-z)\,\Gamma(1+a-d+z)} \exp(2i\theta z)\,dz, $$

the contour of integration being a circle with centre at the origin, the radius R being chosen so that the circle does not pass through any of the poles of the integrand. So long as the real part of z is positive, the integrand may conveniently be written

$$ \frac{\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c+z)\,\Gamma(d+z)}{\Gamma(1+z)\,\Gamma(1+a-b+z)\,\Gamma(1+a-c+z)\,\Gamma(1+a-d+z)} $$

$$ \times \frac{\pi^2 \sin\pi(d+z)\exp(2i\theta z)}{\sin\pi z\,\sin\pi(1+a-b+z)\,\sin\pi(1+a-c+z)}. $$

When we write $Re^{i\phi}$ for z with $-\frac{1}{2}\pi \leqslant \phi \leqslant \frac{1}{2}\pi$, we see that the first fraction in this product is $O(R^{-(4+2a-2b-2c-2d)})$. For the second fraction, we shall need two results: (i) the second fraction is a function of z which remains bounded when $R \to \infty$, provided that $-\pi \leqslant \theta \leqslant \pi$, and (ii) the second fraction is $O[\exp(-2R\theta\sin\phi - 2R\pi|\sin\phi|)]$ when $R\to\infty$.

From (i) we see that the integral round the semicircle on the right of the imaginary axis tends to zero when $R\to\infty$, provided that the conditions

$$ s > 0, \quad -\pi \leqslant \theta \leqslant \pi $$

are satisfied, s denoting the convergency indicator defined by the equation*

$$ s = 3+2a-2b-2c-2d. $$

---

* It may be remarked that the analysis is easily modified to cover the extension to the case of complex parameters a, b, c, ... The only changes which have to be made in the results are that the expressions by which the convergency indicators s are defined have to be replaced by their real parts.

Again, from (ii) combined with Jordan's lemma*, we see that the integral round the same semicircle tends to zero, provided that the conditions

$$s > -1, \quad -\pi < \theta < \pi$$

are satisfied.

Similarly, when z is on the semicircle on the left of the imaginary axis, we write the integrand in the form

$$\frac{\Gamma(-z)\,\Gamma(b-a-z)\,\Gamma(c-a-z)\,\Gamma(d-a-z)}{\Gamma(1-a-z)\,\Gamma(1-b-z)\,\Gamma(1-c-z)\,\Gamma(1-d-z)}$$

$$\times \frac{\pi^2 \sin\pi(d-a-z)\exp(2i\theta z)}{\sin\pi(a+z)\sin\pi(b+z)\sin\pi(c+z)},$$

and similar statements hold, subject to the same conditions.

We now write $r_{n,1}, r_{n,2}, \ldots, r_{n,6}$ for the residues of the integrand at $z = +n,\ -n-a,\ b-a+n,\ -n-b,\ c-a+n,\ -n-c$, for brevity. We first suppose that the conditions

$$s > 0, \quad -\pi \leqslant \theta \leqslant \pi$$

are satisfied. It is easily verified that, in these circumstances, each of the six series

$$\sum_{n=0}^{\infty} r_{n,m} \quad (m = 1, 2, \ldots, 6)$$

is absolutely convergent. Since the integral round the circle of radius R tends to zero as $R \to \infty$, it follows, without difficulty, from Cauchy's theorem that

$$\sum_{m=1}^{6} \sum_{n=0}^{\infty} r_{n,m} = 0.$$

Now, by combining like terms in the two series on the left, we find that

$$\sum_{n=0}^{\infty} r_{n,1} + \sum_{n=0}^{\infty} r_{n,2} = -2i\exp(-ia\theta)\,\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(1-d)\,\Gamma(1+a-d)}$$

$$\times\,{}_4\Phi_3\left[\begin{matrix} a,\ b,\ c,\ d;\ \theta \\ 1+a-b,\ 1+a-c,\ 1+a-d \end{matrix}\right].$$

---

* See, *e.g.*, Whittaker and Watson, *Modern analysis* (1927), 115–116.

By treating the other pairs of series of residues in the same way, and then dividing by the common factor $2i\exp(-ia\theta)$, we find that

$$\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(1-d)\,\Gamma(1+a-d)}\,{}_4\Phi_3\left[\begin{matrix} a,\ b,\ c,\ d;\ \theta \\ 1+a-b,\ 1+a-c,\ 1+a-d\end{matrix}\right]$$

$$+\frac{\Gamma(2b-a)\,\Gamma(b)\,\Gamma(b+c-a)\,\Gamma(a-b)\,\Gamma(c-b)}{\Gamma(1+a-b-d)\,\Gamma(1+b-d)}$$

$$\times\,{}_4\Phi_3\left[\begin{matrix} 2b-a,\ b,\ b+c-a,\ b+d-a;\ \theta \\ 1+b-a,\ 1+b-c,\ 1+b-d\end{matrix}\right]$$

$$+\ \text{idem}\ (b \leftrightarrows c) = 0. \tag{3.1}$$

We next suppose that the conditions

$$3+2a-2b-2c-2d > -1, \quad -\pi < \theta < \pi$$

are satisfied. In these circumstances, each of the six series

$$\sum_{n=0}^{\infty} r_{n,m} \quad (m = 1,\ 2,\ \ldots,\ 6)$$

converges, but not absolutely; a little more care in making $R \to \infty$ is necessary.

For any particular value of R, let N_m+1 be the number of poles of the m-th sequence which lie inside the circle $|z| = R$. We then have

$$\lim_{R\to\infty}\left(\sum_{m=1}^{6}\ \sum_{n=1}^{N_m} r_{n,m}\right) = 0.$$

Now N_2-N_1, N_4-N_3, and N_6-N_5 all remain bounded as $R\to\infty$; and so, since $r_{n,m}\to 0$ as $n\to\infty$, we have

$$\lim_{R\to\infty}\left(\sum_{n=0}^{N_1}(r_{n,1}+r_{n,2})+\sum_{n=0}^{N_3}(r_{n,3}+r_{n,4})+\sum_{n=0}^{N_5}(r_{n,5}+r_{n,6})\right) = 0.$$

Hence, since the limits

$$\sum_{n=0}^{\infty}(r_{n,1}+r_{n,2}), \quad \sum_{n=0}^{\infty}(r_{n,3}+r_{n,4}), \quad \sum_{n=0}^{\infty}(r_{n,5}+r_{n,6})$$

exist, we have

$$\sum_{n=0}^{\infty}(r_{n,1}+r_{n,2})+\sum_{n=0}^{\infty}(r_{n,3}+r_{n,4})+\sum_{n=0}^{\infty}(r_{n,5}+r_{n,6}) = 0,$$

that is to say, (3.1) has now been proved subject to the conditions

$$3+2a-2b-2c-2c-2d > -1, \quad -\pi < \theta < \pi.$$

If, as usual, we write s for the convergency indicator of series under consideration, for the series involved in (3.1) we have

$$s = 3+2a-2b-2c-2d.$$

The conditions concerning (3.1) may now be summarised by saying that it has been established when $s>0$, $-\pi \leqslant \theta \leqslant \pi$, and also when $0 \geqslant s > -1$, $-\pi < \theta < \pi$.

In the same way we can prove that

$$\frac{\Gamma(a)\Gamma(b)\Gamma(b-a)}{\Gamma(1-c)\Gamma(1+a-c)}\,{}_3\Phi_2\left[\begin{matrix} a,\ b,\ c;\ \theta \\ 1+a-b,\ 1+a-c \end{matrix}\right]$$

$$+\frac{\Gamma(2b-a)\Gamma(b)\Gamma(a-b)}{\Gamma(1+a-b-c)\Gamma(1+b-c)}\,{}_3\Phi_2\left[\begin{matrix} 2b-a,\ b,\ b+c-a;\ \theta \\ 1+b-a,\ 1+b-c \end{matrix}\right] = 0 \qquad (3.2)$$

provided that either $s>0$, $-\pi \leqslant 2\theta \leqslant \pi$, or else $0 \geqslant s > -1$, $-\pi < 2\theta < \pi$, where

$$s = 2+a-2b-2c;$$

and that

$$\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(b-a)\Gamma(c-a)\,{}_3\Phi_2\left[\begin{matrix} a,\ b,\ c;\ \theta \\ 1+a-b,\ 1+a-c \end{matrix}\right]$$

$$+\Gamma(2b-a)\Gamma(b)\Gamma(b+c-a)\Gamma(a-b)\Gamma(c-b)\,{}_3\Phi_2\left[\begin{matrix} 2b-a,\ b,\ b+c-a;\ \theta \\ 1+b-a,\ 1+b-c \end{matrix}\right]$$

$$+\ \text{idem}\ (b \rightleftarrows c) = 0, \qquad (3.3)$$

provided that either $s>0$, $-3\pi \leqslant 2\theta \leqslant 3\pi$, or else $0 \geqslant s > -1$, $-3\pi < 2\theta < 3\pi$.

It can be seen that so long as θ is between 0 and $\frac{1}{2}\pi$ three ${}_3\Phi_2$'s are related in pairs, but that if θ is between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ there is just one relation.

Extensions to series of higher orders are easy. Between $r+1$ series of type ${}_{2r}\Phi_{2r-1}$ there is a relation subject to the same conditions as (3.1). For ${}_{2r+1}\Phi_{2r}$ there are relations between $r+1$ series and between $r+2$ series subject to the conditions satisfied by (3.2) and (3.3) respectively.

The restrictions are essential, as may be exemplified in the very simple case in which the Φ series can be summed by the binomial theorem. In that case

$${}_1\Phi_0(a;\ \theta) = 0 \qquad (3.4)$$

so long as $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, but the more general formula is

$${}_1\Phi_0(a;\ \theta) = \sin am\pi (2\cos\theta)^{-a} \qquad (3.41)$$

with $(m-\frac{1}{2})\pi < \theta < (m+\frac{1}{2})\pi$, where m is an integer, positive or negative.

It may be noted here that the Φ's satisfy differential equations analogous to that of the hypergeometric function. Thus the Φ's in (3.1) are solutions of the equation

$$\left\{\sin\theta\left[c_0-c_2\left(\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\theta}\right)^4\right]-\cos\theta\left[c_1\frac{d}{d\theta}-c_3\left(\frac{d}{d\theta}\right)^3\right]\right\}\Phi=0, \qquad (3.51)$$

where the c's are defined to be the coefficients in the equation

$$x^4-c_3x^3+c_2x^2-c_1x+c_0=0, \qquad (3.51)$$

which has a, $2b-a$, $2c-a$, $2d-a$ for its roots.

4. By differentiating the Φ's with respect to θ, series of cosines are obtained. If we denote by

$${}_r\Psi_{r-1}\left[\begin{matrix} a,\ b,\ c,\ \ldots;\ \theta \\ 1+a-b,\ 1+a-c,\ \ldots\end{matrix}\right]$$

the series with typical term

$$(-1)^{(r+1)n}\,u_n\cos(a+2n)\,\theta,$$

then the result of differentiating equation (3.2) is

$$\frac{\Gamma(a+1)\,\Gamma(b)\,\Gamma(b-a)}{\Gamma(1-c)\,\Gamma(1+a-c)}\,{}_4\Psi_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c;\ \theta \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c\end{matrix}\right]$$
$$+\frac{\Gamma(2b-a+1)\,\Gamma(b)\,\Gamma(a-b)}{\Gamma(1+a-b-c)\,\Gamma(1+b-c)}\,{}_4\Psi_3\left[\begin{matrix} 2b-a,\ 1+b-\frac{1}{2}a,\ b,\ b+c-a;\ \theta \\ b-\frac{1}{2}a,\ 1+b-a,\ 1+b-c\end{matrix}\right]=0. \qquad (4.1)$$

It is easy to justify the differentiation term-by-term when the series involved in (4.1) are absolutely convergent, *i.e.* when $s>0$, $-\pi\leqslant 2\theta\leqslant\pi$, where

$$s=1+a-2b-2c,$$

so that in these circumstances (4.1) is valid. It can also be proved that (4.1) is valid when $0\geqslant s>-1$, $-\pi<2\theta<\pi$; to do this, we take the contour integral by which (3.2) is established, differentiate with respect to θ under the integral sign, and investigate *ab initio* the integral so obtained.

Similar equations can be deduced from (3.1) and (3.3).

5. By giving θ the values $\frac{1}{2}\pi$ and π in turn, we deduce from the Φ equations relations between well-poised series. It is not necessary to deal in this way with relations involving only two series, since in each case the series can be summed by known formulae in terms of gamma functions.

The formula (2.3) which has been proved already is obtained by substituting π for θ in (3.1). By writing $\frac{1}{2}\pi$ for θ in the same equation, we find that

$$\sin\tfrac{1}{2}\pi a\,\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(1-d)\,\Gamma(1+a-d)}\;{}_4F_3\left[\begin{matrix}a,\ b,\ c,\ d;\ -1\\ 1+a-b,\ 1+a-c,\ 1+a-d\end{matrix}\right]$$

$$+\sin\tfrac{1}{2}\pi(2b-a)\,\frac{\Gamma(2b-a)\,\Gamma(b)\,\Gamma(b+c-a)\,\Gamma(a-b)\,\Gamma(c-b)}{\Gamma(1+a-b-d)\,\Gamma(1+b-d)}$$

$$\times{}_4F_3\left[\begin{matrix}2b-a,\ b,\ b+c-a,\ b+d-a;\ -1\\ 1+b-a,\ 1+b-c,\ 1+b-d\end{matrix}\right]$$

$$+\text{ idem }(b\rightleftarrows c)=0. \tag{5.1}$$

The analogue of (3.2) gives, with $\theta=\frac{1}{2}\pi$,

$$\sin\tfrac{1}{2}\pi a\,\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(1-d)\,\Gamma(1-e)\,\Gamma(1+a-d)\,\Gamma(1+a-e)}$$

$$\times{}_5F_4\left[\begin{matrix}a,\ b,\ c,\ d,\ e;\ 1\\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e\end{matrix}\right]$$

$$+\sin\tfrac{1}{2}\pi(2b-a)\,\frac{\Gamma(2b-a)\,\Gamma(b)\,\Gamma(b+c-a)\,\Gamma(a-b)\,\Gamma(c-b)}{\Gamma(1+a-b-d)\,\Gamma(1+a-b-e)\,\Gamma(1+b-d)\,\Gamma(1+b-e)}$$

$$\times{}_5F_4\left[\begin{matrix}2b-a,\ b,\ b+c-a,\ b+d-a,\ b+e-a;\ 1\\ 1+b-a,\ 1+b-c,\ 1+b-d,\ 1+b-e\end{matrix}\right]$$

$$+\text{ idem }(b\rightleftarrows c)=0, \tag{5.2}$$

and (3.3) yields, with $\theta=\pi$,

$$\frac{\Gamma(b)\,\Gamma(c)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(1-a)}\;{}_3F_2\left[\begin{matrix}a,\ b,\ c;\ -1\\ 1+a-b,\ 1+a-c\end{matrix}\right]$$

$$+\frac{\Gamma(b)\,\Gamma(b+c-a)\,\Gamma(a-b)\,\Gamma(c-b)}{\Gamma(1-2b+a)}\;{}_3F_2\left[\begin{matrix}2b-a,\ b,\ b+c-a;\ -1\\ 1+b-a,\ 1+b-c\end{matrix}\right]$$

$$+\text{ idem }(b\rightleftarrows c)=0. \tag{5.3}$$

In the same way the Ψ equations lead to special relations, found by giving θ the values 0 or $\frac{1}{2}\pi$.

The analogue of (4.1) yields

$$\begin{aligned}&\frac{\Gamma(a+1)\Gamma(b)\Gamma(c)\Gamma(b-a)\Gamma(c-a)}{\Gamma(1-d)\Gamma(1-e)\Gamma(1+a-d)\Gamma(1+a-e)}\\ &\qquad\times{}_6F_5\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ e;\ -1\\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e\end{matrix}\right]\\ &+\frac{\Gamma(2b-a+1)\Gamma(b)\Gamma(b+c-a)\Gamma(a-b)\Gamma(c-b)}{\Gamma(1+a-b-d)\Gamma(1+a-b-e)\Gamma(1+b-d)\Gamma(1+b-e)}\\ &\qquad\times{}_6F_5\left[\begin{matrix}2b-a,\ 1+b-\frac{1}{2}a,\ b,\ b+c-a,\ b+d-a,\ b+e-a;\ -1\\ b-\frac{1}{2}a,\ 1+b-a,\ 1+b-c,\ 1+b-d,\ 1+b-e\end{matrix}\right]\\ &+\text{idem }(b\rightleftarrows c)=0, \end{aligned}\tag{5.4}$$

and

$$\begin{aligned}&\cos\tfrac{1}{2}\pi a\,\frac{\Gamma(a+1)\Gamma(b)\Gamma(c)\Gamma(b-c)\Gamma(c-a)}{\Gamma(1-d)\Gamma(1-e)\Gamma(1+a-d)\Gamma(1+a-e)}\\ &\qquad\times{}_6F_5\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ b,\ c,\ d,\ e;\ 1\\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e\end{matrix}\right]\\ &+\cos\tfrac{1}{2}\pi(2b-a)\,\frac{\Gamma(2b-a+1)\Gamma(b)\Gamma(b+c-a)\Gamma(a-b)\Gamma(c-b)}{\Gamma(1+a-b-d)\Gamma(1+a-b-e)\Gamma(1+b-d)\Gamma(1+b-e)}\\ &\qquad\times{}_6F_5\left[\begin{matrix}2b-a,\ 1+b-\frac{1}{2}a,\ b,\ b+c-a,\ b+d-a,\ b+e-a;\ 1\\ b-\frac{1}{2}a,\ 1+b-a,\ 1+b-c,\ 1+b-d,\ 1+b-e\end{matrix}\right]\\ &+\text{idem }(b\rightleftarrows c)=0. \end{aligned}\tag{5.5}$$

Differentiating (3.1) and writing $\theta=\frac{1}{2}\pi$ leads to

$$\begin{aligned}&\cos\tfrac{1}{2}\pi a\,\frac{\Gamma(a+1)\Gamma(b)\Gamma(c)\Gamma(b-a)\Gamma(c-a)}{\Gamma(1-d)\Gamma(1+a-d)}\\ &\qquad\times{}_5F_4\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ b,\ c,\ d;\ -1\\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d\end{matrix}\right]\\ &+\cos\tfrac{1}{2}\pi(2b-a)\,\frac{\Gamma(2b-a+1)\Gamma(b)\Gamma(b+c-a)\Gamma(a-b)\Gamma(c-b)}{\Gamma(1+a-b-d)\Gamma(1+b-d)}\\ &\qquad\times{}_5F_4\left[\begin{matrix}2b-a,\ 1+b-\frac{1}{2}a,\ b,\ b+c-a,\ b+d-a;\ -1\\ b-\frac{1}{2}a,\ 1+b-a,\ 1+b-c,\ 1+b-d\end{matrix}\right]\\ &+\text{idem }(b\rightleftarrows c)=0. \end{aligned}\tag{5.6}$$

The equation which is needed to complete the set cannot be obtained in the same way from a relation between cosine series but may be deduced from the ${}_7F_6$ analogue of (5.2). This analogue contains four series and if

$b=1+\tfrac{1}{2}a$ one of these drops out. The result is equivalent to

$$\frac{\Gamma(a+1)\,\Gamma(b)\,\Gamma(c)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(1-d)\,\Gamma(1-e)\,\Gamma(1-f)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}$$

$$\times{}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e,\ f;\ 1 \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f \end{matrix}\right]$$

$$+\frac{\Gamma(2b-a+1)\,\Gamma(b)\,\Gamma(b+c-a)\,\Gamma(a-b)\,\Gamma(c-b)}{\Gamma(1+a-b-d)\,\Gamma(1+a-b-e)\,\Gamma(1+a-b-f)\times\Gamma(1+b-d)\,\Gamma(1+b-e)\,\Gamma(1+b-f)}$$

$$\times{}_7F_6\left[\begin{matrix} 2b-a,\ 1+b-\tfrac{1}{2}a,\ b,\ b+c-a,\ b+d-a,\ b+e-a,\ b+f-a;\ 1 \\ b-\tfrac{1}{2}a,\ 1+b-a,\ 1+b-c,\ 1+b-d,\ 1+b-e,\ 1+b-f \end{matrix}\right]$$

$$+\text{idem}\ (b\rightleftarrows c)=0. \tag{5.7}$$

It will be seen that the direct proof of this equation would depend on a discussion of the integral

$$\frac{1}{2\pi i}\int\frac{(\tfrac{1}{2}a+z)\,\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c+z)\,\Gamma(-z)\times\Gamma(b-a-z)\,\Gamma(c-a-z)\exp(i\pi z)}{\Gamma(1-d-z)\,\Gamma(1-e-z)\,\Gamma(1-f-z)\,\Gamma(1+a-d+z)\times\Gamma(1+a-e+z)\Gamma(1+a-f+z)\sin\pi(\tfrac{1}{2}a+z)}\,dz.$$

The residues at those poles which are the zeros of $\sin\pi(\tfrac{1}{2}a+z)$ cancel in pairs.

The last equation (5.7) is one of the group* from which this investigation started. The equation (5.4) is the correct form of an identity given in an earlier paper† as a deduction from the relation between a well-poised ${}_6F_5(-1)$ and a series ${}_3F_2(+1)$.

The analogue of (5.7) for well-poised series ${}_9F_8(+1)$ involves four series. Bailey has found relations‡ connecting certain series of this order in fours, the parameters of the series being subject to the condition that the convergency indicator is 2. It can be verified that his formulae are consistent with the analogue of (5.7); the analogue implies no such restriction.

In the statement of the results of the present section, the conditions of validity of the various results have not been appended to them. It is easy to see from the analysis of § 3 that the results involving hypergeometric series of argument $+1$, namely (5.2), (5.5), and (5.7), are valid when the convergency indicator is positive, while the results involving hypergeometric series of argument -1, namely (5.1), (5.3), (5.4), and (5.5), are valid when the convergency indicator exceeds -1.

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* *Proc. London Math. Soc.* (2), 40 (1936), 336, equation (3.4).

† *Proc. London Math. Soc.* (2), 24 (1926), 262, equation (15.51).

‡ *Generalized hypergeometric series*, Cambridge Tract, 32 (1935), 64.

6. I am much indebted to the Referees for several suggestions with regard to the form of this paper and especially for pointing out the need for some discussion of the conditions for convergence of the trigonometric series. The application of Jordan's lemma is due to Prof. G. N. Watson, who very kindly revised the paper so as to include the cases in which the series were conditionally convergent.

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ON THE UNIQUENESS OF THE REPRESENTATION OF A FUNCTION BY A TRIGONOMETRIC INTEGRAL

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[Received 26 February, 1936.—Read 20 February, 1936.—Received in revised form 17 June, 1936†.]

1. *Introduction.*

1.1. The theory of the uniqueness of a function represented by a trigonometric integral has been developed by various writers‡. The main problem may be stated, broadly, as follows. We consider a trigonometric integral

$$(1.1.1) \qquad (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(u)\, e^{ixu}\, du$$

with which we associate in some sense either a sum function $\Phi(x)$ or, more generally, upper and lower sum functions $\overline{\Phi}(x)$ and $\underline{\Phi}(x)$. To begin with, let us suppose that

$$(1.1.2) \qquad \overline{\Phi}(x) = \overline{\lim_{\omega\to\infty}}\, (2\pi)^{-\frac{1}{2}} \int_{-\omega}^{\omega} \left(1-\frac{|u|}{\omega}\right) \phi(u)\, e^{ixu}\, du\S,$$

$$(1.1.3) \qquad \underline{\Phi}(x) = \underline{\lim_{\omega\to\infty}}\, (2\pi)^{-\frac{1}{2}} \int_{-\omega}^{\omega} \left(1-\frac{|u|}{\omega}\right) \phi(u)\, e^{ixu}\, du$$

are finite for all x. We wish to show that under certain conditions these sum functions are equivalent to a function $\Phi(x)$ and that the inverse relation

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† I am much indebted to Dr. J. Cossar for reading the manuscript and calling my attention to a number of obscurities.

‡ See Pollard (9), Zygmund (18), Grimshaw (2), Jacob (5), and Verblunsky (16). I have already given some account in (8) of the connection between the theory for integrals and the corresponding theory for series.

§ *I.e.* the upper limits of the real and imaginary parts of the expression taken separately are equal to the real and imaginary parts of $\overline{\Phi}(x)$ respectively. Similarly for (1.1.3).

$$\phi(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \Phi(x)\, e^{-iux}\, dx \tag{1.1.4}$$

is true in some sense.

The conditions under which we may hope to obtain any result of this kind may be divided into three sets, viz.:

(i) restrictions inherent in the definitions of $\overline{\Phi}(x)$ and $\underline{\Phi}(x)$,

(ii) restrictions on $\overline{\Phi}(x)$ and $\underline{\Phi}(x)$,

(iii) restrictions on $\phi(u)$.

Thus, in (1.1.2) and (1.1.3), we have assumed that (1.1.1) is finite $(C, 1)$ for all x. This is, of course, a condition of type (i).

The conditions of types (ii) and (iii) are of two distinct kinds. In the first place, it is very natural to suppose that the three functions concerned are integrable L in every finite range, and indeed, with the exception of Theorem 4, my results deal only with Lebesgue integrals. But these are not the only conditions which belong to these groups. It is necessary to introduce further restrictions, such as restrictions on the order or average order of these functions for large values of the variables.

If, for the moment, we set aside the question of integrability, then in Theorems 1–5 and 7 the hypotheses are of types (i) and (ii); in Theorem 6 the hypotheses are of types (i) and (iii). That is to say, when we restrict the behaviour of $\overline{\Phi}(x)$, $\underline{\Phi}(x)$ for large x, no restrictions are required on $\phi(u)$ other than those inherent in (i)† and the obvious condition of integrability. On the other hand, when, as in Theorem 6, the behaviour of $\phi(u)$ for large u is restricted, the restrictions on $\overline{\Phi}(x)$, $\underline{\Phi}(x)$ are unnecessary.

1.2. I now state my main theorems. In the first place I assume that $\overline{\Phi}(x) = \underline{\Phi}(x) = \Phi(x)$ for all x and that the integrals are summable by Cesàro means. The trigonometric integral (1.1.1) is said to be summable (C, k), for a particular value of x, if

$$\lim_{\omega \to \infty} (2\pi)^{-\frac{1}{2}} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right)^{k} \phi(u)\, e^{ixu}\, du \tag{1.2.1}$$

exists.

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† *E.g.*, the hypotheses of Theorem 1 require

$$\int_{-\infty}^{\infty} \phi(u)\, du = 0 \quad (C, 1).$$

THEOREM 1. *If $\phi(u)$ is integrable L in every finite range and if* (1.1.1) *is summable* $(C, 1)$ *to zero for all x, then $\phi(u)$ vanishes almost everywhere.*

As I have already remarked[†], this theorem is a best possible theorem in the sense that one exceptional point is sufficient to render the conclusion false and it is not possible to replace $(C, 1)$ by $(C, 1+\delta)$ for any positive δ.

We say that $\Phi(x)$ belongs to the class L^p, where $1 \leqslant p < \infty$, if $\Phi(x)$ is measurable and

$$\int_{-\infty}^{\infty} |\Phi(x)|^p \, dx < \infty,$$

while $\Phi(x)$ belongs to L^∞ if it is equivalent to a bounded function. The following theorem includes Theorem 1.

THEOREM 2. *Suppose that $\phi(u)$ is integrable L in every finite range and that*

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(u)\, e^{ixu}\, du \quad (C, 1) \tag{1.2.2}$$

for all x. Suppose, further, that $\Phi(x)$ belongs to L^p, where $1 \leqslant p \leqslant \infty$. Then

$$\phi(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \Phi(x)\, e^{-iux}\, dx \quad (C, 1) \tag{1.2.3}$$

almost everywhere, and the integrals in (1.2.2) *and* (1.2.3) *are equi-summable* $(C, 1)$ *with Fourier series in every finite interval. If, further, $1 < p < \infty$, then the integral in* (1.2.2) *converges strongly*[‡] *with exponent p to $\Phi(x)$.*

The notion of equi-summability is explained in § 2. The proof of this theorem is given in § 5.

The case $p = 1$ of Theorem 2 is in some respects the analogue of Verblunsky's[§] theorem that, if a trigonometric series is summable $(C, 1)$ for all values of the argument to an integrable function $f(x)$, then it is the Fourier series of this function. The result for series remains true when, instead of supposing that $f(x)$ is integrable L in $(0, 2\pi)$, we suppose that it is integrable in the sense of Denjoy-Perron in $(0, 2\pi)$. The following special case of the analogue for integrals is particularly interesting. Suppose that $\Phi(x)$ is integrable L in every finite range and that its integral exists as a Cauchy integral at infinity, *i.e.*

[†] Offord, **8**. This paper also contains a simple proof of a special case of Theorem 1.

[‡] *Cf.* Hobson, **3**, 251, and Wiener, **17**, 28.

[§] *Cf.* Verblunsky, **13**, 84, and **14**, II 468.

$$\lim_{a,\,b\to\infty}\int_{-b}^{a}\Phi(x)\,dx \tag{1.2.4}$$

exists when a, b tend to infinity independently. The following theorem includes Theorem 1 and the special case of Theorem 2 when $p=1$.

THEOREM 3. *Suppose that* (1.2.2) *holds for all x, and that $\Phi(x)$ is integrable L in every finite range and satisfies* (1.2.4). *Then* (1.2.3) *holds almost everywhere and the integrals* (1.2.2) *and* (1.2.3) *are equi-summable* $(C, 1)$ *with Fourier series.*

The proof of this theorem is given in § 5. For the sake of completeness, I state the following theorem. A sketch of the proof will be found in § 5. We define summability (C, k) for Denjoy-Perron integrals by (1.2.1) just as in the case of Lebesgue integrals.

THEOREM 4. *Suppose that $\phi(u)$ is integrable in the sense of Denjoy-Perron in every finite range and that* (1.2.2) *holds for all x. Suppose, further, that $\Phi(x)$ is integrable in the sense of Denjoy-Perron in* $(-\infty, \infty)$, *i.e.*

$$\lim_{a,\,b\to\infty}(D)\int_{-b}^{a}\Phi(x)\,dx$$

exists when (a, b) *tend to infinity independently. Then*

$$\phi(u)=(2\pi)^{-\frac{1}{2}}(D)\int_{-\infty}^{\infty}\Phi(x)\,e^{-iux}\,dx \quad (C,\ 1+\delta)$$

almost everywhere and the integrals (1.2.2) *and* (1.2.3), *interpreted in the Denjoy-Perron sense, are equi-summable* $(C, 1)$ *with Fourier series.*

The following theorem includes Theorems 1–3, apart from the question of strong convergence in Theorem 2. The proof is given in § 4.

THEOREM 5. *Suppose that $\phi(u)$ is integrable L in every finite range. Let $\overline{\Phi}(x)$ and $\underline{\Phi}(x)$ be defined by* (1.1.2) *and* (1.1.3). *Suppose that $\overline{\Phi}(x)$ and $\underline{\Phi}(x)$ satisfy the following conditions:* (i) *they are finite for all x,* (ii) *they are equal at an everywhere dense set in* $(-\infty, \infty)$, (iii) *they are integrable L in every finite range,* (iv) *that*

$$\int_{0}^{x}\overline{\Phi}(t)\,dt=O(|x|^{\kappa}),\qquad \int_{0}^{x}\underline{\Phi}(t)\,dt=O(|x|^{\kappa}), \tag{1.2.5}$$

where $0\leqslant\kappa<1$, as x approaches infinity positive or negative.

Then $\overline{\Phi}(x) \equiv \underline{\Phi}(x) \equiv \Phi(x)$ *and* (1.2.2) *and* (1.2.3) *hold almost everywhere, both integrals being equi-summable* $(C, 1)$ *with Fourier series in every finite interval.*

1.3. As I have already remarked, the hypotheses of Theorems 1–5 assume nothing about the behaviour of $\phi(u)$ for large u. These theorems are, however, subject to a severe limitation. They deal only with summability $(C, 1)$ and indeed become false when we replace $(C, 1)$ by (C, k), $k > 1$. This objection can be partly removed if restrictions are made on $\phi(u)$. In order to replace Cesàro by Poisson summability, it is sufficient to suppose that $\phi(u)$ satisfies any one of the three following conditions. It is still unnecessary to restrict $|\phi(u)|$.

$$(1.3.1) \qquad \int_0^\omega (\omega-u)\,\phi(u)\,du = o(\omega), \qquad \int_0^\omega (\omega-u)\,\phi(-u)\,du = o(\omega),$$

$$(1.3.2) \qquad \int_0^\omega \phi(u)\,du = o(\omega), \qquad \int_0^\omega \phi(-u)\,du = o(\omega),$$

$$(1.3.3) \qquad \int_0^\omega \{|\phi(u)|+|\phi(-u)|\}\,du = o(\omega^2).$$

Some discussion of these conditions is given in § 6. They are rather more general than those employed in earlier investigations.

If $\phi(u)$ is even and satisfies the hypotheses of Theorems 1 or 2, then, putting $x = 0$ in (1.2.2), we get

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\omega (\omega-u)\,\phi(u)\,du = \omega\Phi(0)+o(\omega),$$

which is essentially (1.3.1). Hence in the case of an even function, but only in this case, Theorems 1, 2, 4, 5 are included in Theorem 6 below.

Consider the integral

$$(1.3.4) \qquad \Phi(\sigma, x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\sigma|u|}\,\phi(u)\,e^{ixu}\,du,$$

which we assume to exist in some sense for $\sigma > 0$ and all x.

If $\phi(u)$ satisfies (1.3.2) or (1.3.3) the integral converges for $\sigma > 0$ and all x. If $\phi(u)$ satisfies (1.3.1) it is summable $(C, 1)$ for $\sigma > 0$ and all x. If, for a particular value of x, $\Phi(\sigma, x)$ tends to a finite limit as σ approaches zero, we say that (1.3.4) is Poisson summable for the value of x. The proof of the following theorem is given in § 7.

THEOREM 6. *Suppose that* $\phi(u)$ *satisfies any one of the conditions* (1.3.1), (1.3.2), *or* (1.3.3), *and that*

$$\overline{\Phi}(x) = \overline{\lim_{\sigma \to 0}}\, \Phi(\sigma, x), \tag{1.3.5}$$

$$\underline{\Phi}(x) = \underline{\lim_{\sigma \to 0}}\, \Phi(\sigma, x) \tag{1.3.6}$$

are finite for all x. *Suppose, further, that* $\underline{\Phi}(x) \geqslant \psi(x)$, *where* $\psi(x)$ *is integrable* L *in every finite range.*

Then $\overline{\Phi}(x) \equiv \underline{\Phi}(x) \equiv \Phi(x)$ *and*

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(u)\, e^{ixu}\, du \quad (C, 2), \tag{1.3.7}$$

$$\phi(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \Phi(x)\, e^{-iux}\, dx \quad (C, 2) \tag{1.3.8}$$

almost everywhere. Further, in any finite range both these integrals are equi-summable $(C, 2)$ *with Fourier series.*

1.4. It is now possible to extend Theorem 5 to the case of Poisson summability, and I conclude the paper with a proof of the following theorem. In this theorem we do not restrict $\phi(u)$.

THEOREM 7. *Let* $\phi(u)$ *be integrable* L *in every finite range. Suppose that*

(i) *the integral* (1.3.4) *is summable* $(C, 1)$ *for* $\sigma > 0$ *and all* x *to a function* $\Phi(\sigma, x)$,

(ii) *the functions* $\overline{\Phi}(x)$ *and* $\underline{\Phi}(x)$, *defined by* (1.3.5) *and* (1.3.6), *are finite for all* x, *integrable* L *in every finite range, and satisfy* (1.2.5),

(iii) *there exists a set* E, *everywhere dense in* $(-\infty, \infty)$, *such that* (1.1.1) *is summable* $(C, 1)$ *to a finite sum for all* x *of* E.

Then $\overline{\Phi}(x) \equiv \underline{\Phi}(x) \equiv \Phi(x)$ *and the conclusions of Theorem* 5 *are true.*

1.5. The proofs of all the above theorems depend on two theorems which I have called the fundamental uniqueness theorems for Cesàro and Poisson summability. These are Theorems 8 and 11. Theorem 8 is included in Theorem 11, but I give its proof in detail as it is much simpler.

My warmest thanks are due to Prof. G. H. Hardy for his very valuable advice and encouragement throughout this investigation. I am also deeply indebted to the University of London for the award of a Keddey-Fletcher Warr studentship. This work was done during my tenure of the studentship.

2. *Preliminary lemmas.*

2.1. We frequently employ the following notion. Suppose that the integral (1.1.1) is summable (C, k) for almost all x in an interval $a < x < b$, $(b-a \leqslant 2\pi)$ to a function $\Phi(x)$. Let

$$\sum_{-\infty}^{\infty} c_n e^{in\theta}$$

be the Fourier series of the function equal to $\Phi(x)$ in (a, b), zero in the remainder of the interval $\{\frac{1}{2}(a+b)-\pi, \frac{1}{2}(a+b)+\pi\}$, and periodic outside. Suppose that the difference

$$\sum_{-\omega \leqslant n \leqslant \omega} \left(1-\frac{|n|}{\omega}\right)^k c_n e^{inx} - (2\pi)^{-\frac{1}{2}} \int_{-\omega}^{\omega} \left(1-\frac{|u|}{\omega}\right)^k \phi(u) e^{ixu} du$$

tends uniformly to zero in every interval strictly contained in (a, b). Then we say that (1.1.1) is equi-summable† (C, k) in (a, b) with a Fourier series.

The following property is of great importance to us. It is known that a Fourier series‡ can be multiplied by a function of bounded variation and integrated term by term. Now, denote (1.2.1) by $I(x, \omega)$ and suppose that (1.1.1) is equi-summable (C, k) with a Fourier series in a finite interval (a, b). Let $g(x)$ be a function of bounded variation in (a, b); then, in virtue of the above property,

$$\int_a^b g(x) \lim_{\omega \to \infty} I(x, \omega)\, dx = \lim_{\omega \to \infty} \int_a^b g(x) I(x, \omega)\, dx.$$

This is a result of which we make repeated application.

The following result is due to Pollard§:

LEMMA 1. *Let $\psi(t)$ be integrable in the sense of Denjoy-Perron in $(-\pi, \pi)$, and let $\Sigma c_n e^{int}$ be its Fourier series. Then, if $\delta > 0$,*

$$(2.1.1) \quad \sum_{-\omega \leqslant n \leqslant \omega} \left(1-\frac{|n|}{\omega}\right)^k c_n e^{int}$$

$$= \frac{1}{\pi} \int_{t-\delta}^{t+\delta} \psi(\xi)\, d(\xi) \int_{-\omega}^{\omega} \left(1-\frac{|u|}{\omega}\right)^k e^{iu(t-\xi)} du + o(1),$$

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† *Cf.* Zygmund, **19**, 23.

‡ Hobson, **4**, 582.

§ Pollard, **10**, 213, 216. Pollard proves his theorem in the case when $\psi(t)$ is integrable in the general Denjoy sense.

where $k \geqslant 1$ *and* $o(1)$ *denotes an expression which tends to zero as* ω *tends to infinity uniformly in* t *for each fixed* δ.

If $\psi(\xi)$ *is integrable* L *in* $(-\pi, \pi)$, *then the limits, as* ω *tends to infinity, of each member of* (2.1.1) *exist for almost all* x *in* $(-\pi, \pi)$. *This remains true when* $k > 1$ *and* $\psi(\xi)$ *is integrable in the sense of Denjoy-Perron in* $(-\pi, \pi)$.

2.2. Let $A(u, \lambda)$ be an integrable function of u for each fixed λ. Write

$$A_1(u, \lambda; a) = \int_a^u A(v, \lambda)\, dv,$$

$$A_k(u, \lambda;\ a) = \int_a^u A_{k-1}(v, \lambda;\ a)\, dv,$$

where $u \geqslant a \geqslant 0$, and k is any positive integer.

LEMMA 2. *Let* m *be positive or zero and suppose that, for each fixed* λ,

$$A_{k+1}(u, \lambda;\ a) = O(u^{k+m}) \tag{2.2.1}$$

as u *tends to infinity. Let* $b(u, \mu)$ *be an indefinitely differentiable function of* u *such that*

$$b^{(\nu)}(u, \mu) = O\left(\frac{1}{u^{\nu+m+1}}\right) \quad (0 \leqslant \nu \leqslant k) \tag{2.2.2}$$

for each fixed μ.

Then the integral

$$\int_a^\infty A(u, \lambda)\, b(u, \mu)\, du \tag{2.2.3}$$

is summable (C, k) *for all* λ *and* μ *to*

$$(-)^{k+1} \int_a^\infty A_{k+1}(u, \lambda;\ a)\, b^{(k+1)}(u, \mu)\, du. \tag{2.2.4}$$

If, further, (2.2.1) *holds uniformly in* λ *for* $0 \leqslant \lambda \leqslant \Lambda$, *then* (2.2.3) *is uniformly summable* (C, k) *with respect to* λ *for* $0 \leqslant \lambda \leqslant \Lambda$. *Similarly, the conclusion holds uniformly in* μ *when* (2.2.2) *is uniform in* μ.

Proof. Consider

$$I = \int_a^\omega \left(1 - \frac{u}{\omega}\right)^k A(u, \lambda)\, b(u, \mu)\, du.$$

Integrate this by parts k times and use the fact that $A_\nu(a, \lambda;\ a) = 0$ for $\nu \geqslant 1$. We obtain

$$I = (-)^k \int_a^\omega A_k(u, \lambda;\ a) \left(\frac{d}{du}\right)^k \left\{\left(1-\frac{u}{\omega}\right)^k b(u, \mu)\right\} du.$$

Integrate once more by parts and apply Leibniz's theorem and so obtain

$$\begin{aligned} I &= \omega^{-k} A_{k+1}(\omega, \lambda;\ a)\, b(\omega, \mu) \\ &\quad + \sum_{\nu=1}^{k} (-)^\nu \binom{k+1}{\nu} \omega^{-k+\nu-1} \int_a^\omega A_{k+1}(u, \lambda;\ a)\left(1-\frac{u}{\omega}\right)^{\nu-1} b^{(\nu)}(u, \mu)\, du \\ &\quad + (-)^{k+1} \int_a^\omega \left(1-\frac{u}{\omega}\right)^k A_{k+1}(u, \lambda;\ a)\, b^{(k+1)}(u, \mu)\, du \\ &= I_0 + \sum_{\nu=1}^{k} I_\nu + I_{k+1}. \end{aligned}$$

By (2.2.1) and (2.2.2), $I_0 = O(\omega^{-1})$ for all λ, μ and for $1 \leqslant \nu \leqslant k-1$,

$$I_\nu = O(\omega^{-k+\nu-1}) \int_a^\omega O(u^{k+m})\, O(u^{-\nu-m-1})\, du = O(\omega^{-1}),$$

$$I_k = O(\omega^{-1} \log \omega)$$

for all λ, μ. Therefore

$$I = I_{k+1} + O(\omega^{-1} \log \omega). \tag{2.2.5}$$

Finally, observe that, by (2.2.1) and (2.2.2), the integral (2.2.4) converges absolutely. Hence, by the consistency theorem for Cesàro summation,

$$\lim_{\omega \to \infty} I_{k+1} = (-)^{k+1} \int_0^\infty A_{k+1}(u, \lambda;\ a)\, b^{(k+1)}(u, \mu)\, du.$$

Combining this with (2.2.5) we obtain the desired result.

It is also plain that this conclusion will hold uniformly in λ whenever (2.2.1) holds uniformly in λ. Similarly, it holds uniformly in μ if (2.2.2) holds uniformly in μ.

3. *The fundamental uniqueness theorem for Cesàro summability.*

3.1. In this section I state and prove a uniqueness theorem for even functions which is employed in the proofs of Theorems 1–5, *i.e.* the theorems which involve only Cesàro summability.

THEOREM 8. *Let $f(u)$ be integrable L in every finite range and suppose that*

$$\bar{F}(x) = \overline{\lim_{\omega\to\infty}} \int_0^\omega \left(1-\frac{u}{\omega}\right) f(u)\cos xu\,du, \tag{3.1.1}$$

$$\underline{F}(x) = \underline{\lim}_{\omega\to\infty} \int_0^\omega \left(1-\frac{u}{\omega}\right) f(u)\cos xu\,du \tag{3.1.2}$$

are finite for all x and integrable L in every finite range. Further, suppose that $f(u)$ satisfies

$$\int_0^\omega (\omega-u) f(u)\,du = L\omega + o(\omega) \tag{3.1.3}$$

as ω tends to infinity.

Then there exists a function $F(x)$ such that

$$F(x) = \int_0^\infty f(u)\cos xu\,du \quad (C,\ 2) \tag{3.1.4}$$

almost everywhere and

$$f(u) = \frac{2}{\pi}\int_0^\infty F(x)\cos ux\,dx \quad (C,\ 2)\dagger \tag{3.1.5}$$

almost everywhere. Further, in any finite interval the integrals (3.1.4) *and* (3.1.5) *are equi-summable* (*C*, 2) *with Fourier series.*

The reciprocal formulae (3.1.4) and (3.1.5) are stated in a slightly unsymmetric form in order to avoid a constantly recurring factor $\sqrt{(2/\pi)}$.

To prove Theorem 8 we have recourse to a series of lemmas. Some of these lemmas are used again in the proof of Theorem 11 (the fundamental uniqueness theorem for Poisson summability), and on this account it is necessary to state the hypotheses of each lemma explicitly. Lemmas 4–10 have much in common with their analogues for series. In these lemmas considerable use is made of the generalized third derivative introduced by Verblunsky (**12**, **13**). It will be observed that it is only in Lemma 6, the analogue of Fatou's theorem‡, that the hypothesis (3.1.3) is employed. Lemmas 11–16 have no analogues in the corresponding problem

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† (*Added* 14 *November*, 1936.) Macphail and Titchmarsh, *Journal London Math. Soc.*, 11 (1936), 313–318, show that, in the case of the function

$$f(u) = 0 \quad (0 \leqslant u < \pi),$$

$$f(u) = 2^{2^\nu}\nu^{-2}\cos 2^\nu u \quad [\nu\pi \leqslant u < (\nu+1)\pi],$$

(3.1.5) is not summable (C, α) for any $\alpha < 2$, although (3.1.4) is uniformly summable (*C*, 1) in every finite range.

‡ *Cf.* Zygmund, **19**, 272.

for series. We prove the relation (3. 1 . 5) (Lemma 12) first and employ it in the proof of (3 . 1 . 4) (Lemma 16). Theorem 8 follows at once from Lemmas 9, 11, and 15, by taking $F(x) = \gamma(x)$.

Before giving any lemmas we make some obvious simplifications. In the first place we may, without loss of generality, suppose $f(u)$ to be real. Secondly we may suppose that $f(u)$ is zero for $0 \leqslant u \leqslant 1$. To see this write $f(u) = f^{(1)}(u) + f^{(2)}(u)$, where $f^{(1)}(u) = f(u)$ if $0 \leqslant u \leqslant 1$ and $f^{(1)}(u) = 0$ if $u > 1$, and let

$$F^{(1)}(x) = \int_0^1 f(u) \cos xu \, du,$$

$$\bar{F}^{(2)} = \bar{F}(x) - F^{(1)}(x), \; \ldots .$$

Then $f^{(2)}(u)$ satisfies the hypotheses of Theorem 8. It will be assumed throughout that this reduction has been made.

3 . 2. The first lemma of this paragraph is very straightforward and the second lemma introduces the generalized third derivative. We write

$$f_1(\omega) = \int_0^\omega f(u) \, du, \qquad f_2(\omega) = \int_0^\omega (\omega - u) f(u) \, du. \tag{3.2.1}$$

In virtue of the preliminary reduction, $f_1(u)$ and $f_2(u)$ vanish for $0 \leqslant u < 1$.

LEMMA 3. *Suppose that* $f(u)$ *vanishes for* $0 \leqslant u < 1$ *and satisfies*

$$f_2(\omega) = O(\omega) \tag{3.2.2}$$

as ω *tends to infinity. Then*

$$G(x) = -\int_1^\infty \frac{f(u)}{u^3} \sin xu \, du \quad (C,\ 1) \tag{3.2.3}$$

exists for all x, *the integral being uniformly summable* $(C,\ 1)$ *for* $0 \leqslant x \leqslant X$. *Further,*

$$G(x) = -\int_1^\infty f_2(u) \left(\frac{d}{du}\right)^2 \left(\frac{\sin xu}{u^3}\right) du \tag{3.2.4}$$

and

$$G(x) = o(x^{\frac{1}{2}} \tag{3.2.5}$$

as x *tends to infinity.*

Proof. In Lemma 2 take $a=1$, $k=1$, $m=0$, and

$$A(u)=f(u), \qquad b(u, x)=u^{-3}\sin xu.$$

Then (2.2.1) is satisfied in virtue of (3.2.2), and (2.2.2) is satisfied uniformly in x for $0 \leqslant x \leqslant X$. Hence the integral in (3.2.3) is uniformly summable $(C, 1)$ for $0 \leqslant x \leqslant X$ and (3.2.4) is true.

The relation (3.2.5) follows from (3.2.4) for, plainly,

$$G(x)=x^2\int_1^\infty \frac{f_2(u)}{u^3}\sin xu\,du+O(x)=o(x^2)\cdot$$

by the Riemann-Lebesgue theorem[†]. This completes the proof.

The generalized third derivative[‡] is defined as follows. Let

$$(3.2.6)\qquad \Delta^3 G(x)=G(x+3h)-3G(x+h)+3G(x-h)-G(x-3h),$$

then

$$\overline{D}^3 G(x)=\varlimsup_{h\to 0}\frac{\Delta^3 G(x)}{8h^3}, \qquad \underline{D}^3 G(x)=\varliminf_{h\to 0}\frac{\Delta^3 G(x)}{8h^3}$$

are respectively the upper and lower generalized third derivatives of $G(x)$ at the point x. The generalized third derivative exists when the two are equal.

LEMMA 4. *Suppose that $f(u)$ is integrable L in every finite range and vanishes for $0 \leqslant u < 1$. Suppose, further, that the functions $\overline{F}(x)$ and $\underline{F}(x)$, defined by* (3.1.1) *and* (3.1.2), *are finite for all finite x and integrable L in every finite range. Let $G(x)$ be given by* (3.2.3), *then*

$$(3.2.7)\qquad -\chi(x)\leqslant \underline{D}^3 G(x)\leqslant \overline{D}^3 G(x)\leqslant \chi(x),$$

where[§]

$$\chi(x)=C\max\{|\overline{F}(x)|, |\underline{F}(x)|\},$$

and is therefore finite for all finite x and integrable L in every finite range.

Proof. It follows at once from (3.1.1) and (3.1.2) that $f(u)$ satisfies (3.2.2). Write

$$\chi_1(x)=\max\{|\overline{F}(x)|, |\underline{F}(x)|\},$$

$$(3.2.8)\qquad f_1(x, \omega)=\int_0^\omega f(u)\cos xu\,du,$$

$$(3.2.9)\qquad f_2(x, \omega)=\int_0^\omega f_1(x, u)\,du=\int_0^\omega (\omega-u)f(u)\cos xu\,du,$$

[†] *Cf.* Hobson, **4**, 515; Wiener, **17**, 14.

[‡] Verblunsky, **12**, 387.

[§] Throughout the paper we use C to denote an absolute constant.

then, from (3.1.1) and (3.1.2),

$$(3.2.10)\quad -\chi_1(x) \leqslant \varliminf_{\omega\to\infty} \omega^{-1} f_2(x,\omega) \leqslant \varlimsup_{\omega\to\infty} \omega^{-1} f_2(x,\omega) \leqslant \chi_1(x),$$

and $\chi_1(x)$ is finite for all x and integrable L in every finite range.

By (3.2.6) and an elementary identity,

$$\Delta^3 G(x) = 8 \int_1^\infty \frac{f(u)}{u^3} \cos xu \sin^3 hu\, du \quad (C,\ 1).$$

Apply Lemma 2, with $a=1$, $k=1$, $m=0$,

$$A(u,x) = f(u)\cos xu, \qquad b(u,h) = \left(\frac{\sin hu}{hu}\right)^3.$$

The hypotheses of Lemma 2 are satisfied in virtue of (3.2.10). Hence

$$\frac{\Delta^3 G(x)}{8h^3} = \int_1^\infty f_2(x,u) \left(\frac{d}{du}\right)^2 \left(\frac{\sin hu}{hu}\right)^2 du.$$

Choose $\omega_0(\epsilon)$ so that

$$|\omega^{-1} f_2(x,\omega)| \leqslant \chi_1(x) + \epsilon$$

for $\omega \geqslant \omega_0(\epsilon)$. Then

$$\left|\frac{\Delta^3 G(x)}{8h^3}\right| \leqslant \max_{0\leqslant\omega\leqslant\omega_0} |f_2(x,\omega)| \int_0^{\omega_0} \left| \left(\frac{d}{du}\right)^2 \left(\frac{\sin hu}{hu}\right)^3 \right| du$$
$$+ \{\chi_1(x)+\epsilon\} \int_{\omega_0}^\infty u \left| \left(\frac{d}{du}\right)^2 \left(\frac{\sin hu}{hu}\right)^3 \right| du$$
$$\leqslant \max_{0\leqslant\omega\leqslant\omega_0} |f_2(x,\omega)|\, h \int_0^\infty \left| \left(\frac{d}{d\theta}\right)^2 \left(\frac{\sin\theta}{\theta}\right)^3 \right| d\theta$$
$$+ \{\chi_1(x)+\epsilon\} \int_0^\infty \theta \left| \left(\frac{d}{d\theta}\right)^2 \left(\frac{\sin\theta}{\theta}\right)^3 \right| d\theta.$$

But $\theta^{-1}\sin\theta$ is an analytic function of θ regular at $\theta=0$. Hence

$$\varlimsup_{h\to 0} \left|\frac{\Delta^3 G(x)}{8h^3}\right| \leqslant \chi_1(x) \int_0^\infty \theta \left| \left(\frac{d}{d\theta}\right)^2 \left(\frac{\sin\theta}{\theta}\right)^3 \right| d\theta = \chi(x).$$

This is the desired result.

3.3. From (3.1.1) and (3.1.2) it readily follows that the function $g(x)$, defined by (3.3.1) below, exists for all x. In this paragraph we show that $G(x)$ has everywhere a differential coefficient equal to $g(x)$. We begin by showing in Lemma 5 that $g(x)$ is everywhere the generalized first derivative of $G(x)$.

LEMMA 5. *Under the hypotheses of Lemma* 4,

$$g(x) = -\int_0^\infty \frac{f(u)}{u^2} \cos xu \, du \quad (C, 1) \tag{3.3.1}$$

exists for all x *and*

$$g(x) = \lim_{h \to 0} \frac{G(x+h) - G(x-h)}{2h}$$

for all x.

Proof. In Lemma 2 take $a = 1$, $k = 1$, $m = 0$,

$$A(u, x) = f(u) \cos xu, \quad b(u) = u^{-2}.$$

The hypotheses of Lemma 2 are satisfied in virtue of (3.2.10) and hence

$$g(x) = -\lim_{\omega \to \infty} \int_1^\omega \left(1 - \frac{u}{\omega}\right) \frac{f(u)}{u^2} \cos xu \, du = -6 \int_1^\infty \frac{f_2(x, u)}{u^4} \, du \tag{3.3.2}$$

exists for all x, where $f_2(x, u)$ is given by (3.2.9).

Again,

$$\frac{G(x+h) - G(x-h)}{2h} = -\int_0^\infty \frac{f(u)}{u^2} \cos xu \frac{\sin hu}{hu} \, du \quad (C, 1).$$

Apply Lemma 2 with $a = 1$, $b = 1$, $m = 0$,

$$A(u, x) = f(u) \cos xu, \quad b(u, h) = u^{-3} \sin hu.$$

We obtain

$$\begin{aligned} \frac{G(x+h) - G(x-h)}{2h} & \\ &= -\int_1^\infty f_2(x, u) \left(\frac{d}{du}\right)^2 \left\{\frac{1}{u^2} \frac{\sin hu}{hu}\right\} du \\ &= -6 \int_1^\infty u^{-4} f_2(x, u) \frac{\sin hu}{hu} \, du + 4 \int_1^\infty u^{-3} f_2(x, u) \frac{d}{du} \left(\frac{\sin hu}{hu}\right) du \\ &\qquad - \int_1^\infty u^{-2} f_2(x, u) \left(\frac{d}{du}\right)^2 \left(\frac{\sin hu}{hu}\right) du \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Now, by (3.2.10) and (3.3.2),

$$\lim_{h \to 0} I_1 = -6 \int_1^\infty u^{-4} f_2(x, u) \, du = g(x)$$

for all x. Again, in virtue of (3.2.10), we may write

$$|u^{-1} f_2(x, u)| \leqslant \chi_2(x) \quad (u \geqslant 1),$$

where $\chi_2(x)$ is finite for all x. Then

$$|I_2| \leqslant \chi_2(x) \int_1^\infty \frac{1}{u^2} \left| \frac{d}{du} \left(\frac{\sin hu}{hu} \right) \right| du = \chi_2(x) h^2 \int_h^\infty \frac{1}{\theta^2} \left| \frac{d}{d\theta} \left(\frac{\sin \theta}{\theta} \right) \right| d\theta$$

$$\leqslant M \chi_2(x) h.$$

Therefore $$\lim_{h \to 0} |I_2| = 0.$$

Finally,

$$|I_3| \leqslant \chi_2(x) \int_1^\infty \frac{1}{u} \left| \left(\frac{d}{du} \right)^2 \left(\frac{\sin hu}{hu} \right) \right| du = \chi_2(x) h^2 \int_h^\infty \frac{1}{\theta} \left| \left(\frac{d}{d\theta} \right)^2 \left(\frac{\sin \theta}{\theta} \right) \right| d\theta$$

$$\leqslant M \chi_2(x) h^2 \log h^{-1},$$

and so $$\lim_{h \to 0} |I_3| = 0.$$

Hence

$$\lim_{h \to 0} \frac{G(x+h) - G(x-h)}{2h} = \lim_{h \to 0} I_1 = g(x)$$

for all x. This is the desired result.

In the next lemma we show that $\{G(x+2h)+G(x-2h)-2G(x)\}/4h$ tends to zero with h. This is the second step towards showing that $G(x)$ has everywhere a differential coefficient. The hypothesis (3.1.3) of Theorem 8 is employed in the proof of this lemma and nowhere else.

LEMMA 6. *If $f(u)$ satisfies (3.1.3) and if $G(x)$ is defined as in Lemma 3, then*

$$\lim_{h \to 0} \{G(x+2h)+G(x-2h)-2G(x)\}/4h = 0$$

for all x.

Proof. The weaker hypothesis (3.2.2) is sufficient for all purposes but one, where we must use (3.1.3). Write

$$\Delta^2 G(x) = G(x+2h)+G(x-2h)-2G(x),$$

so that $$\Delta^2 G(x) = 4 \int_1^\infty \frac{f(u)}{u^3} \sin xu \sin^2 hu \, du \quad (C, 1).$$

Apply Lemma 2, with $a = 1$, $k = 1$, $m = 0$,

$$A(u) = f(u), \qquad b(u, h) = u^{-3} \sin xu \sin^2 hu.$$

We obtain, in virtue of (3.2.2),

$$\frac{\Delta^2 G(x)}{4h} = \int_1^\infty f_2(u) \left(\frac{d}{du}\right)^2 \left\{\sin xu \, \frac{\sin^2 hu}{hu^3}\right\} du$$

$$= -x^2 \int_1^\infty f_2(u) \sin xu \, \frac{\sin^2 hu}{hu^3} \, du + 2x \int_1^\infty f_2(u) \cos xu \left(\frac{d}{du}\right) \left(\frac{\sin^2 hu}{hu^3}\right) du$$

$$+ \int_1^\infty f_2(u) \sin xu \left(\frac{d}{du}\right)^2 \left(\frac{\sin^2 hu}{hu^3}\right) du$$

$$= I_1 + I_2 + I_3.$$

We now show that I_2 and I_3 tend to zero as h tends to zero; for these (3.2.2) is sufficient. We have

$$|I_2| \leqslant xM \int_1^\infty u \left| \frac{d}{du} \left\{\frac{1}{u} \, \frac{\sin^2 hu}{hu^2}\right\} \right| du$$

$$\leqslant xM \int_1^\infty \left| \frac{d}{du} \, \frac{\sin^2 hu}{hu^2} \right| du + xM \int_1^\infty \frac{1}{u} \left| \frac{\sin^2 hu}{hu^2} \right| du$$

$$= Mxh \int_h^\infty \left| \left(\frac{d}{d\theta}\right) \left(\frac{\sin\theta}{\theta}\right)^2 \right| d\theta + Mxh \int_h^\infty \frac{1}{\theta} \left(\frac{\sin\theta}{\theta}\right)^2 d\theta$$

$$\leqslant Mxh \log(1/h) \rightarrow 0$$

as h tends to zero. Again,

$$|I_3| \leqslant Mx \int_1^\infty u \left| \left(\frac{d}{du}\right)^2 \left\{\frac{1}{u} \, \frac{\sin^2 hu}{hu^2}\right\} \right| du$$

$$\leqslant Mxh^2 \int_h^\infty \left| \left(\frac{d}{d\theta}\right)^2 \left(\frac{\sin\theta}{\theta}\right)^2 \right| d\theta + Mxh^2 \int_h^\infty \frac{1}{\theta} \left| \left(\frac{d}{d\theta}\right) \left(\frac{\sin\theta}{\theta}\right)^2 \right| d\theta$$

$$+ Mx^2 h^2 \int_h^\infty \frac{1}{\theta^2} \left(\frac{\sin\theta}{\theta}\right)^2 d\theta$$

$$\leqslant Mx^2 h \rightarrow 0$$

as h tends to zero.

This leaves only I_1, and for this we require (3.1.3). We have

$$I_1 = -x^2 \int_1^K f_2(u) \sin xu \, \frac{\sin^2 hu}{hu^3} \, du - Lx^2 \int_K^\infty u \sin xu \, \frac{\sin^2 hu}{hu^3} \, du$$

$$-x^2 \int_K^\infty o(u) \sin xu \, \frac{\sin^2 hu}{hu^3} \, du$$

$$= I_4 + I_5 + I_6.$$

Choose K so that

$$|I_6| \leqslant \epsilon x^2 \int_K^\infty \frac{\sin^2 hu}{hu^2}\, du \leqslant \epsilon x^2 \int_0^\infty \left(\frac{\sin\theta}{\theta}\right)^2 d\theta \leqslant \epsilon'.$$

K being fixed,

$$\lim_{h\to 0} I_4 = \lim_{h\to 0} \left\{-x^2 h \int_1^K f_2(u) \sin xu \frac{\sin^2 hu}{h^2 u^3}\, du\right\}$$

$$= \lim_{h\to 0} \left\{-x^2 h \int_1^K f_2(u)\, u^{-1} \sin xu\, du\right\} = 0.$$

Finally,

$$|I_5| \leqslant Lx^2 \left| \int_{hK}^\infty \sin\left(\frac{x\theta}{h}\right) \left(\frac{\sin\theta}{\theta}\right)^2 d\theta \right|$$

$$\leqslant Lx^2 \int_0^{hK} \left(\frac{\sin\theta}{\theta}\right)^2 d\theta + Lx^2 \left| \int_0^\infty \sin\left(\frac{x\theta}{h}\right) \left(\frac{\sin\theta}{\theta}\right)^2 d\theta \right|.$$

Since $(\sin\theta/\theta)^2$ belongs to $L(0, \infty)$, the second term on the right tends to zero with h, by the Riemann-Lebesgue theorem. This completes the proof of the lemma.

Combining Lemmas 5 and 6, we deduce

LEMMA 7. *Under the hypotheses of Theorem* 8, $G'(x)$ *exists and is equal to* $g(x)$ *for all* x.

Proof.

$$\frac{G(x+2h)-G(x)}{2h} = \frac{G(x+2h)+G(x-2h)-2G(x)}{4h} + \frac{G(x+2h)-G(x-2h)}{4h}.$$

Hence, by Lemmas 5 and 6,

$$\lim_{h\to 0} \frac{G(x+2h)-G(x)}{2h} = g(x)$$

for all x.

3.4. Our next object is to show that $G(x)$ is of the form (3.4.1). For this we require the following lemma.

LEMMA 8. (Saks-Verblunsky.) *Let* $(-a, a)$ *be a finite interval and suppose that, for* $|x| < a$, (i) $G(x)$ *is continuous,* (ii) $G'(x)$ *exists for all* x *and is finite,* (iii) $\underline{D}^3 G(x) \geqslant \psi(x)$, *where* $\psi(x)$ *is integrable* L.

Then $\underline{D}^3 G(x)$ *is integrable L in every interval strictly contained in* (a, b) *and*

$$G(x)=\int_0^x dy \int_0^y dz \int_0^z \underline{D}^3 G(t)\,dt+px^2+qx+r$$

for $-a<-a'\leqslant x\leqslant a'<a$.

This theorem is included in Verblunsky's (**15**, 178) fundamental theorem for the generalized third derivative. It also follows readily from a much easier theorem of Saks (**12**, 249).

It follows at once from Lemmas 4 and 7 that the hypotheses of Lemma 8 are satisfied in our case. Now, from (3.2.4), we see that both $G(x)$ and $\underline{D}^3 G(x)$ are odd functions. Hence both p and r must be zero and we obtain the following result.

LEMMA 9. *Under the hypotheses of Theorem* 8, $G(x)$ *is of the form*

$$G(x)=\int_0^x dy \int_0^y dz \int_0^z \gamma(t)\,dt+qx \tag{3.4.1}$$

for all finite x.

3.5. We now consider the problem of obtaining an expression for $f(u)$ in terms of $\gamma(t)$. This is given in Lemma 11. The first step (Lemma 10) is to invert (3.2.3). This inversion depends simply on (3.2.2) and we do not require any of Lemmas 4–9.

LEMMA 10. *Under the hypotheses of Lemma* 3,

$$\frac{f(u)}{u^3}=-\frac{2}{\pi}\int_0^\infty G(x)\sin ux\,dx \quad (C,\ 2) \tag{3.5.1}$$

almost everywhere. Further, in any finite interval, the integral in (3.5.1) *is equi-summable* $(C,\ 2)$ *with a Fourier series.*

Proof. Since the integral in (3.2.4) converges absolutely, we have

$$\begin{aligned} I&=\int_0^X\left(1-\frac{x}{X}\right)^2 G(x)\sin tx\,dx\\ &=-\int_1^\infty f_2(u)\,du\int_0^X\left(1-\frac{x}{X}\right)^2\sin tx\left(\frac{d}{du}\right)^2\left(\frac{\sin ux}{u^3}\right)dx\\ &=-\tfrac{1}{2}\int_1^\infty f_2(u)\,du\left(\frac{d}{du}\right)^2\left[\frac{1}{u^3}\int_0^X\left(1-\frac{x}{X}\right)^2\{\cos(u-t)x-\cos(u+t)x\}\,dx\right]. \end{aligned}$$

Write

$$K_1(\xi)=K_1(\xi,\ X)=\tfrac{1}{2}\int_0^X\left(1-\frac{x}{X}\right)^2\cos\xi x\,dx=\frac{1}{X\xi^2}-\frac{\sin X\xi}{X^2\xi^3}.$$

Then

$$I=\int_1^\infty f_2(u)\left(\frac{d}{du}\right)^2\left\{\frac{K_1(u+t)}{u^3}\right\}du$$
$$-\left\{\int_1^{t-\delta}+\int_{t-\delta}^{t+\delta}+\int_{t+\delta}^{\infty}\right\}f_2(u)\left(\frac{d}{du}\right)^2\left\{\frac{K_1(u-t)}{u^3}\right\}du$$
$$=I_1-I_2-I_3-I_4.$$

Hence, in virtue of (3.2.2),

$$I_1=\int_1^\infty \frac{f_2(u)}{u^3}\,\frac{\sin X(u+t)}{(u+t)^3}\,du+O\left(\frac{1}{X}\right),$$

where the expression $O(X^{-1})$ is uniform in t provided that $0\leqslant t\leqslant T$. Therefore, by the Riemann-Lebesgue theorem, I_1 tends to zero uniformly in t for $0\leqslant t\leqslant T$ as X tends to infinity.

Similarly, I_2 and I_4 tend to zero uniformly for $0\leqslant t\leqslant T$ as X tends to infinity.

Integrating I_3 by parts twice, we get

$$I_3=\left[f_2(u)\frac{d}{du}\left\{\frac{K_1(u-t)}{u^3}\right\}-f_1(u)\frac{K_1(u-t)}{u^3}\right]_{t-\delta}^{t+\delta}+\int_{t-\delta}^{t+\delta}\frac{f(u)}{u^3}K_1(u-t)\,du$$
$$=\int_{t-\delta}^{t+\delta}\frac{f(u)}{u^3}K_1(u-t)\,du+O\left(\frac{1}{X}\right)$$

uniformly in t provided that $0\leqslant t\leqslant T$. The desired result now follows from Lemma 1.

LEMMA 11. *If the hypotheses of Lemma 3 are satisfied and if $G(x)$ is of the form (3.4.1), then*

(3.5.2) $$f(t)=\frac{2}{\pi}\int_0^\infty \gamma(x)\cos tx\,dx \quad (C,\ 2)$$

almost everywhere. Further, in any finite interval, the integral in (3.5.2) is equi-summable $(C,\ 2)$ with a Fourier series.

Proof. We integrate

$$I=\int_0^X (X-x)^2\gamma(x)\cos tx\,dx$$

three times by parts using (3.4.1), from which it follows that

$$G(0)=G''(0)=0,\quad G'(0)=q.$$

We have

$$I = -\int_0^X G''(x)\,\frac{d}{dx}\,\{(X-x)^2\cos tx\}\,dx$$

$$= -2qX + \int_0^X G'(x)\left(\frac{d}{dx}\right)^2\{(X-x)^2\cos tx\}\,dx$$

$$= \{-2qX + 2G(X)\cos tX\} + 6t\int_0^X G(x)\sin tx\,dx$$

$$-6t^2\int_0^X (X-x)\,G(x)\cos tx\,dx - t^3\int_0^X (X-x)^2\,G(x)\sin tx\,dx$$

$$= I_1 + I_2 + I_3 + I_4.$$

Now, by Lemma 10,

$$\lim_{X\to\infty} X^{-2} I_4 = f(t)$$

almost everywhere, and the integral which occurs here is equi-summable $(C, 2)$ with a Fourier series. Hence, to complete the proof, it is sufficient to show that

$$I_1 + I_2 + I_3 = o(X^2)$$

uniformly in t for $0 \leqslant t \leqslant T$. But, by $(3.2.5)$, $I_1 = o(X^2)$ uniformly in t. Therefore we have only to show that I_2 and I_3 are both $o(X^2)$ for large X.

Substitute in I_2 the expression for $G(x)$ from $(3.2.4)$. We get, on account of absolute convergence,

$$I_2 = -6t\int_1^\infty f_2(u)\,du\left(\frac{d}{du}\right)^2\left\{\frac{1}{u^3}\int_0^X \sin tx\,\sin ux\,dx\right\}$$

$$= -3t\int_1^\infty f_2(u)\left(\frac{d}{du}\right)^2\left\{\frac{\sin(u-t)X}{u^3(u-t)} - \frac{\sin(u+t)X}{u^3(u+t)}\right\}du.$$

Take $f_2(-u) = f_2(u)$ and remember that $f(u)$, and therefore $f_2(u)$, is zero for $-1 \leqslant u \leqslant 1$. We have

$$I_2 = -3t\left\{\int_{-\infty}^{t-1} + \int_{t-1}^{t+1} + \int_{t+1}^{\infty}\right\} f_2(u)\left(\frac{d}{du}\right)^2\left\{\frac{\sin(u-t)X}{u^3(u-t)}\right\}du$$

$$= I_5 + I_6 + I_7.$$

Now, for $0 \leqslant t \leqslant T$,

$$I_7 = 3tX^2\int_{t+1}^\infty \frac{f_2(u)}{u^3}\,\frac{\sin(u-t)X}{u-t}\,du + O(X) = o(X^2)$$

uniformly, by the Riemann-Lebesgue theorem. Similarly $I_5 = o(X^2)$. Integrating I_6 twice by parts, we get

$$I_6 = -3t\,{}_{t-1}^{t+1}\!\left[f_2(u)\frac{d}{du}\left\{\frac{\sin(u-t)X}{u^3(u-t)}\right\} - f_1(u)\,\frac{\sin(u-t)X}{u^3(u-t)}\right]$$

$$-3t\int_{t-1}^{t+1}\frac{f(u)}{u^3}\,\frac{\sin(u-t)X}{u-t}\,du$$

$$= O(X)$$

uniformly for $0 \leqslant t \leqslant T$. Hence $I_2 = o(X^2)$.

Again, by using (3.2.4),

$$I_3 = -6t^2\int_1^\infty f_2(u)\,du\left(\frac{d}{du}\right)^2\left\{\frac{1}{u^3}\int_0^X (X-x)\cos tx\sin ux\,dx\right\}$$

$$= -3t^2\left\{\int_{-\infty}^{t-\delta} + \int_{t-\delta}^{t+\delta} + \int_{t+\delta}^{\infty}\right\} f_2(u)\left(\frac{d}{du}\right)^2\left\{\frac{X}{u^3(u-t)} - \frac{\sin(u-t)X}{u^3(u-t)^2}\right\}du$$

$$= I_8 + I_9 + I_{10}.$$

Now, by the Riemann-Lebesgue theorem,

$$I_{10} = -3t^2X^2\int_{t+\delta}^{\infty}\frac{f_2(u)}{u^3}\,\frac{\sin(u-t)X}{(u-t)^2}\,du + O(X) = o(X^2)$$

uniformly for $0 \leqslant t \leqslant T$. Similarly $I_8 = o(X^2)$, while, integrating by parts, we have

$$I_9 = -3t^2\int_{t-\delta}^{t+\delta}\frac{f(u)}{u^3}\left\{\int_0^X (X-x)\cos tx\sin ux\,dx\right\}du + O(X).$$

Hence $$|I_9| \leqslant 3t^2X^2\int_{t-\delta}^{t+\delta}\frac{|f(u)|}{u^3}\,du + O(X) = o(X^2),$$

by suitable choice of δ. This completes the proof of the lemma.

3.6. We began with the hypothesis

$$\int_0^\infty f(u)\cos xu\,du = O(1) \quad (C,\ 1) \tag{3.6.1}$$

for all x, though not uniformly in x. We shall now show that this integral is, in fact, equi-summable $(C,\ 2)$ with a Fourier series to $\gamma(x)$ in every finite range. For this we require three lemmas, viz. Lemmas 12–14. The final result is given by Lemma 15.

LEMMA 12. *If* $X > 0$ *and* $t > 0$, *then*

$$\left| \int_0^X (X-x)^2 x^2 \cos tx \, dx \right| \leqslant CX^2 t^{-3}, \tag{3.6.2}$$

$$\left| \int_0^X (X-x)^2 \, x \sin tx \, dx \right| \leqslant CX t^{-3}, \tag{3.6.3}$$

where C *is an absolute constant.*

These inequalities are elementary and we leave the proofs to the reader.

LEMMA 13. *Under the hypotheses of Lemma* 11,

$$\left| \int_a^X (X-x)^2 \, \gamma(x) \cos \omega x \, dx \right| \leqslant MX^4 \omega$$

or $\omega \geqslant 1$, $X \geqslant 1$ *and* $0 \leqslant a \leqslant X$. *Here* $M = M(a)$ *is independent of* ω, X, *and is bounded in* a *for* $0 \leqslant a \leqslant A$.

Proof. Without loss of generality we may suppose that $a = 0$. Using (3.4.1), we get

$$\begin{aligned} I &= \int_0^X (X-x)^2 \gamma(x) \cos \omega x \, dx \\ &= \int_0^X \gamma_2(x) \left(\frac{d}{dx}\right)^2 \{(X-x)^2 \cos \omega x\} \, dx \\ &= 2G(X) \cos \omega X - \int_0^X G(x) \left(\frac{d}{dx}\right)^3 \{(X-x)^2 \cos \omega x\} \, dx - 2qX. \end{aligned}$$

Substitute for $G(x)$ from (3.2.4). The second term on the right becomes, on account of absolute convergence,

$$\int_1^\infty f_2(u) \, du \int_0^X \left(\frac{d}{dx}\right)^3 \{(X-x)^2 \cos \omega x\} \left(\frac{d}{du}\right)^2 \left(\frac{\sin ux}{u^3}\right) dx$$

$$= \int_1^\infty f_2(\omega) \, du \left(\frac{d}{du}\right)^2 \left[\frac{1}{u^3} \int_0^X \sin ux \left(\frac{d}{dx}\right)^3 \{(X-x)^2 \cos \omega x\} \, dx\right].$$

Now, by a simple calculation,

$$\begin{aligned} &\int_0^X \sin ux \left(\frac{d}{dx}\right)^3 \{(X-x)^2 \cos \omega x\} \, dx \\ &\qquad = 2 \sin uX \cos \omega X - 2uX + u^3 \int_0^X (X-x)^2 \cos ux \cos \omega x \, dx. \end{aligned} \tag{3.6.4}$$

Therefore, using $f_2(u) = O(u)$ and (3.2.5), we get

$$I = \int_1^\infty f_2(u)\,du \left(\frac{d}{du}\right)^2 \int_0^X (X-x)^2 \cos ux \cos \omega x\,dx + o(X^2)$$

$$= -\tfrac{1}{2}\int_1^\infty f_2(u)\,du \int_0^X (X-x)^2 x^2 \{\cos(u-\omega)x + \cos(u+\omega)x\}\,dx + o(X^2)$$

$$= -\tfrac{1}{2}\left\{\int_{-\infty}^{\omega-1/X} + \int_{\omega-1/X}^{\omega+1/X} + \int_{\omega+1/X}^{\infty}\right\} f_2(u)\,du \times \int_0^X (X-x)^2 x^2 \cos(u-\omega)x\,dx + o(X^2)$$

$$= \tfrac{1}{2}\{I_1 + I_2 + I_3\} + o(X^2).$$

By (3.6.2),

$$|I_3| \leqslant CX^2 \int_{\omega+1/X}^\infty \frac{|f_2(u)|}{(u-\omega)^3}\,du \leqslant MX^2 \int_{\omega+1/X}^\infty \frac{u\,du}{(u-\omega)^3} = MX^4\omega.$$

There is a similar inequality for I_1. Also

$$|I_2| \leqslant MX^5 \int_{\omega-1/X}^{\omega+1/X} u\,du = MX^4\omega,$$

and the lemma is proved.

LEMMA 14. *Under the hypotheses of Lemma* 11,

$$\left| \int_a^X (X-x)^2 \gamma(x) \frac{\sin \omega x}{x}\,dx \right| \leqslant MX^3\omega$$

for $\omega \geqslant 1$, $X \geqslant 1$, *and* $0 \leqslant a \leqslant X$. *Here* $M = M(a)$ *is independent of* ω, X, *and is bounded in* a *for* $0 \leqslant a \leqslant A$.

Proof. The proof of this lemma is very similar to that of Lemma 13. There is again no loss of generality in supposing that $a = 0$.

We can show, just as in Lemma 13, that

$$\int_0^X (X-x)^2 \gamma(x) \frac{\sin \omega x}{x}\,dx$$

$$= \int_1^X f_2(u) \left(\frac{d}{du}\right)^2 \int_0^X (X-x)^2 \cos ux \frac{\sin \omega x}{x}\,dx + O(X),$$

and the desired result follows by using (3.6.3) in the place of (3.6.2).

LEMMA 15. *Under the hypotheses of Lemma* 11,

$$\gamma(x) = \int_1^\infty f(u) \cos xu\,du \quad (C,\ 2)$$

almost everywhere. Further, in any finite range, this integral is equi-summable (*C*, 2) *with a Fourier series.*

Proof. By Lemma 11,

$$\int_0^\omega (\omega-u)^2 f(u)\cos yu\,du$$

$$=\frac{2}{\pi}\lim_{X\to\infty}\int_0^X\left(1-\frac{x}{X}\right)^2\gamma(x)\,dx\int_0^\omega(\omega-u)^2\cos xu\cos yu\,du$$

$$=\frac{1}{\pi}\{I(\omega,\,x-y)+I(\omega,\,x+y)\},$$

where

$$I(\omega,\,\xi)=\lim_{X\to\infty}\int_0^X\left(1-\frac{x}{X}\right)^2\gamma(x)\,dx\int_0^\omega(\omega-u)^2\cos\xi u\,du.$$

Now

$$(3.6.4)\quad \int_0^\omega(\omega-u)^2\cos(x-y)u\,du$$

$$=\frac{2\omega}{(x-y)^2}-\frac{2\sin(x-y)\omega}{(x-y)^3}$$

$$=\frac{2\omega}{(x-y)^2}+\frac{2\cos x\omega\sin y\omega}{(x-y)^3}-\frac{2\sin x\omega\cos y\omega}{(x-y)^3}.$$

Write

$$I_1=\int_{y+\delta}^X\left(1-\frac{x}{X}\right)^2\frac{\gamma(x)}{(x-y)^2}\,dx,$$

$$I_2=\int_{y+\delta}^X\left(1-\frac{x}{X}\right)^2\gamma(x)\,\frac{\cos\omega x}{(x-y)^3}\,dx,$$

$$I_3=\int_{y+\delta}^X\left(1-\frac{x}{X}\right)^2\gamma(x)\,\frac{\sin\omega x}{(x-y)^3}\,dx.$$

We now apply Lemma 2 to each of the terms I_1, I_2, and I_3. Observe that, in this application, we say nothing about uniformity. Let

$$G(x,\,\delta)=\tfrac{1}{2}\int_{y+\delta}^x(x-t)^2\gamma(t)\,dt.$$

Then, by (3.2.5), $G(x,\,\delta)=o(x^2)$. Apply Lemma 2 with $a=y+\delta$, $k=2$, $m=0$,

$$A(u)=\gamma(u),\qquad b(u,\,y)=(u-y)^{-2};$$

then

$$\lim_{X\to\infty}I_1=24\int_{y+\delta}^\infty G(x,\,\delta)(x-y)^{-5}\,dx.$$

But $G(x, \delta) = o(x^2)$ uniformly in y for $0 \leqslant y \leqslant Y$. Therefore

$$\left| \lim_{X \to \infty} I_1 \right| \leqslant M,$$

where M is independent of ω.

Write

$$J(x) = \int_{y+\delta}^{x} \gamma(t) \cos \omega t \, dt, \quad J_1(x) = \int_{y+\delta}^{x} J(t) \, dt, \; \ldots.$$

Then, by Lemma 13,

$$|J_3(x)| \leqslant M x^4 \omega.$$

Apply Lemma 2 with $a = y+\delta$, $k = 2$, $m = 2$,

$$A(t) = \gamma(t) \cos \omega t, \quad b(t) = \frac{\cos \omega t}{(t-y)^3},$$

and we obtain
$$\left| \lim_{X \to \infty} I_2 \right| \leqslant M\omega$$

uniformly in y for $0 \leqslant y \leqslant Y$. Similarly, in virtue of Lemma 14,

$$\left| \lim_{X \to \infty} I_3 \right| \leqslant M\omega$$

uniformly in y for $0 \leqslant y \leqslant Y$.

We have therefore proved that

$$I(\omega, x-y) = \int_0^{y+\delta} \gamma(x) \, dx \int_0^{\omega} (\omega-u)^2 \cos u(x-y) \, du + O(\omega)$$

$$= \int_{y-\delta}^{y+\delta} \gamma(x) \, dx \int_0^{\omega} (\omega-u)^2 \cos u(x-y) \, du + O(\omega),$$

by (3.6.4). This holds uniformly in y for $0 \leqslant y \leqslant Y$. It can be shown in a similar manner that $I(x+y) = O(\omega)$, and hence we obtain

$$\int_0^{\omega} (\omega-u)^2 f(u) \cos yu \, du = \frac{1}{\pi} \int_{y-\delta}^{y+\delta} \gamma(x) \, dx \int_0^{\omega} (\omega-u)^2 \cos u\,(x-y) \, du + O(\omega)$$

uniformly in y for $0 \leqslant y \leqslant Y$. This, in virtue of Lemma 1, is the desired result.

As we have already remarked, Theorem 8 follows from Lemmas 9, 11, and 15.

4. *Proof of Theorem* 5.

4.1. We now prove some of the theorems stated in the introduction. Theorem 5 includes Theorems 1, 2, and 3, apart from the question of strong convergence in Theorem 2. We return to these theorems in § 5.

Before beginning the proof of Theorem 5 let us look, for a moment, at the corresponding result for series. Write

$$A_n(x) = a_n \cos nx + b_n \sin nx,$$

$$P(r, x) = \sum_1^\infty A_n(x) r^n, \tag{4.1.1}$$

and suppose that

$$\overline{P}(x) = \varlimsup_{r\to 1} P(r, x), \quad \underline{P}(x) = \varliminf_{r\to 1} P(r, x) \tag{4.1.2}$$

are finite for all x and integrable L in $(0, 2\pi)$. Suppose also that the series

$$\sum_1^\infty A_n(x) \tag{4.1.3}$$

is summable $(C, 1)$ for all x of a set E, which is everywhere dense in $(0, 2\pi)$. We wish to show, under these conditions, that (4.1.3) is a Fourier series.

This can be proved in the following way. Let y be a point of the everywhere dense set E at which (4.1.3) is summable $(C, 1)$. Then

$$A_n(y) = o(n) \tag{4.1.4}$$

for large n. Let

$$Q(r, x) = \sum_1^\infty A_n(y)\, r^n \cos nx.$$

Then, by (4.1.1) and (4.1.2),

$$\tfrac{1}{2}\{\underline{P}(y+x)+\underline{P}(y-x)\} \leqslant \varliminf_{r\to 1} Q(r, x) \leqslant \varlimsup_{r\to 1} Q(r, x) \tag{4.1.5}$$

$$\leqslant \tfrac{1}{2}\{\overline{P}(y+x)+\overline{P}(y-x)\}.$$

Hence, by Verblunsky's theorem[†], for fixed y, the series

$$\sum_1^\infty A_n(y) \cos nx$$

[†] Verblunsky, **14**, II, 468. *Cf.* Zygmund, **19**, 298.

is a Fourier series and there exists a function $F_y(x)$ such that

$$A_n(y) = \frac{1}{\pi}\int_{-\pi}^{\pi} F_y(x)\cos nx\,dx.$$

Also, from (4.1.5), for almost all x,

$$\tag{4.1.6} \tfrac{1}{2}\{\underline{P}(y+x)+\underline{P}(y-x)\} \leqslant F_y(x) \leqslant \tfrac{1}{2}\{\bar{P}(y+x)+\bar{P}(y-x)\}.$$

Therefore, for all n and all y of E,

$$\tag{4.1.7} |A_n(y)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi}\{|\bar{P}(x)|+|\underline{P}(x)|\}\,dx \leqslant M,$$

where M is independent of n and y. But E is everywhere dense and $A_n(y)$ is a continuous function of y for each n. Therefore (4.1.7) must hold for all y, *i.e.* $A_n(y)$ is bounded for all n and y. Hence, by Cantor's theorem[†], a_n and b_n are bounded for all n. Another application of Verblunsky's theorem completes the proof.

This argument is unsuitable for integrals, but it illustrates the part played by the everywhere dense set E. We do indeed establish an inequality similar to (4.1.6), though there is nothing to correspond to (4.1.7). Observe that, when $\bar{P}(x)$ and $\underline{P}(x)$ are equivalent, considerable simplification can be made in the above argument. A similar simplification is possible, in the same circumstances, in the proof of Theorem 5, and this is explained in § 5.

4.2. We first reduce the problem to one in which only real functions occur. To do this we divide (1.1.2) and (1.1.3) into their real and imaginary parts and consider each part separately. Write

$$\tag{4.2.1} \alpha(u) = \mathfrak{R}(2\pi)^{-\frac{1}{2}}\{\phi(u)+\phi(-u)\},$$

$$\tag{4.2.2} \beta(u) = -\mathfrak{I}(2\pi)^{-\frac{1}{2}}\{\phi(u)-\phi(-u)\},$$

$$\tag{4.2.3} \bar{H}(x) = \mathfrak{R}\bar{\Phi}(x), \quad \underline{H}(x) = \mathfrak{R}\underline{\Phi}(x),$$

$$\tag{4.2.4} J(x,\omega) = \int_0^{\omega}\left(1-\frac{u}{\omega}\right)\{\alpha(u)\cos xu+\beta(u)\sin xu\}\,du;$$

then, on taking real parts, (1.1.2) and (1.1.3) yield

$$\tag{4.2.5} \bar{H}(x) = \varlimsup_{\omega\to\infty} J(x,\omega), \quad \underline{H}(x) = \varliminf_{\omega\to\infty} J(x,\omega).$$

Plainly, if $\alpha(u)$ and $\beta(u)$ are defined appropriately, we should obtain the same expressions on taking the imaginary parts of (1.1.2) and (1.1.3). Theorem 5 is therefore equivalent to the following theorem.

[†] Hobson, 4, 658.

THEOREM 9. *Suppose that* $\alpha(u)$ *and* $\beta(u)$ *are real and integrable* L *in every finite range, and let* $\bar{H}(x)$, $\underline{H}(x)$ *be defined by* (4.2.4) *and* (4.2.5). *Suppose that* (i) $\bar{H}(x)$ *and* $\underline{H}(x)$ *are finite for all* x, (ii) $\bar{H}(x) = \underline{H}(x)$ *at all points of a set which is everywhere dense in* $(-\infty, \infty)$, (iii) $\bar{H}(x)$ *and* $\underline{H}(x)$ *are integrable* L *in every finite range, and* (iv)

$$(4.2.6) \qquad \int_0^x \bar{H}(t)\,dt = O(|x|^\kappa), \quad \int_0^x \underline{H}(t)\,dt = O(|x|^\kappa),$$

where $0 \leqslant \kappa < 1$, *as* $|x|$ *tends to infinity.*

Then $\bar{H}(x) \equiv \underline{H}(x) \equiv H(x)$ *and*

$$(4.2.7) \qquad H(x) = \int_0^\infty \{\alpha(u)\cos xu + \beta(u)\sin xu\}\,du \quad (C, 1),$$

$$(4.2.8) \qquad \alpha(u) = \frac{1}{\pi}\int_{-\infty}^\infty H(x)\cos ux\,dx \qquad (C, 1),$$

$$(4.2.9) \qquad \beta(u) = \frac{1}{\pi}\int_{-\infty}^\infty H(x)\sin ux\,dx \qquad (C, 1).$$

almost everywhere. Further, in any finite interval these three integrals are equi-summable $(C, 1)$ *with Fourier series.*

To prove this theorem we require the three following lemmas. The first of these is of a general character.

4.3. LEMMA 16. *Let* (a, b) *be any finite interval and suppose that, for* $a \leqslant x \leqslant b$, *the sequence* $\{f_n(x)\}$ *converges weakly with exponent* 1 *to the weak limit* $f(x)$. *Suppose further that*

$$\bar{f}(x) = \varlimsup_{n\to\infty} f_n(x), \quad \underline{f}(x) = \varliminf_{n\to\infty} f_n(x)$$

are finite for almost all x *and integrable* L *in* (a, b). *Then*

$$\int_a^b \underline{f}(x)\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b \bar{f}(x)\,dx.$$

Proof. It will be sufficient to prove the second inequality. Write

$$g_n(x) = \max_{m \geqslant n} f_m(x).$$

Then $\{g_n(x)\}$ is a monotone decreasing sequence which converges to $\bar{f}(x)$ almost everywhere in (a, b). Now $g_n(x)$ need not be integrable L in (a, b). However, by Egoroff's theorem†, there exists a set E whose measure is arbitrarily near $b-a$ and such that $g_n(x)$ converges uniformly to $\bar{f}(x)$ in E. Consequently, $g_n(x)$ is integrable L in E for all n and we have

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† Hobson, **4**, 144.

$$\lim_{n\to\infty}\int_E g_n(x)\,dx=\int_E \bar{f}(x)\,dx.$$

But, by hypothesis,

$$\int_E f(x)\,dx=\lim_{n\to\infty}\int_E f_n(x)\,dx\leqslant\lim_{n\to\infty}\int_E g_n(x)\,dx.$$

Hence
$$\int_E f(x)\,dx\leqslant\int_E \bar{f}(x)\,dx.$$

But $f(x)$ and $\bar{f}(x)$ are both integrable L in (a,b) and, since the measure of E can be made as near to $b-a$ as we please, we infer that

$$\int_a^b f(x)\,dx\leqslant\int_a^b \bar{f}(x)\,dx,$$

which is the desired result.

4.4. Write

$$\underline{H}_1(x)=\int_0^x \underline{H}(t)\,dt,\quad \overline{H}_1(x)=\int_0^x \overline{H}(t)\,dt. \tag{4.4.1}$$

Then, in virtue of (4.2.6), the integral

$$I^*(y,\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\overline{H}_1(x)\,\frac{d}{dx}\left\{\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\right\}dx \tag{4.4.2}$$

converges absolutely for all y. Integrating once by parts, we see that

$$I^*(y,\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\overline{H}(x)\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dx, \tag{4.4.3}$$

where the integral in the second member converges for all y. There are corresponding integrals in which $\overline{H}_1(x)$ and $\overline{H}(x)$ are replaced by $\underline{H}_1(x)$ and $\underline{H}(x)$. We denote these by $I_*(y,\omega)$.

The following inequality plays a fundamental part in the proof of Theorem 9.

Lemma 17. *Under the hypotheses of Theorem* 9,

$$I_*(y,\omega)\leqslant J(y,\omega)\leqslant I^*(y,\omega) \tag{4.4.4}$$

for all y.

Proof. Write

$$(4.4.5) \qquad \mathfrak{h}_*(x,\, y) = \tfrac{1}{2}\{\underline{H}(y+x)+\underline{H}(y-x)\},$$

$$(4.4.6) \qquad \mathfrak{h}^*(x,\, y) = \tfrac{1}{2}\{\overline{H}(y+x)+\overline{H}(y-x)\}.$$

Then, from (4.4.3), in virtue of (4.4.1),

$$(4.4.7) \qquad I^*(y,\, \omega) = \frac{1}{\pi}\int_0^\infty \mathfrak{h}^*(x,\, y)\, \frac{1-\cos\omega x}{\omega x^2}\, dx.$$

Similarly,

$$(4.4.8) \qquad I_*(y,\, \omega) = \frac{1}{\pi}\int_0^\infty \mathfrak{h}_*(x,\, y)\, \frac{1-\cos\omega x}{\omega x^2}\, dx.$$

Also, from (4.2.4) and (4.2.5),

$$\begin{aligned}(4.4.9) \quad \mathfrak{h}_*(x,\, y) &\leqslant \varliminf_{\omega\to\infty} \int_0^\omega \left(1-\frac{u}{\omega}\right)\{\alpha(u)\cos yu+\beta(u)\sin yu\}\, du \\ &\leqslant \varlimsup_{\omega\to\infty} \int_0^\omega \left(1-\frac{u}{\omega}\right)\{\alpha(u)\cos yu+\beta(u)\sin yu\}\, du \\ &\leqslant \mathfrak{h}^*(x,\, y).\end{aligned}$$

Let E be the everywhere dense set at which $\overline{H}(x) = \underline{H}(x)$. Suppose that y belongs to E. Then, putting $x = 0$ in (4.4.9), we get

$$(4.4.10) \qquad H(y) = \int_0^\infty \{\alpha(u)\cos yu+\beta(u)\sin yu\}\, du \quad (C,\, 1).$$

In Theorem 8 take

$$f(u) = \alpha(u)\cos yu+\beta(u)\sin yu.$$

The hypotheses of this theorem are satisfied in virtue of condition (i) and (4.4.10), and hence, for each y of E, there exists a function $F_y(x)$ such that

$$(4.4.11) \qquad F_y(x) = \int_0^\infty \{\alpha(u)\cos yu+\beta(u)\sin yu\}\cos xu\, du \quad (C,\, 2),$$

$$(4.4.12) \qquad \alpha(u)\cos yu+\beta(u)\sin yu = \frac{2}{\pi}\int_0^\infty F_y(x)\cos ux\, dx \quad (C,\, 2)$$

almost everywhere. Also, in every finite interval, the integrals in (4.4.11) and (4.4.12) are equi-summable $(C, 2)$ with Fourier series. Hence, from (4.4.12),

$$(4.4.13) \qquad J(y,\, \omega) = \frac{2}{\pi}\int_0^\infty F_y(x)\, \frac{1-\cos\omega x}{\omega x^2}\, dx \quad (C,\, 2)$$

for all ω and all y of E.

Since $(C, 2)$ sums must lie between the upper and lower $(C, 1)$ sums, it follows, from (4.4.9) and (4.4.11), that

$$\mathfrak{h}_*(x, y) \leqslant F_y(x) \leqslant \mathfrak{h}^*(x, y) \tag{4.4.14}$$

whenever $F_y(x)$ exists. Now $(1-\cos\omega x)/(\omega x^2)$ is positive, and therefore, from (4.4.8), (4.4.13), and (4.4.14),

$$I_*(y, \omega) \leqslant J(y, \omega) \leqslant I^*(y, \omega)$$

for all y of E. But, from (4.2.6) and (4.4.2), it is obvious that $I^*(y, \omega)$ is a continuous function of y for fixed ω. Similarly $I_*(y, \omega)$ is a continuous function of y. Hence, since we have proved (4.4.4) for all y of an everywhere dense set, it follows that it must hold for all y and the lemma is proved.

Lemma 18. *Under the hypotheses of Theorem* 9,

$$\left|\int_a^\omega (\omega-u)\,\alpha(u)\cos yu\,du\right| \leqslant M\eta^\kappa\omega^2, \tag{4.4.15}$$

$$\left|\int_a^\omega (\omega-u)\,\beta(u)\sin yu\,du\right| \leqslant M\eta^\kappa\omega^2, \tag{4.4.16}$$

where $0 \leqslant a \leqslant A$, $\omega \geqslant 1$, *and* $\eta = \max(|y|, 1)$.

Proof. Without loss of generality, we may suppose that $a = 0$. From (4.2.6) and (4.4.2),

$$|I^*(x, y)| \leqslant \left|\int_{y-1}^{y+1} \bar{H}_1(x)\frac{d}{dx}\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dx\right| + \int_{y+1}^\infty O(|x|^\kappa)\,O\left\{\frac{1}{(x-y)^2}\right\}dx$$

$$= O(|y|)^\kappa \int_{y-1}^{y+1}\left|\frac{d}{dx}\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\right|dx + O(|\eta|^\kappa) \leqslant M\eta^\kappa\omega,$$

and there is a corresponding inequality for $I_*(x, y)$. Therefore

$$|J(y, \omega)| \leqslant M|\eta|^\kappa\omega.$$

Since this holds for all y, positive and negative, the inequalities (4.4.15) and (4.4.16) follow at once.

Lemma 19. *Under the hypotheses of Theorem* 9,

$$\left|\int_a^\omega (\omega-u)\,\alpha(u)\sin yu\,du\right| \leqslant M|y|^\kappa\omega^2, \tag{4.4.17}$$

$$\left|\int_a^\omega (\omega-u)\,\beta(u)(1-\cos yu)\,du\right| \leqslant M|y|^\kappa\omega^2, \tag{4.4.18}$$

where $0 \leqslant a \leqslant A$, $|y| \geqslant 1$, $\omega \geqslant 1$.

Proof. Without loss of generality we may suppose that $a = 0$. From (4.4.12), for all y of the everywhere dense set E and almost all x,

$$\alpha(u) \sin 2yu + \beta(u)\, 2 \sin^2 yu = \frac{2}{\pi}\, 2 \sin yu \int_0^\infty F_y(x) \cos ux\, dx \quad (C,\, 2).$$

Hence, on account of the equi-summability of the integral on the right,

$$J_1(y,\, \omega) = \int_0^\omega (\omega - u)\{\alpha(u) \sin 2yu + \beta(u)\, 2 \sin^2 yu\}\, du$$

$$= \frac{2}{\pi} \int_0^\infty F_y(x)\, dx \int_0^\omega (\omega - u)\, 2 \sin yu \sin xu\, du.$$

The integral on the right converges in virtue of (4.2.6) and (4.4.13).

Write $$K_2(t) = \frac{\omega}{t} - \frac{\sin \omega t}{t^2}.$$

Then, since $$F_y(x) = F_y(-x),$$

we have $$J_1(y,\, \omega) = -\frac{2}{\pi} \int_{-\infty}^{\infty} F_y(x)\, K_2(x-y)\, dx.$$

We now employ (4.4.14). Taking account of the sign of $K_2(t)$, we get

$$J_1(y,\, \omega) \leqslant -\frac{2}{\pi} \int_y^\infty \mathfrak{h}_*(x,\, y)\, K_2(x-y)\, dx - \frac{2}{\pi} \int_{-\infty}^{y} \mathfrak{h}^*(x,\, y)\, K_2(x-y)\, dx.$$

Write

$$2\mathfrak{H}_*(x,\, y) = \underline{H}_1(y+x) - \underline{H}_1(y-x), \quad 2\mathfrak{H}^*(x,\, y) = \overline{H}_1(y+x) - \overline{H}_1(y-x).$$

Then

$$J_1(y,\, \omega) \leqslant \frac{2}{\pi} \int_y^\infty \mathfrak{H}_*(x,\, y) \frac{d}{dx} K_2(x-y)\, dx + \frac{2}{\pi} \int_{-\infty}^{y} \mathfrak{H}^*(x,\, y) \frac{d}{dx} K_2(x-y)\, dx$$

$$= M\, |y|^\kappa\, \omega^2,$$

in virtue of (4.2.6). There is a similar inequality with the sign reversed and $\mathfrak{H}^*$ and $\mathfrak{H}_*$ interchanged. From these two inequalities we deduce that, for all y of E,

$$|J_1(y,\, \omega)| \leqslant M\, |y|^\kappa\, \omega^2.$$

But, since $J_1(y, \omega)$ is a continuous function of y, the above inequality must hold for all y. The desired result follows as before.

4.5. *Proof of Theorem* 9. We first show that there is a sequence $\{\omega_\nu\}$ and a function $H(y)$ such that $J(y, \omega_\nu)$ tends weakly, with exponent 1, to $H(y)$ in every finite interval.

From (4.4.2) we get, on integrating by parts,

$$\left|\int_{-\infty}^{\infty} \overline{H}_1(x)\,\frac{d}{dx}\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dx\right| \leqslant \left|\int_{y-1}^{y+1} \overline{H}(x)\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dx\right| + M|y|^{\kappa}.$$

There is a corresponding inequality for $\underline{H}(x)$. Let

$$\mathfrak{H}(x) = |\overline{H}(x)| + |\underline{H}(x)|.$$

Then, from Lemma 17,

$$|J(y, \omega)| \leqslant \int_{y-1}^{y+1} \mathfrak{H}(x)\,\frac{2\sin^2\frac{1}{2}\omega(x-y)}{\omega(x-y)^2}\,dx + M|y|^{\kappa}$$

$$= 2\int_{-1}^{1} \mathfrak{H}(x+y)\,\frac{\sin^2\frac{1}{2}\omega x}{\omega x^2}\,dx + M|y|^{\kappa}.$$

Let $a \leqslant y \leqslant b$ be any finite interval. Then

$$\text{(4.5.1)} \qquad \int_a^b |J(y, \omega)|\,dy \leqslant 2\int_{-1}^{1} \frac{\sin^2\frac{1}{2}\omega x}{\omega x^2}\,dx \int_a^b \mathfrak{H}(x+y)\,dy + M$$

$$\leqslant 2\int_{-1}^{1} \frac{\sin^2\frac{1}{2}\omega x}{\omega x^2}\,dx \int_{a-1}^{b+1} \mathfrak{H}(y)\,dy + M$$

$$\leqslant M,$$

where M depends on (a, b).

Again, if e is any measurable set,

$$\text{(4.5.2)} \qquad \int_e |J(y, \omega)|\,dy \leqslant 2\int_{-1}^{1} \frac{\sin^2\frac{1}{2}\omega x}{\omega x^2}\,dx \int_e \mathfrak{H}(x+y)\,dy + M\int_e |y|^{\kappa}\,dy,$$

and this can be made less than any positive ϵ by taking the measure of e sufficiently small.

Let $\{a_m\}$ be an increasing sequence of positive numbers tending to infinity. Consider the interval $-a_1 \leqslant y \leqslant a_1$. By a well known theorem[†], since $J(y, \omega)$ satisfies (4.5.1) and (4.5. 2), there exists a sequence $\{\omega_{1n}\}$ and an integrable function $H(y)$ such that the sequence $\{J(y, \omega_{1n})\}$ converges weakly† with exponent 1 to $H(y)$ in $(-a_1, a_1)$. Now let $-a_2 \leqslant y \leqslant a_2$. Then, from the sequence $\{\omega_{1n}\}$, we can choose a subsequence $\{\omega_{2n}\}$ such that $J(y, \omega_{2n})$ converges weakly in $(-a_2, a_2)$. Since $\{\omega_{2n}\}$ belongs to $\{\omega_{1n}\}$, the weak limit of $J(y, \omega_{2n})$ is identical with $H(y)$ in $(-a_1, a_1)$ and so we can denote it by $H(y)$ throughout $(-a_2, a_2)$. Take any finite interval $(-a_m, a_m)$. Then, proceeding in the above way, we can show that there exists a sequence $\{\omega_{m,\, n}\}$ such that, in $(-a_m, a_m)$, the sequence $\{J(y, \omega_{m,n})\}$ converges weakly with exponent 1 to a function $H(y)$, where $H(y)$ is independent of m.

[†] *Cf.* Hobson, **4**, 168.

Let $\sigma_n = \omega_{nn}$. Then, by what we have just proved, the sequence of functions $\{J(y, \sigma_n)\}$ converges weakly with exponent 1 to the weak limit $H(y)$ in any finite interval.

We can now establish the inversion formulae (4.2.8) and (4.2.9), but we give only the proof of (4.2.9). The proof of (4.2.8) is similar, but rather simpler than that of (4.2.9). We therefore leave it to the reader.

By weak convergence‡,

$$(4.5.3) \quad \int_{-Y}^{Y} \left(1-\frac{|y|}{Y}\right) H(y) \sin ty\, dy$$

$$= \lim_{n\to\infty} \int_{-Y}^{Y} \left(1-\frac{|y|}{Y}\right) J(y, \sigma_n) \sin ty\, dy$$

$$= \lim_{n\to\infty} \int_0^{\sigma_n} \left(1-\frac{u}{\sigma_n}\right) du$$

$$\times \int_{-Y}^{Y} \left(1-\frac{|y|}{Y}\right) \{\alpha(u) \cos uy + \beta(u) \sin uy\} \sin ty\, dy$$

$$= \lim_{n\to\infty} \int_0^{\sigma_n} \left(1-\frac{u}{\sigma_n}\right) \beta(u)\, du \int_0^{Y} \left(1-\frac{y}{Y}\right) 2 \sin uy \sin ty\, dy$$

$$= \lim_{n\to\infty} \int_0^{\sigma_n} \left(1-\frac{u}{\sigma_n}\right) \beta(u) \left\{\frac{1-\cos Y(u-t)}{Y(u-t)^2} - \frac{1-\cos Y(u+t)}{Y(u+t)^2}\right\} du.$$

It is important to notice that we already know of the existence of the limit on the right. This fact is employed later.

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† *Cf.* Banach, **1**, 136; Hobson, **4**, 431.

‡ *Cf.* Banach, **1**, 7; Hobson, **4**, 431.

Let

$$K_3(u, t;\ Y) = \frac{1-\cos Y(u-t)}{(u-t)^2} - \frac{1-\cos Y(u+t)}{(u+t)^2},$$

$$k_1(u, t) = (u-t)^{-2}-(u+t)^{-2},$$

$$k_2(u, t) = (u-t)^{-2}+(u+t)^{-2}.$$

Then

$$K_3(u, t;\ Y) = (1-\cos Yt)\, k_1(u, t)+\cos Yt(1-\cos Yu)\, k_1(u, t) \\ -\sin Yt \sin Yu\, k_2(u, t).$$

Let
$$I_1 = \int_{t+1}^{\sigma_n} \left(1-\frac{u}{\sigma_n}\right) \beta(u)\, k_1(u, t)\, du,$$

$$I_2 = \int_{t+1}^{\sigma_n} \left(1-\frac{u}{\sigma_n}\right) \beta(u)(1-\cos Yu)\, k_1(u, t)\, du,$$

$$I_3 = \int_{t+1}^{\sigma_n} \left(1-\frac{u}{\sigma_n}\right) \beta(u) \sin Yu\, k_2(u, t)\, du.$$

Further, let

$$B_1(\omega, Y) = \int_{t+1}^{\omega} \beta(u)(1-\cos Yu)\, du, \quad B_2(\omega, Y) = \int_{t+1}^{\omega} B_1(u, Y)\, du,$$

$$\mathfrak{B}_1(\omega, Y) = \int_{t+1}^{\omega} \beta(u) \sin Yu\, du, \quad \mathfrak{B}_2(\omega, Y) = \int_{t+1}^{\omega} \mathfrak{B}_1(u, Y)\, du.$$

Then, by Lemmas 18 and 19,

$$|B_2(\omega, Y)| \leqslant MY^{\kappa} \omega^2, \quad |\mathfrak{B}_2(\omega, Y)| \leqslant MY^{\kappa} \omega^2,$$

provided that $0 \leqslant t \leqslant T$.

Apply Lemma 2 to the integral I_2, taking $k=1$, $m=1$, $a=t+1$,

$$A(u, Y) = \beta(u)(1-\cos Yu), \quad b(u, t) = k_1(u, t).$$

Then

$$\lim_{n\to\infty} I_2 = \int_{t+1}^{\infty} B_2(u, Y) \left(\frac{d}{du}\right)^2 k_1(u, t)\, du \tag{4.5.4}$$

uniformly in t, provided that $0 \leqslant t \leqslant T$. Therefore

$$\left| \lim_{n\to\infty} I_2 \right| \leqslant MY^{\kappa} \int_{t+1}^{\infty} u^2\, O\{(u-t)^{-4}\}\, du \leqslant MY^{\kappa}. \tag{4.5.5}$$

Similarly, applying Lemma 2 to I_3 with $k=1$, $m=1$, $a=t+1$,

$$A(u,\,Y)=\beta(u)\sin Yu,\quad b(u,\,t)=k_2(u,\,t),$$

we get

$$(4.5.6)\qquad \lim_{n\to\infty} I_3=\int_{t+1}^{\infty}\mathfrak{B}_2(u,\,Y)\left(\frac{d}{du}\right)^2 k_2(u,\,t)\,du$$

uniformly in t, provided that $0\leqslant t\leqslant T$. Hence

$$(4.5.7)\qquad \left|\lim_{n\to\infty} I_3\right|\leqslant MY^{\kappa}\int_{t+1}^{\infty}u^2\,O\{(u-t)^{-4}\}\,du\leqslant MY^{\kappa}.$$

We are not able to treat I_1 in a similar manner, and it is to dispose of this term that we use our information about the existence of the limit in the second member of (4.5.3). From (4.5.3),

$$(4.5.8)\qquad \int_{-Y}^{Y}(Y-|y|)\,H(y)\sin ty\,dy=\int_{0}^{t+1}\beta(u)\,K_3(u,\,t;\;Y)\,du$$
$$+\lim_{n\to\infty}\{(1-\cos Yt)\,I_1+\cos Yt\,I_2-\sin Yt\,I_3\}.$$

Now the limit on the right is known to exist and we have proved in (4.5.4) and (4.5.6) that the limits of I_2 and I_3 exist. Hence the limit of the remaining term must also exist, *i.e.*

$$\lim_{n\to\infty} I_1=\chi(t)$$

exists for all t.

Suppose, in (4.5.8), that $0\leqslant t\leqslant T$, $1\leqslant Y\leqslant\pi/\delta$, where $0<\delta<1$. Then, by (4.5.5) and (4.5.7), we infer that

$$(4.5.9)\qquad |(1-\cos Yt)\,\chi(t)|\leqslant M$$

for $0\leqslant t\leqslant T$ and $1\leqslant Y\leqslant\pi/\delta$. This implies

$$(4.5.10)\qquad |\chi(t)|\leqslant 2M$$

for $0<\delta\leqslant t\leqslant T$. For suppose, if possible, that there is a point t_0 such that $|\chi(t_0)|>2M$. Since $t_0>\delta$ and $1\leqslant Y\leqslant\pi/\delta$, it is plain that we can choose Y_0, $1\leqslant Y_0\leqslant\pi/\delta$, so that $\cos Y_0t_0\leqslant\frac{1}{2}$. This means that $|(1-\cos Y_0t_0)\,\chi(t_0)|>M$, which contradicts (4.5.9). Since $\chi(t)$ is independent of Y, (4.5.10) asserts that $\chi(t)$ is a bounded function of t for $\delta \leqslant t \leqslant T$.

From (4.5.5), (4.5.7), (4.5.8), and (4.5.10), we have

$$\int_{-Y}^{Y} \left(1-\frac{|y|}{Y}\right) H(y) \sin ty\, dy = Y^{-1} \int_0^{t+1} \beta(u)\, K_3(u, t;\ Y)\, du + O(Y^{\kappa-1})$$

$$= \int_{t-1}^{t+1} \beta(u)\, \frac{1-\cos Y(u-t)}{Y(u-t)^2}\, du + O(Y^{\kappa-1})$$

uniformly in t for $\delta \leqslant t \leqslant T$. Hence, by Lemma 1,

$$\beta(t) = \lim_{Y\to\infty} \frac{1}{\pi} \int_{-Y}^{Y} \left(1-\frac{|y|}{Y}\right) H(y) \sin ty\, dy \tag{4.5.11}$$

for almost all t in (δ, T), and the second member is equi-summable $(C, 1)$ with a Fourier series. Since δ is arbitrary, the above limit exists for almost all t. We have not, however, proved that the integral in (4.5.11) is equi-summable $(C, 1)$ in the neighbourhood of $t = 0$, and the above argument does not seem to establish this. We complete this step in the following way.

Consider

$$\int_0^{\omega} (\omega-u)(u-\delta)\beta(u)\, du \tag{4.5.12}$$

$$= \int_0^{\delta} (\omega-u)(u-\delta)\beta(u)\, du + \int_{\delta}^{\omega} (\omega-u)(u-\delta)\beta(u)\, du$$

$$= I_4 + I_5.$$

Plainly
$$|I_4| \leqslant \omega^2 \int_0^{\delta} |\beta(u)|\, du \leqslant M\omega^2.$$

Substitute for $\beta(u)$ from (4.5.11) into I_5. On account of equi-summability for $\delta \leqslant u \leqslant \omega$, we get

$$I_5 = \lim_{Y\to\infty} \frac{1}{\pi} \int_{-Y}^{Y} \left(1-\frac{|y|}{Y}\right) H(y) \int_{\delta}^{\omega} (\omega-u)(u-\delta) \sin yu\, du. \tag{4.5.13}$$

Now

$$\int_{\delta}^{\omega} (\omega-u)(u-\delta) \sin yu\, du = -(\omega-\delta)\, \frac{\sin y\omega - \sin y\delta}{y^2} - 2\, \frac{\cos y\omega - \cos y\delta}{y^3}$$

$$= K_4(y, \omega).$$

Plainly

$$(4.5.14) \qquad |K_4(y, \omega| \leqslant \omega^3, \quad |K_4(y, \omega)| \leqslant M \frac{\omega}{y^2},$$

$$(4.5.15) \qquad \left| \frac{d}{dx} K_4(y, \omega) \right| \leqslant M \frac{\omega^2}{y^2},$$

provided that $|y| \geqslant 1$ and $\omega \geqslant \delta$. Now

$$\begin{aligned} I_6 &= \int_1^Y (Y-y) H(y) K_4(y, \omega)\, dy \\ &= -(Y-1) H_1(1) K_4(1, \omega) + \int_1^Y H_1(y) \frac{d}{dy} \{(Y-y) K_4(y, \omega)\}\, dy. \end{aligned}$$

Hence, by (4.2.6) and (4.5.15),

$$|I_6| \leqslant MY\omega^2 + MY\omega^2 \int_1^Y y^{\kappa-2}\, dy \leqslant MY\omega^2.$$

Similarly

$$\left| \int_{-Y}^{-1} (Y-|y|) H(y) K_4(y, \omega)\, dy \right| \leqslant MY\omega^2.$$

Also, from (4.5.14),

$$\left| \int_{-1}^{1} (Y-|y|) H(y) K_4(y, \omega)\, dy \right| \leqslant MY\omega^3 \int_{-1}^{1} |H(y)|\, dy \leqslant MY\omega^3.$$

Therefore, from (4.5.13),

$$|I_5| \leqslant M\omega^3.$$

Consequently we have

$$\left| \int_0^\omega (\omega-u)(u-\delta) \beta(u)\, du \right| \leqslant M\omega^3.$$

Let a be some positive number such that $0 \leqslant a \leqslant A$. Then, plainly,

$$(4.5.16) \qquad \left| \int_a^\omega (\omega-u)(u-\delta) \beta(u)\, du \right| \leqslant M\omega^3.$$

We now return to the integral I_1 and, using (4.5.16), we are now able to prove that $\chi(t)$ is bounded in the neighbourhood of $t=0$. Let

$$B_1(\omega)=\int_{t+1}^{\omega}(u-\delta)\beta(u)\,du, \qquad B_2(\omega)=\int_{t+1}^{\omega}B_1(u)\,du.$$

Then, by (4.5.16),

$$|B_2(\omega)|\leqslant M\omega^3, \tag{4.5.17}$$

provided that $0\leqslant t\leqslant T$. We may plainly suppose that $t\geqslant 0$. Apply Lemma 2 to I_1 with $k=1$, $m=2$, $a=t+1\geqslant 1$,

$$A(u)=(u-\delta)\beta(u), \quad b(u,t)=\frac{k_1(u,t)}{u-\delta}.$$

Then
$$\chi(t)=\lim_{n\to\infty}I_1=\int_{t+1}^{\infty}B_2(u)\left(\frac{d}{du}\right)^2\left\{\frac{k_1(u,t)}{u-\delta}\right\}du,$$

so that, by (4.5.17),

$$|\chi(t)|\leqslant M\int_{t+1}^{\infty}u^3\,O(u^{-5})\,du\leqslant M$$

for $0\leqslant t\leqslant T$, as desired. This proves (4.2.9) completely.

The proof of (4.2.8) is similar except that the term corresponding to I_1 can be much more easily dealt with by using the special case of (4.4.15) when $y=0$.

We have now only to prove (4.2.7). Substituting for $\alpha(u)$ and $\beta(u)$ from (4.2.8) and (4.2.9) into $J(x,\omega)$, we obtain

$$J(x,\omega)=\lim_{Y\to\infty}\frac{1}{\pi}\int_0^{\omega}\left(1-\frac{u}{\omega}\right)du\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)H(y)\cos u(x-y)\,dy \tag{4.5.18}$$

$$=\lim_{Y\to\infty}\frac{1}{\pi}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)H(y)\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dy.$$

By Lemma 16, if (a,b) is any finite interval,

$$\int_a^b \underline{H}(y)\,dy\leqslant\int_a^b H(y)\,dy\leqslant\int_a^b \overline{H}(y)\,dy.$$

Hence, by (4.2.6),

$$\int_0^y H(t)\,dt=O(|y|^{\kappa}) \quad (0\leqslant\kappa<1).$$

Therefore the last integral in (4.5.18) converges, and

$$J(x,\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}H(y)\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dy$$

$$=-\frac{1}{\pi}\int_{-\infty}^{\infty}H_1(y)\,\frac{d}{dy}\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dy$$

$$=-\frac{1}{\pi}\int_{x-\delta}^{x+\delta}H_1(y)\,\frac{d}{dy}\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dy+o(1),$$

by the Riemann-Lebesgue theorem. Hence

$$J(x,\omega)=\frac{1}{\pi}\int_{x-\delta}^{x+\delta}H(y)\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dy+o(1)$$

uniformly in x for $0\leqslant x\leqslant X$. The desired result follows by Lemma 1, and the proof of Theorem 9 is complete.

5. *Proofs of Theorems* 2, 3, *and* 4.

5.1. To deduce Theorem 2 from Theorem 5, we have only to prove that, under the hypotheses of Theorem 2, (1.2.2) converges strongly with exponent p, $1<p<\infty$, to $\Phi(x)$. Now, by Theorem 5,

$$(5.1.1)\quad (2\pi)^{-\frac{1}{2}}\int_{-\omega}^{\omega}\phi(u)\,e^{ixu}\,du$$

$$=\frac{1}{2\pi}\lim_{Y\to\infty}\int_{-Y}^{Y}\left(1-\frac{|y|}{Y}\right)\Phi(y)\,dy\int_{-\omega}^{\omega}e^{iu(x-y)}\,du$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\Phi(y)\,\frac{\sin\omega(x-y)}{x-y}\,dy.$$

Since $\Phi(y)$ belongs to $L^p\,(-\infty,\infty)$, this integral converges for all ω. The desired conclusion now follows from known results[†].

When $p=1,\infty$, this result is no longer true. However, from Theorem 5, it follows that

$$(2\pi)^{-\frac{1}{2}}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)\phi(u)\,e^{ixu}\,du=\frac{1}{\pi}\int_{-\infty}^{\infty}\Phi(y)\,\frac{1-\cos\omega(x-y)}{\omega(x-y)^2}\,dy,$$

[†] Hille and Tamarkin, 3, 770. *Cf.* Offord, 7, 260.

and this expression converges strongly[†] with exponent p, $1 \leqslant p \leqslant \infty$, to $\Phi(x)$.

5.2. Theorem 3 and *a fortiori* Theorems 1 and 2 can be proved more easily than Theorem 5. Indeed, a very considerable simplification can be made when $\bar{\Phi}(x)$ and $\underline{\Phi}(x)$ are equivalent. For, if in the proof of Theorem 9 we assume that $\bar{H}(x)$ and $\underline{H}(x)$ are equivalent, then, instead of the inequality (4.4.11), we obtain the equality

$$J(y, \omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} H(x+y) \frac{1-\cos \omega x}{\omega x^2} dx. \tag{5.2.1}$$

We can now proceed at once to the proof of (4.2.7) and we do not need the theory of weak convergence.

5.3. Theorem 4 may be proved on the same lines as Theorem 3. The first step is to establish (5.2.1). In proving this we employ Theorem 8, which was proved only for Lebesgue integrals. Now an examination of the proof of Theorem 8 shows that Lemmas 2, 3, 5–7, 10–15 remain true when the functions are integrable in the sense of Denjoy-Perron and the proofs require only slight alterations, if any. In Lemma 4, it is no longer necessarily true that $\chi(x)$ is integrable, as the fact that $\bar{F}(x)$ and $\underline{F}(x)$ are integrable in the sense of Denjoy-Perron does not imply that $\chi(x)$ is integrable in this sense. However, in the present case, $\bar{F}(x) = \underline{F}(x)$ for all x, and it is easy to see that $D^3 G(x) = F(x)$ for all x. Hence $D^3 G(x)$ is integrable in the sense of Denjoy-Perron.

Lemma 8 has been proved by Verblunsky for functions integrable in the sense of Denjoy-Perron and so Lemma 9 follows as before. The remaining changes required in the proof of Theorem 9 present no difficulty.

Note that our principal weapon, integration by parts, works just as well with Denjoy-Perron integrals as it does with Lebesgue integrals. Also, observe that $f_2(u)$ and $G(x)$ are integrable L and therefore (3.2.4) is a Lebesgue integral. This means that the double integrals which occur in Lemmas 10, 11, 13, and 14 are Lebesgue integrals, and so inversion presents no difficulty.

6. *Conditions on $\phi(u)$ and $f(u)$.*

6.1. It has already been remarked that the hypothesis (3.1.3) is employed once only in the proof of Theorem 8, namely in Lemma 6. For all the other lemmas (3.2.2) is sufficient and this is, of course, implied by (3.1.1) and (3.1.2). In some respects it is rather unfortunate that we were unable to prove Theorem 8 under (3.1.1) and (3.1.2) alone. As a consequence of our failure, it is necessary, when dealing with the general trigonometric integral in Theorem 9, to assume that the integral is summable $(C, 1)$, not merely at a point, but at points of an everywhere dense set. Again, a function $f(u)$ may satisfy (3.1.1), (3.1.2), and

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[†] *Cf.* Offord, 7, 259.

$$\int_0^\infty \frac{|f(u)|}{u^2}\,du < \infty, \tag{6.1.1}$$

but not (3.1.3). Yet the conclusions of Theorem 8 hold under these conditions. This means that we must consider alternatives to (3.1.3). The following hypothesis is more general than (6.1.1).

$$\int_0^\omega |f(u)|\,du = o(\omega^2). \tag{6.1.2}$$

For, if (6.1.1) is satisfied, then

$$\omega^{-2}\int_K^\omega |f(u)|\,du \leqslant \int_K^\omega \frac{|f(u)|}{u^2}\,du < \epsilon$$

for large K.

Another suitable hypothesis is

$$\int_0^\omega f(u)\,du = o(\omega). \tag{6.1.3}$$

These conditions have a further interest. It will be shown later that, if $f(u)$ satisfies one of the conditions (3.1.3), (6.1.2), or (6.1.3), then it is possible to replace the condition that the trigonometric integral should be finite $(C, 1)$ for all x by the condition that it should be finite (P) for all x. This is given in Theorem 11. At the moment we shall just show that the conclusions of Lemma 6, and so of Theorem 8, remain true when (3.1.3) is replaced by (6.1.2) or (6.1.3).

It is interesting to compare conditions (3.1.3), (6.1.2), and (6.1.3) with those considered by other writers. Pollard (**9**) assumes that $f(u)$ satisfies (6.1.1), but his results are restricted to convergent integrals. Cossar† has extended Pollard's results to the case when the trigonometric integral is finite (P) for all x. He retains, however, condition (6.1.1).

Zygmund (**18**) employs the even stronger hypothesis

$$\int_\omega^{\omega+1} |f(u)|\,du = o(1).$$

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† J. Cossar, *Cambridge Ph.D. thesis* (1933), unpublished.

However, he introduces sets of uniqueness of trigonometric integrals summable in the Poisson sense and consequently requires a strong condition. His results are in some respects best possible.

Verblunsky (**16**) deals with Poisson summability and employs (6.1.1). He also introduces an alternative condition

$$(6.1.5) \qquad \int_{\omega}^{\omega+1} |f(u)|^p \, du = o(\omega^p) \quad (1 < p \leqslant 2).$$

It is easy to see that (6.1.2) will be satisfied whenever (6.1.5) is satisfied. For, if $f(u)$ satisfies (6.1.5), then, by Hölder's inequality,

$$(6.1.6) \qquad \int_{\omega}^{\omega+1} |f(u)| \, du = o(\omega),$$

and so
$$\int_0^{\omega} |f(u)| \, du = \sum_{n \leqslant \omega+1} \int_{n-1}^{n} |f(u)| \, du = o(\omega^2).$$

Indeed (6.1.5) is a sufficient condition for $1 \leqslant p < \infty$.

Some writers have obtained results in which they employ less stringent conditions on $|f(u)|$ at the expense of introducing a supplementary condition. Thus Verblunsky (**16**, 16) assumes that

$$(6.1.7) \qquad \int_1^{\infty} \left\{ \frac{|f(u)|}{u^2} \right\}^p du < \infty \quad (1 < p \leqslant 2),$$

but then requires

$$\lim_{\eta \to 0} \int_1^{\infty} e^{-\eta u} \frac{f(u)}{u^2} \cos xu \, du$$

to be continuous for $-\infty < x < \infty$. Condition (6.1.7) is insufficient by itself since it is satisfied when $f(u) = u \sin u$.

6.2. It is convenient to replace (6.1.2) and (6.1.3) by the condition

$$(6.2.1) \qquad \int_1^{\omega} f(u) \sin xu \, du = o(\omega^2),$$

uniformly in x, for $0 \leqslant x \leqslant X$ and all finite X. Obviously (6.2.1) is satisfied whenever either (6.1.2) or (6.1.3) is satisfied.

LEMMA 20. *Suppose that* $f(u)$ *is integrable* L *in every finite range and satisfies* (6.2.1) *uniformly in* x *for* $0 \leqslant x \leqslant X$ *and all finite* X. *Then*

$$G(x) = -\int_1^\infty \frac{f(u)}{u^3} \sin xu\, du \tag{6.2.2}$$

exists for all x *and*

$$\lim_{h \to 0} \{G(x+2h)+G(x-2h)-2G(x)\}/4h = 0$$

for all x.

Proof. Write

$$\Psi(x, \omega) = \int_1^\omega f(u) \sin xu\, du.$$

Then
$$\int_1^\omega \frac{f(u)}{u^3} \sin xu\, du = \frac{\Psi(x, \omega)}{\omega^3} + 3\int_1^\omega \frac{\Psi(x, u)}{u^4}\, du.$$

Therefore, by (6.2.1), (6.2.2) converges for all x. Hence

$$\begin{aligned}\Delta^2 G(x) &= G(x+2h)+G(x-2h)-2G(x)\\ &= -\int_1^{1/h} \frac{f(u)}{u^3} \sin xu \sin^2 hu\, du - \Delta^2 \int_{1/h}^\infty \frac{f(u)}{u^3} \sin xu\, du\\ &= I_1 + I_2.\end{aligned}$$

Choose $\omega_0(\epsilon)$ so that $|\Psi| \leqslant \epsilon\omega^2$ for $\omega \geqslant \omega_0$, $0 \leqslant x \leqslant X$. Then

$$I_1 = -\Psi\left(x, \frac{1}{h}\right) h^3 \sin^2 1 + \left\{\int_1^{\omega_0} + \int_{\omega_0}^{1/h}\right\} \Psi(x, u) \frac{d}{du}\left(\frac{\sin^2 hu}{u^3}\right) du.$$

Therefore, using (6.2.1),

$$\begin{aligned}|I_1| &\leqslant o(h) + M\int_1^{\omega_0} \left|\frac{d}{du}\left(\frac{\sin^2 hu}{u^3}\right)\right| du + \epsilon \int_{\omega_0}^{1/h} u^2 \left|\frac{d}{du}\left(\frac{\sin^2 hu}{u^3}\right)\right| du\\ &\leqslant o(h) + Mh^3 \int_h^{\omega_0 h} \left|\frac{d}{d\theta}\left(\frac{\sin^2\theta}{\theta^3}\right)\right| d\theta + \epsilon h \int_0^1 \theta^2 \left|\frac{d}{d\theta}\left(\frac{\sin^2\theta}{\theta^3}\right)\right| d\theta\\ &= o(h) + \epsilon' h.\end{aligned}$$

Hence
$$\overline{\lim_{h \to 0}} \frac{|I_1|}{h} \leqslant \epsilon'.$$

Again,

$$I_2 = -\Delta^2 \lim_{\omega\to\infty} \left\{ \frac{\Psi(x, \omega)}{\omega^3} - h^3 \Psi\left(x, \frac{1}{h}\right) + 3 \int_{1/h}^{\omega} \frac{\Psi(x, u)}{u^4}\, du \right\}$$

$$= \Delta^2 \left\{ h^3 \Psi\left(x, \frac{1}{h}\right) - 3 \int_{1/h}^{\infty} u^{-4} \Psi(x, u)\, du \right\}$$

$$= h^3 \Delta^2 \Psi\left(x, \frac{1}{h}\right) - 3 \int_{1/h}^{\infty} u^{-4} \Delta^2 \Psi(u, u)\, du$$

$$= o(h),$$

since (6.2.1) holds uniformly in x. This completes the proof of the lemma.

We have therefore proved the following theorem.

THEOREM 10. *Suppose that $f(u)$ satisfies the hypotheses of Lemma* 20 *and that the functions $\overline{F}(x)$ and $\underline{F}(x)$, defined by* (3.1.1) *and* (3.1.2), *are finite for all x and integrable L in every finite range. Then the conclusions of Theorem* 8 *hold.*

7. *The fundamental theorem for Poisson summability.*

7.1. We shall now prove Theorem 6. We begin by reducing it to the case in which all the functions are real. Let $\alpha(u)$ and $\beta(u)$ be defined by (4.2.1) and (4.2.2). Then

$$\tfrac{1}{2}\mathfrak{R}\{\Phi(x, \sigma) + \Phi(-x, \sigma)\} = \int_0^{\infty} e^{-\sigma u} \alpha(u) \cos xu\, du \quad (C, 1), \tag{7.1.1}$$

$$\tfrac{1}{2}\mathfrak{R}\{\Phi(x, \sigma) - \Phi(-x, \sigma)\} = \int_0^{\infty} e^{-\sigma u} \beta(u) \sin xu\, du \quad (C, 1). \tag{7.1.2}$$

We obtain two similar expressions on taking imaginary parts. Suppose that $\phi(u)$ satisfies any one of the conditions (1.3.1), (1.3.2), or (1.3.3). Then $\alpha(u)$ and $\beta(u)$ will satisfy the corresponding condition out of (3.1.3), (6.1.3), and (6.1.2).

We need consider only one of these integrals. It is convenient to have a condition on $\alpha(u)$ which tells us something about the convergence or summability of the second member of (7.1.1) for $\sigma > 0$. When $\alpha(u)$ satisfies (3.1.3), it follows at once that this integral is summable $(C, 1)$ for $\sigma > 0$ and all x. The hypothesis (6.2.1) employed in Lemma 19 does not tell us anything about the existence of this integral; but the hypotheses (6.1.3) and (6.1.2) give respectively convergence or absolute convergence.

Observe that we must not restrict our theorems to the case in which

$$\int_0^\infty e^{-\sigma u} |\alpha(u)| \, du < \infty$$

for all $\sigma > 0$. For this is not satisfied when

$$\alpha(u) = e^{-\delta u} \frac{d}{du} \{\exp(ie^u)\} \quad (0 < \delta < \tfrac{1}{2})$$

and $0 < \sigma < \frac{1}{2}$. Yet in this case

$$\int_0^\infty \alpha(u) \cos xu \, du$$

converges for all x.

Theorem 6 is equivalent to the following theorem.

THEOREM 11. *Suppose that* $f(u)$ *satisfies any one of the conditions* (3.1.3), (6.1.2), *or* (6.1.3). *Let*

$$F(\sigma, x) = \int_0^\infty e^{-\sigma u} f(u) \cos xu \, du \quad (C, 1) \tag{7.1.3}$$

for $\sigma > 0$ *and suppose that*

$$\bar{F}(x) = \overline{\lim_{\sigma \to \infty}} F(\sigma, x), \qquad \underline{F}(x) = \underline{\lim_{\sigma \to \infty}} F(\sigma, x) \tag{7.1.4}$$

are finite for all x. *Further suppose that, for all* x, $\underline{F}(x) \geqslant \psi(x)$, *where* $\psi(x)$ *is integrable* L *in every finite range. Then the conclusions of Theorem* 8 *hold.*

We give only the proof of Theorem 11 in the case where $f(u)$ satisfies (3.1.3). The proofs for the other cases can be obtained by making a few trivial changes in the proof given.

The proof of this theorem is rather more difficult than that of Theorem 8. The hypotheses of Lemmas 4 and 5 are not now satisfied and our main task is to establish analogues of these lemmas.

We first prove a preliminary lemma, Lemma 21, and use this in the proof of Lemma 22, which corresponds to Lemma 5. This leads at once to Lemma 23, the analogue of Lemma 7. Lemma 24 takes the place of Lemma 4. We employ Lemma 22 in the proof of Lemma 24. As one might expect, Lemma 24 is decidedly harder to prove than Lemma 4. Combining Lemmas 23, 24, and 25, we deduce Lemma 26, which asserts that the conclusions of Lemma 9 hold under the new hypotheses. When $f(u)$ satisfies (3.1.3), Theorem 11 is a consequence of Lemma 26 and Lemmas 11 and 15, since the hypotheses of the last two are satisfied in virtue of (3.1.3) and Lemma 26.

7.2. We make the following preliminary reductions. As in the proof of Theorem 8, we may suppose that $f(u)$ is zero for $0 \leqslant u < 1$. Further, without loss of generality, we may take $L = 0$. For consider the function

$$f(u) = \begin{cases} L & (1 \leqslant u \leqslant 2), \\ 0 & \text{otherwise}. \end{cases}$$

Plainly, if $\omega \geqslant 2$,

$$\int_0^\omega (\omega - u) f(u)\, du = L\omega - \tfrac{3}{2}L.$$

Hence, if we subtract this function from the original one, we reduce the problem to one in which L is zero.

In Lemma 21 we establish, under the new hypotheses, the existence of the function $g(x)$ which occurred first in (3.3.1). This is no longer trivial, as it was in Lemma 5.

LEMMA 21. *Suppose that* $f(u)$ *satisfies* (3.1.3) *and let* $F(\sigma, x)$, $\bar{F}(x)$, *and* $\underline{F}(x)$ *be defined by* (7.1.3) *and* (7.1.4). *Suppose that* $\bar{F}(x)$ *and* $\underline{F}(x)$ *are finite for all* x. *Then*

$$g(x) = -\lim_{\omega \to \infty} \int_1^\omega \left(1 - \frac{u}{\omega}\right) \frac{f(u)}{u^2} \cos xu\, du \tag{7.2.1}$$

exists for all x.

Proof. In the first place, since $f(u)$ satisfies (3.1.3), the integral in (7.1.3) is uniformly summable $(C, 1)$ for $\sigma \geqslant \sigma_0 > 0$ and $|x| \leqslant X$. To see this, apply Lemma 2 to this integral, taking $k = 1$, $m = 0$, $a = 0$,

$$A(u) = f(u), \quad b(u, x) = e^{-\sigma u} \cos xu.$$

The hypotheses of the lemma are satisfied in virtue of (3.1.3) and we see that

$$F(\sigma, x) = \int_0^\infty f_2(u) \left(\frac{d}{du}\right)^2 \{e^{-\sigma u} \cos xu\}\, du.$$

This integral converges uniformly for $\sigma \geqslant \sigma_0 > 0$ and $|x| \leqslant X$.

In the same way it can be shown that the integral

$$I = \int_1^\infty \frac{f(u)}{u^2} e^{-\sigma u} \cos xu\, du \tag{7.2.2}$$

is also uniformly summable $(C, 1)$ in the same region.

We now show that

$$g^{(1)}(x) = -\lim_{\sigma \to 0} I \tag{7.2.3}$$

exists for all x. Write

$$F_1(\sigma, x) = \int_\sigma^1 F(\rho, x)\, d\rho,$$

$$F_2(\sigma, x) = \int_\sigma^1 F_1(\rho, x)\, d\rho.$$

Then, since (7.1.3) is uniformly summable $(C, 1)$ for $\sigma \geqslant \sigma_0 > 0$ and fixed x, it follows that

$$(7.2.4) \quad F_2(\sigma, x) = \int_1^\infty \frac{f(u)}{u^2} e^{-\sigma u} \cos xu\, du - \int_1^\infty \frac{f(u)}{u^2} e^{-u} \cos xu\, du - (1-\sigma) \int_1^\infty \frac{f(u)}{u} e^{-u} \cos xu\, du,$$

where all the integrals are interpreted in the $(C, 1)$ sense.

Now, by hypothesis, for fixed x, $F(\sigma, x)$ is bounded as σ approaches zero. Therefore both $F_1(\sigma, x)$ and $F_2(\sigma, x)$ tend to limits as σ approaches zero, for each x. So, by (7.2.4), $g^{(1)}(x) = -\lim I$ exists for all x.

We shall now establish (7.2.1). Using the notation (7.2.2), write

$$(7.2.5) \quad I = \int_1^{1/\sigma} \frac{f(u)}{u^2} e^{-\sigma u} \cos xu\, du + \lim_{\omega \to \infty} \int_{1/\sigma}^\omega \left(1 - \frac{u}{\omega}\right) \frac{f(u)}{u^2} e^{-\sigma u} \cos xu\, du$$

$$= \int_1^{1/\sigma} (1-\sigma u) \frac{f(u)}{u^2} \cos xu\, du$$

$$- \int_1^{1/\sigma} (1-\sigma u) \frac{f(u)}{u^2} (1-e^{-\sigma u}) \cos xu\, du$$

$$+ \sigma \int_1^{1/\sigma} \frac{f(u)}{u} e^{-\sigma u} \cos xu\, du + \lim_{\omega \to \infty} \int_{1/\sigma}^\omega \left(1 - \frac{u}{\omega}\right) \frac{f(u)}{u^2} e^{-\sigma u} \cos xu\, du$$

$$= I_1 + I_2 + I_3 + I_4.$$

We now show that I_2 and $I_3 + I_4$ tend to zero as σ approaches zero. Integrating twice by parts, we see that

$$I_2 = -\int_1^{1/\sigma} f_2(u) \left(\frac{d}{du}\right)^2 \left\{(1-\sigma u)(1-e^{-\sigma u}) \frac{\cos xu}{u^2}\right\} du.$$

Now, if $0 \leqslant u \leqslant 1/\sigma$,

$$|1 - e^{-\sigma u}| \leqslant \sigma u, \qquad \left| \frac{d}{du} (1 - e^{-\sigma u}) \right| \leqslant \sigma.$$

Hence, by (3.1.3),

$$|I_2| \leqslant \sigma \int_1^{1/\sigma} o(u)\, O(u)\, O\left(\frac{1}{u^2}\right) du = o(1)$$

as σ tends to zero. Again, using (3.1.3), we have

$$\begin{aligned}
I_3 &= \frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} - \sigma \int_1^{1/\sigma} f_1(u) \frac{d}{du}\left\{\frac{e^{-\sigma u}}{u} \cos xu\right\} du \\
&= \frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} - \sigma f_2\left(\frac{1}{\sigma}\right)\left[\frac{d}{du}\left\{\frac{e^{-\sigma u}}{u} \cos xu\right\}\right]_{u=1/\sigma} \\
&\qquad\qquad + \sigma \int_1^{1/\sigma} f_2(u) \left(\frac{d}{du}\right)^2 \left\{\frac{e^{-\sigma u}}{u} \cos xu\right\} du \\
&= \frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} + o(1)\, O(\sigma) + \sigma \int_1^{1/\sigma} o(u)\, O\left(\frac{1}{u}\right) du \\
&= \frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} + o(1)
\end{aligned}$$

as σ tends to zero. Finally,

$$\begin{aligned}
I_4 &= -\frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} - \lim_{\omega\to\infty} \int_{1/\sigma}^{\omega} f_1(u) \frac{d}{du}\left\{\left(1-\frac{u}{\omega}\right)\frac{e^{-\sigma u}}{u^2} \cos xu\right\} du \\
&= -\frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} + f_2\left(\frac{1}{\sigma}\right)\left[\frac{d}{du}\left\{\frac{e^{-\sigma u}}{u^2} \cos xu\right\}\right]_{u=1/\sigma} \\
&\qquad\qquad + \int_{1/\sigma}^{\infty} f_2(u) \left(\frac{d}{du}\right)^2 \left\{\frac{e^{-\sigma u}}{u^2} \cos xu\right\} du \\
&= -\frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} + \int_{1/\sigma}^{\infty} o(u)\, O\left(\frac{1}{u^2}\right) e^{-\sigma u}\, du + o(1) \\
&= -\frac{\sigma^2}{e} f_1\left(\frac{1}{\sigma}\right) \cos\frac{x}{\sigma} + o(1).
\end{aligned}$$

Comparing this with I_3, we see that the terms involving $f_1(1/\sigma)$ cancel and we obtain

$$I_3 + I_4 = o(1).$$

Hence, from (7.2.5),

$$I = \int_1^{1/\sigma} (1-\sigma u) \frac{f(u)}{u^2} \cos xu\, du + o(1),$$

where $o(1)$ denotes terms which tend to zero as σ approaches zero, uniformly in x for $|x| \leqslant X$. Now we have already shown that the limit, as σ approaches zero, of the first member exists. Hence the limit of the second member exists for all x and we have

$$g(x) = -\lim_{\sigma \to 0} \int_1^{1/\sigma} (1-\sigma u) \frac{f(u)}{u^2} \cos xu \, du$$

for all x. This is (7.2.1).

LEMMA 22. *Under the hypotheses of Lemma* 21,

$$g(x) = \lim_{h=0} \frac{G(x+h)-G(x-h)}{2h}$$

for all x, where $G(x)$ is given by (3.2.3).

Proof. We have

$$\begin{aligned} \frac{G(x+h)-G(x-h)}{2h} \\ &= -\int_1^{\infty} \frac{f(u)}{u^2} \cos xu \frac{\sin hu}{hu} \, du \quad (C,\ 1) \\ &= -\int_1^{1/h} (1-hu) \frac{f(u)}{u^2} \cos xu \frac{\sin hu}{hu} \, du - h \int_1^{1/h} \frac{f(u)}{u} \cos xu \frac{\sin hu}{hu} \, du \\ &\qquad - \lim_{\omega \to \infty} \int_{1/h}^{\omega} \left(1-\frac{u}{\omega}\right) \frac{f(u)}{u^2} \cos xu \frac{\sin hu}{hu} \, du \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Write $$\omega g(x,\ \omega) = -\int_1^{\omega} (\omega-u) \frac{f(u)}{u^2} \cos xu \, du.$$

Then, by Lemma 21, $g(x,\ \omega)$ tends to $g(x)$ as ω tends to infinity. Integrating I_1 twice by parts, we get

$$\begin{aligned} I_1 &= \Big[-ug(x,\ u) \frac{d}{du} \left\{ (1-hu) \frac{\sin hu}{hu} \right\} \Big]_1^{1/h} \\ &\qquad + \int_1^{1/h} ug(x,\ u) \left(\frac{d}{du}\right)^2 \left\{ (1-hu) \frac{\sin hu}{hu} \right\} du \\ &= I_4 + I_5. \end{aligned}$$

Let us write $$g(x,\ u) = g(x) + o(1).$$

Suppose that x is fixed. Then it is easy to see that

$$I_5 = g(x)\Big[u\,\frac{d}{du}\Big\{(1-hu)\,\frac{\sin hu}{hu}\Big\} - (1-hu)\,\frac{\sin hu}{hu}\Big]_1^{1/h} + \int_1^{1/h} o(u)\left|\left(\frac{d}{du}\right)^2\Big\{(1-hu)\,\frac{\sin hu}{hu}\Big\}\right| du$$

$$= g(x)\Big[u\,\frac{d}{du}\Big\{(1-hu)\,\frac{\sin hu}{hu}\Big\}\Big]_1^{1/h} + g(x)(1-h)\,\frac{\sin h}{h} + o(1)$$

as h tends to zero. Hence

$$I_1 = \Big[\{g(x)-g(x,\,u)\}\,u\,\frac{d}{du}\Big\{(1-hu)\,\frac{\sin hu}{hu}\Big\}\Big]_1^{1/h} + g(x) + o(1)$$

$$= g(x) + o(1).$$

Write

$$\mu(u) = \begin{cases} h/u & (1 \leqslant u \leqslant 1/h), \\ 1/u^2 & (u \geqslant 1/h). \end{cases}$$

Then

$$I_2 + I_3 = -\lim_{\omega\to\infty}\int_1^{\omega}\left(1-\frac{u}{\omega}\right) f(u)\,\mu(u)\cos xu\,\frac{\sin hu}{hu}\,du$$

$$= \lim_{\omega\to\infty}\Big[f_2(u)\,\frac{d}{du}\Big\{\left(1-\frac{u}{\omega}\right)\mu(u)\cos xu\,\frac{\sin hu}{hu}\Big\}\Big]_{1,\ 1/h+0}^{1/h-0,\ \omega} - \lim_{\omega\to\infty}\int_1^{\omega} f_2(u)\left(\frac{d}{du}\right)^2\Big\{\left(1-\frac{u}{\omega}\right)\mu(u)\cos xu\,\frac{\sin hu}{hu}\Big\}\,du$$

$$= -\Big[f_2(u)\,\frac{d}{du}\Big\{\mu(u)\cos xu\,\sin\frac{hu}{hu}\Big\}\Big]_{1/h-0}^{1/h+0} - \int_1^{\infty} f_2(u)\left(\frac{d}{du}\right)^2\Big\{\mu(u)\cos xu\,\frac{\sin hu}{hu}\Big\}\,du.$$

Now let h tend to zero. We get

$$I_2 + I_3 = o(1) + h\int_1^{1/h} o(1)\,du + \frac{1}{h}\int_{1/h}^{\infty} o\left(\frac{1}{u^2}\right) du = o(1).$$

Hence

$$\lim_{h\to 0}\frac{G(x+h)-G(x-h)}{2h} = \lim_{h\to 0} I_1 = g(x),$$

as desired.

LEMMA 23. *Under the hypotheses of Lemma* 21, *$G'(x)$ exists for all x and is equal to $g(x)$.*

If the hypotheses of Lemma 21 are satisfied, then so also are those of Lemma 6. Lemma 23 follows at once from Lemmas 6 and 22.

When $f(u)$ satisfies either (6.1.2) or (6.1.3), we substitute Lemma 20 for Lemma 6.

7.3. In this paragraph we establish the properties of $g(x)$ needed for the proof of Theorem 11. These are given in Lemmas 24 and 26. In Lemma 24 we establish the analogues for trigonometric integrals of the Rajchman-Zygmund inequalities for trigonometric series. The proof of this lemma has, however, more in common with Verblunsky's (**14**, I, 445) proof of these inequalities than with the original proof given by Rajchman and Zygmund.

LEMMA 24. *Under the hypotheses of Lemma* 21,

$$\overline{D}^2 g(x) \geqslant \underline{F}(x), \quad \underline{D}^2 g(x) \leqslant \overline{F}(x) \tag{7.3.1}$$

for all x.

Proof. Write

$$\begin{aligned} Q(\sigma, y) &= \tfrac{1}{2}\{F(\sigma, x+y)+F(\sigma, x-y)\} \\ &= \int_1^\infty f(u)\, e^{-\sigma u} \cos xu \cos yu \, du \quad (C, 1), \end{aligned} \tag{7.3.2}$$

$$\begin{aligned} T(y) &= \tfrac{1}{2}\{g(x+y)+g(x-y)\} \\ &= -\int_1^\infty \frac{f(u)}{u^2} \cos xu \cos yu \, du \quad (C, 1), \end{aligned} \tag{7.3.3}$$

$$T_1(y) = -\int_1^\infty \frac{f(u)}{u^3} \cos xu \sin yu \, du \quad (C, 1), \tag{7.3.4}$$

$$T_2(y) = \int_1^\infty \frac{f(u)}{u^4} \cos xu \cos yu \, du \quad (C, 1), \tag{7.3.5}$$

so that $T(y) = T_1'(y) = T_2''(y)$.

If x is fixed, we can show, just as in the proof of (3.2.5), that

$$T_1(y) = o(y^2), \quad T_2(y) = o(y^2). \tag{7.3.6}$$

It will be sufficient to prove the second of the inequalities (7.3.1). From (7.3.3) it follows that

$$\Delta^2 T(0) = T(h)+T(-h)-2T(0) = \Delta^2 g(x).$$

Hence it will be sufficient to show that $\underline{D}^2 T(0) \leqslant \bar{Q}(0)$, where

$$\bar{Q}(0) = \varlimsup_{\sigma \to 0} Q(\sigma, 0)$$

and $Q(\sigma, y)$ is given by (7.3.2).

Plainly we may suppose that $\underline{D}^2 T(0) > -\infty$, otherwise there is nothing to prove. There is then a finite number k, depending in general on x, such that $\underline{D}^2 T(0) > k$, *i.e.*

$$\varliminf_{h \to 0} \{T(h) - T(0)\}/2h^2 > k, \tag{7.3.7}$$

since $T(y)$ is even. We show that, for any k satisfying (7.3.7), $\bar{Q}(0) \geqslant k$ and thus establish the lemma.

We first reduce the problem to one in which $k = 0$ and $T(0) = 0$. From now onwards we treat x as fixed. Write

$$s(u) = \begin{cases} 3(2A+B)u^2 - 2(7A+3B) & (1 \leqslant u \leqslant 2), \\ 0 & \text{otherwise}, \end{cases}$$

and let

$$S(y) = -\int_1^2 \frac{s(u)}{u^2} \cos yu \, du.$$

Then

$$S(0) = -\int_1^2 \frac{s(u)}{u^2} \, du = A,$$

and

$$S''(0) = \int_1^2 s(u) \, du = B.$$

Hence, if we choose A and B appropriately and add $s(u)$ to $f(u) \cos xu$, we can so arrange that $k = 0$ and $T(0) = 0$, while the new functions $T_1(y)$ and $T_2(y)$ still satisfy (7.3.6).

It is therefore sufficient to prove that

$$\varliminf_{h \to 0} \frac{T(h)}{h^2} > 0 \tag{7.3.8}$$

implies $\bar{Q}(0) \geqslant 0$. We show that, corresponding to any positive number δ, there exists a number σ', $0 < \sigma' < \delta$, such that $Q(\sigma', 0) > 0$. This implies

$$\bar{Q}(0) \geqslant 0.$$

Write

$$\mu(x) = \int_0^\infty u e^{-\sigma u} \sin xu \, du = \frac{2\sigma x}{(\sigma^2 + x^2)^2}. \tag{7.3.9}$$

Inverting

$$\mu'(x) = \int_0^\infty u^2 e^{-\sigma u} \cos xu\, du,$$

we see that

$$u^2 e^{-\sigma u} = \frac{2}{\pi}\int_0^\infty \mu'(x) \cos xu\, dx. \tag{7.3.10}$$

Consider

$$R(\sigma) = \frac{2}{\pi}\int_0^\infty \mu'(y)\, T_2(y)\, dy,$$

where $T_2(y)$ is given by (7.3.5). Since $f_2(u) = o(u)$, it readily follows, by two partial integrations, that

$$\left| \int_1^\infty \left(1 - \frac{u}{\omega}\right) \frac{f(u)}{u^4} \cos xu \cos yu\, du \right| \leqslant M(y^2+1)$$

for fixed x. Now, by (7.3.9), $\mu'(y) = O(y^{-4})$. Hence

$$\begin{aligned} R(\sigma) &= \lim_{\omega\to\infty} \frac{2}{\pi}\int_0^\infty \mu'(y)\, dy \int_0^\omega \left(1 - \frac{u}{\omega}\right) \frac{f(u)}{u^4} \cos xu \cos yu\, du \\ &= \int_1^\infty \frac{f(u)}{u^2} e^{-\sigma u} \cos xu\, du \quad (C,\ 1) \end{aligned} \tag{7.3.11}$$

in virtue of (7.3.10). Also

$$\begin{aligned} R'(\sigma) &= \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial \sigma} \mu'(y)\, T_2(y)\, dy = \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial y} \frac{\partial}{\partial \sigma} \mu(y)\, T_2(y)\, dy \\ &= \frac{2}{\pi}\int_0^\infty \frac{\partial}{\partial y} \left\{ \frac{2y}{(\sigma^2+y^2)^2} - \frac{8\sigma^2 y}{(\sigma^2+y^2)^3} \right\} T_2(y)\, dy \\ &= \sigma^{-1} R(\sigma) + \frac{16}{\pi}\int_0^\infty \frac{\sigma^2 y}{(\sigma^2+y^2)^3}\, T_1(y)\, dy \end{aligned} \tag{7.3.12}$$

on performing one integration by parts.

Compare the functions $R(\sigma)$ and $R'(\sigma)$ with the functions $F_2(\sigma, x)$ and $F_1(\sigma, x)$ of Lemma 21 and, in particular, compare equations (7.2.4) and (7.3.11). Plainly $R(\sigma)$ and $R'(\sigma)$ differ from $F_2(\sigma, x)$ and $F_1(\sigma, x)$ respectively by linear functions of σ. Since we showed in the proof of Lemma 21 that, for all x, $F_2(\sigma, x)$ and $F_1(\sigma, x)$ tend to finite limits as σ

approaches zero, it follows that the same is true of $R(\sigma)$ and $R'(\sigma)$. Also, since $F_1(\sigma, x)$ is differentiable for $\sigma > 0$, so is $R'(\sigma)$.

Since $R'(\sigma)$ is continuous in the closed interval $(0, \delta)$ and differentiable at all interior points, it follows by Taylor's theorem that there exists a number θ, $0 < \theta < 1$, such that

$$R(0) = R(\sigma) - \sigma R'(\sigma) + \tfrac{1}{2}\sigma^2 R''(\theta\sigma).$$

But $R(0) = -T(0) = 0$. Hence, by (7.3.12),

$$R''(\theta\sigma) = \frac{32}{\pi}\int_0^\infty \frac{\sigma y}{(\sigma^2+y^2)^3} T_1(y)\,dy$$

$$= \frac{32}{\pi}\left\{\int_0^\delta + \int_\delta^\infty\right\} dy = I_1 + I_2.$$

Now, using (7.3.6), we have

$$|I_2| \leqslant M\sigma \int_\delta^\infty \frac{y^3}{(\sigma^2+y^2)^3}\,dy \leqslant M\sigma \int_\delta^\infty \frac{dy}{y^3},$$

so that, when δ is fixed,

$$\lim_{\sigma \to 0} |I_2| = 0.$$

Integrating I_1 once by parts, we see that

$$I_1 = -\frac{8}{\pi}\frac{\sigma}{(\sigma^2+\delta^2)^2} T_1(\delta) + \frac{8}{\pi}\int_0^\delta \frac{\sigma}{(\sigma^2+y^2)^2} T(y)\,dy$$

$$= I_3 + I_4.$$

Now, δ being fixed,

$$\lim_{\sigma \to 0} |I_3| = 0.$$

This leaves only I_4. By (7.3.8), we can choose δ so that

$$\frac{T(y)}{y^2} \geqslant \eta > 0$$

for $0 \leqslant y \leqslant \delta$ and some η. Hence, if $0 < \sigma \leqslant \delta$,

$$I_4 \geqslant \eta\,\frac{8}{\pi}\int_0^\delta \frac{\sigma y^2}{(\sigma^2+y^2)^2}\,dy = \eta\,\frac{8}{\pi}\int_0^{\delta/\sigma} \frac{y^2\,dy}{(1+y^2)^2}$$

$$\geqslant \eta\,\frac{8}{\pi}\int_0^1 \frac{y^2\,dy}{(1+y^2)^2} = \eta\left(1 - \frac{2}{\pi}\right).$$

We have thus proved that, for every σ satisfying $0 < \sigma \leqslant \delta$, there exists a number θ, $0 < \theta < 1$, such that

$$R''(\theta\sigma) > \tfrac{1}{3}\eta - |I_2| - |I_3|.$$

Hence, δ being fixed, we can find σ_0 so that, for every σ satisfying $0 < \sigma \leqslant \sigma_0$, there exists a θ, $0 < \theta < 1$, such that

$$R''(\theta\sigma) \geqslant \tfrac{1}{4}\eta.$$

But
$$R''(\sigma) = \int_1^\infty e^{-\sigma u} f(u) \cos xu\, du = Q(\sigma, 0) \quad (C, 1),$$

and so, for some σ' in $(0, \sigma_0)$, $Q(\sigma', 0) \geqslant \tfrac{1}{4}\eta$, however small σ_0 may be. Therefore $\bar{Q}(0) > 0$ and the lemma is proved.

LEMMA 25. *If $\Phi'(x)$ is everywhere a finite differential coefficient and if $\bar{D}^2 \Phi'(x) \geqslant 0$ for all x, then $\Phi'(x)$ is continuous and convex.*

This lemma is included in a lemma of Verblunsky's (**15**, 176). Since $\Phi'(x)$ is everywhere a finite differential coefficient, it is a function of Baire's first class† and is therefore point-wise discontinuous on every perfect set. That is to say, its points of discontinuity with respect to any perfect set P are non-dense in P. This property is described by Verblunsky (**15**, 169) as the property R and his lemma is proved for functions having the more general property R^*.

The lemma also follows readily from Saks's argument (**11**, 249), although it is not stated explicitly by him. Indeed, since $\Phi'(x)$ is point-wise discontinuous on every perfect set, it belongs to Saks's class u^*.

LEMMA 26. *Under the hypotheses of Theorem* 11, *$g(x)$ is of the form*

$$g(x) = \int_0^x dy \int_0^y \gamma(t)\, dt + q,$$

where q is some number.

This lemma follows from Lemmas 24 and 25 by a well known argument of de la Vallée Poussin‡. Since $g(x)$ is an even function, there can be no term of the form px and hence $g(x)$ has the desired form.

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† *Cf.* Lebesgue, **6**, 99 and 202, or Hobson, **4**, 273.

‡ *Cf.* Hobson, **4**, 669–671, or Zygmund, **19**, 276 and 300.

We have now shown that the conclusions of Lemma 9 remain true under the hypotheses of Theorem 11 and consequently this theorem follows in virtue of Lemmas 11 and 15.

8. *Proof of Theorem* 7.

8.1. We divide the integral (1.3.4) into its real and imaginary parts, as in the proof of Theorem 5. Then Theorem 7 is a consequence of the following theorem.

Theorem 12. *Let $\alpha(u)$ and $\beta(u)$ be real and integrable L in every finite range. Suppose that*

(i) *the integral*

$$\int_0^\infty e^{-\sigma u}\{\alpha(u)\cos xu+\beta(u)\sin xu\}\,du$$

is summable $(C, 1)$ for $\sigma>0$ and all x to a function $P(\sigma, x)$;

(ii) *the functions*

$$\overline{P}(x)=\overline{\lim_{\sigma\to 0}}\,P(\sigma, x),\qquad \underline{P}(x)=\underline{\lim_{\sigma\to 0}}\,P(\sigma, x)$$

are finite for all x and integrable L in every finite range;

(iii) $$\int_0^x \overline{P}(t)\,dt=O(|x|^\kappa),\qquad \int_0^x \underline{P}(t)\,dt=O(|x|^\kappa)\quad (0<\kappa<1)$$

as $|x|$ tends to infinity;

(iv) *there exists a set E, everywhere dense in $(-\infty, \infty)$, such that*

$$J(x, \omega)=\int_0^\omega\left(1-\frac{u}{\omega}\right)\{\alpha(a)\cos xu+\beta(u)\sin xu\}\,du$$

tends to a finite limit as ω tends to infinity for all x of E

Then $\overline{P}(x)\equiv\underline{P}(x)\equiv P(x)$ and

$$P(x)=\int_0^\infty\{\alpha(u)\cos xu+\beta(u)\sin xu\}\,du\quad (C, 1),$$

$$\alpha(u)=\frac{1}{\pi}\int_{-\infty}^\infty P(x)\cos xu\,du,\quad \beta(u)=\frac{1}{\pi}\int_{-\infty}^\infty P(x)\sin xu\,du\quad (C, 1),$$

almost everywhere. Further, in any finite interval, these three integrals are equi-summable (C, 1) *with Fourier series.*

Proof. The proof is similar to that of Theorem 9 except that we employ Theorem 11 in the place of Theorem 8. We show that Lemma 17 remains true under the hypotheses of Theorem 12.

By (i), the integral

$$P(\sigma;\ x,\ y) = \int_0^\infty e^{-\sigma u}\{\alpha(u)\cos yu + \beta(u)\sin yu\}\cos xu\,du$$

is summable $(C, 1)$ for $\sigma > 0$ and all x, y. From (ii),

$$(8.1.1)\quad \tfrac{1}{2}\{\underline{P}(y+x)+\underline{P}(y-x)\} \leqslant \varliminf_{\sigma\to 0} P(\sigma;\ x,\ y) \leqslant \varlimsup_{\sigma\to 0} P(\sigma;\ x,\ y)$$

$$\leqslant \tfrac{1}{2}\{\overline{P}(y+x)+\overline{P}(y-x)\}.$$

Suppose that y is a point of E. Then, by (iv),

$$P(y) = \int_0^\infty \{\alpha(u)\cos yu + \beta(u)\sin yu\}\,du \quad (C,\ 1)$$

exists. In Theorem 11 take

$$f(u) = \alpha(u)\cos yu + \beta(u)\sin yu.$$

The hypotheses of this theorem are satisfied in virtue of (ii) and (8.1.1). Hence for each y of E there exists a function $F_y(x)$ such that the relations (4.4.11) and (4.4.12) are true, just as in the proof of Lemma 17. Consequently, (4.4.13) holds as before.

Again, by (i). (4.4.11), and the consistency theorems,

$$F_y(x) = \lim_{\sigma\to 0}\ \lim_{\omega\to\infty} \int_0^\omega \left(1-\frac{u}{\omega}\right)^2 e^{-\sigma u}\{\alpha(u)\cos yu + \beta(u)\sin yu\}\cos xu\,du$$

$$= \lim_{\sigma\to 0} P(\sigma,\ x,\ y),$$

for all y of E and almost all x. Therefore, from (8.1.1),

$$\tfrac{1}{2}\{\underline{P}(y+x)+\underline{P}(y-x)\} \leqslant F_y(x) \leqslant \tfrac{1}{2}\{\overline{P}(y+x)+\overline{P}(y-x)\}$$

for all y of E and almost all x. Lemma 17 now follows just as before. The rest of the proof is the same as that of Theorem 9.

References.

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
2. M. E. Grimshaw, " A contribution to the theory of uniqueness of representation by trigonometric integrals ", *Proc. London Math. Soc.* (2), 29 (1929), 243–256.
3. E. Hille and J. D. Tamarkin, " On the theory of Fourier transforms ", *Bulletin American Math. Soc.*, 39 (1933), 768–774.
4. E. W. Hobson, *The theory of functions of a real variable*, 2 (2nd ed., 1926).
5. M. Jacob, "Über den Eindeutigkeitssatz in der Theorie der verallgemeinerten trigonometrischen Integrale, *Math. Annalen*, 100 (1928), 278–294.
6. H. Lebesgue, *Leçons sur l'intégration* (2me ed., 1928).
7. A. C. Offord, " On Fourier transforms (III) ", *Trans. American Math. Soc.*, 38 (1935), 250–266.
8. ————, " Note on the uniqueness of the representation of a function by a trigonometric integral ", *Journal London Math. Soc.*, 11 (1936), 171–174.
9. S. Pollard, " Identification of the coefficients in a trigonometric integral ", *Proc. London Math. Soc.* (2), 25 (1926), 451–468.
10. ————, " The summation of Denjoy-Fourier series ", *Proc. London Math. Soc.* (2), 27 (1928), 209–222.
11. S. Saks, "On the generalized derivatives ", *Journal London Math. Soc.*, 7 (1932), 247–251.
12. S. Verblunsky, " The generalized third derivative and its application to the theory of trigonometric series ", *Proc. London Math. Soc.* (2), 31 (1930), 387–406.
13. ————, " The generalized fourth derivative ", *Journal London Math. Soc.*, 6 (1931), 82–84.
14. ————, " On the theory of trigonometric series (I), (II) ", *Proc. London Math. Soc.* (2), 34 (1932), 441–456 and 457–491.
15. ————, " On the theory of trigonometric series (V) ", *Fundamenta Math.*, 21 (1933), 168–210.
16. ————, " Trigonometric integrals and harmonic functions ", *Proc. London Math. Soc.* (2), 38 (1935), 1–48.
17. N. Wiener, *The Fourier integral* (Cambridge, 1933).
18. A. Zygmund, "Über die Beziehungen der Eindeutigkeitsfragen in den Theorien der trigonometrischen Reihen und Integrale ", *Math. Annalen*, 99 (1928), 562–589.
19. ————, *Trigonometric series* (Warsaw, 1935).

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THE REPRESENTATION OF A NUMBER AS A SUM OF THREE OR FOUR SQUARES

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[Received 30 May, 1936.—Read 18 June, 1936.]

1. In two former papers (Wright, **1**, **2**) I discussed the problem of the representation of a number n as a sum of squares "almost proportional" to assigned positive numbers. I showed that, if $s \geqslant 5$, $\lambda_1, \ldots, \lambda_s$ are any assigned positive numbers whose sum is unity,

$$\alpha = \frac{s-4}{4(s-3)} \quad (s = 5, 6, 7), \qquad \alpha = \frac{s-2}{2(2s-1)} \quad (s \geqslant 8),$$

and $0 < \beta < \alpha$, then every positive integer n can be expressed in the form

$$n = m_1^2 + \ldots + m_s^2,$$

where

$$|m_i^2 - \lambda_i n| = O(n^{1-\beta}).$$

Here we shall consider a similar problem for $s = 3$ and $s = 4$. The results are not true in these cases for all values of n, but only for a certain number. Elsewhere (Wright, **3**) I have shown that every positive integer n with a sufficiently large odd factor is expressible in the form

$$n = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

with

$$|m_i^2 - \lambda_i n| = o(n).$$

The method of proof is different from that used here.

Throughout this paper we take $s = 3$ or 4,

$$\alpha = \tfrac{1}{6} \quad (s = 3), \qquad \alpha = \tfrac{1}{5} \quad (s = 4),$$

and $\lambda_1, \ldots, \lambda_s$ any assigned positive numbers whose sum is unity. We shall prove

THEOREM 1. *If* $0 < \beta < \frac{1}{5}$, *the number of positive integers* j *less than or equal to* N *which cannot be expressed in the form*

$$j = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

with

$$|m_i^2 - \lambda_i j| = O(j^{1-\beta})$$

is $O(N^{1-c})$, *where* c *is any positive number less than* $\frac{5}{12}(\frac{1}{5} - \beta)$.

THEOREM 2. *If* $0 < \beta < \frac{1}{6}$, *the number of positive integers* j *less than or equal to* N *which are not of the form* $4^a(8l+7)$ *and which cannot be expressed in the form*

$$j = m_1^2 + m_2^2 + m_3^2$$

with

$$|m_i^2 - \lambda_i j| = O(j^{1-\beta})$$

is $O(N^{1-c})$, *where* c *is any positive number less than* $\frac{3}{7}(\frac{1}{6} - \beta)$.

The numbers $4^a(8l+7)$ are, of course, precisely those numbers which cannot be expressed as the sum of three squares.

In the proof of Theorem 2 we have to consider the "truncated" singular series for three squares; special difficulties arise here, since the complete singular series is not absolutely convergent. I have overcome these difficulties by a device which, so far as I know, is new. The details will be found in sections 5 and 6.

Notation.

2. In what follows s, i, n, j, h_i, m_i are positive integers, $s = 3$ or 4,

$$1 \leqslant i \leqslant s, \quad \alpha = \alpha(s), \quad \alpha(3) = \tfrac{1}{6}, \quad \alpha(4) = \tfrac{1}{5}.$$

The numbers $\beta, B, \lambda_1, \ldots, \lambda_s$ are any assigned positive numbers subject only to the conditions

$$0 < \beta < \alpha, \quad \lambda_1 + \ldots + \lambda_s = 1.$$

We write $\delta = \alpha - \beta$ and denote by c any positive number less than $\frac{1}{2}\delta/(1+\alpha)$. It is easily verified that this definition agrees with that of Theorems 1 and 2. We write also

$$\gamma = \tfrac{1}{2}s - 1 - (s-1)\beta, \quad \omega = s - 1 - (2s-1)\beta.$$

We use Π to denote $\prod\limits_{i=1}^{s}$.

The number ϵ is at first an arbitrary positive number, to be thought of as small; ϵ will eventually be chosen to be a suitable function of s, β, and c. C is a positive number whose value varies from one occurrence to another. C depends at most on

$$s,\ \beta,\ \lambda_1,\ \ldots,\ \lambda_s,\ c,\ B,\ \epsilon.$$

The constant implied in the $O(\quad)$ notation is of type C.

We write

$$\rho = \exp(2\pi i p/q) \ \ [(p, q) = 1], \qquad S_\rho = \sum_{h=1}^{q} \rho^{h^k},$$

$$A_s(q, j) = q^{-s} \sum_{\rho} S_\rho^{\,s} \rho^{-j},$$

where ρ runs through all primitive q-th roots of unity;

$$\mathfrak{S}(j, \lambda) = \sum_{1 \leqslant q \leqslant \lambda} A_s(q, j),$$

$$N_i = [\lambda_i n + 2sBn^{1-\beta}], \quad N_i' = [\lambda_i n - 2s\,Bn^{1-\beta} + 1].$$

We take n sufficiently large to ensure that N_i' is positive.

We define $r(j)$ as the number of solutions of

$$j = m_1^2 + \ldots + m_s^2 \tag{2.01}$$

such that

$$m_i \geqslant 0, \quad \lambda_i n - 2sBn^{1-\beta} < m_i^2 \leqslant \lambda_i n + 2sBn^{1-\beta} \quad (1 \leqslant i \leqslant s),$$

or, what is the same thing,

$$N_i' \leqslant m_i^2 \leqslant N_i \quad (1 \leqslant i \leqslant s).$$

We write
$$\mu(j) = 2^{-s} \sum_{h_1, \ldots, h_s} (\Pi h_i^{-\frac{1}{2}}),$$

where the summation is taken over all sets of positive integral values of $h_1, \ldots, h_s$ which satisfy

$$j = h_1 + \ldots + h_s \tag{2.02}$$

and
$$N_i' \leqslant h_i \leqslant N_i \quad (1 \leqslant i \leqslant s). \tag{2.03}$$

Finally we write

$$\sigma(j) = r(j) - \mu(j)\,\mathfrak{S}(j, n^{\alpha}).$$

If j can be expressed in the form (2.01), but cannot be expressed in this form subject to the condition

$$|m_i^2 - \lambda_i j| < 4s\,Bj^{1-\beta} \quad (1 \leqslant i \leqslant s), \tag{2.04}$$

we say that j is *exceptional with respect to* β, or, more shortly, *exceptional.*

If $s = 4$, any j can be expressed in the form (2.01) and so any j which cannot satisfy condition (2.04) is exceptional. On the other hand, if $s = 3$, j can be expressed in the form (2.01) if, and only if, j is not of the form $4^a(8l+7)$; hence, for $s = 3$, numbers of the form $4^a(8l+7)$ are not exceptional.

We say that j is *restricted* if

$$n - Bn^{1-\beta} \leqslant j \leqslant n. \tag{2.05}$$

We assume throughout that n is sufficiently large to satisfy various requirements; that is, all statements involving n bear the implied condition than n is greater than some suitable C.

Proof of Theorems 1 *and* 2.

3.1. With our definition of exceptional j we see that Theorems 1 and 2 are immediate consequences of

THEOREM 3. *The number of exceptional* $j \leqslant N$ *is* $O(N^{1-c})$.

To prove this theorem we need two fundamental lemmas (Lemmas 1 and 2) which we prove in §§4–6.

LEMMA 1. $$\sum_j \sigma^2(j) < Cn^{\omega-\delta+\epsilon}.$$

The summation extends over all positive values of j. Of course, when $j < \sum_{i=1}^{s} N_i'$, or when $j > \sum_{i=1}^{s} N_i$, we have

$$r(j) = \mu(j) = \sigma(j) = 0.$$

LEMMA 2. *The number of restricted quadratfrei j for which*

$$\mathfrak{S}(j, n^{\alpha}) \leqslant Cj^{-\epsilon}$$

and, for $s = 3$ *only*, $j \not\equiv 7 \pmod 8$, *is* $O(n^{1-\alpha+\epsilon})$.

3.2. LEMMA 3. *If j is restricted and exceptional, we have* $r(j) = 0$.

If $r(j) > 0$, j can be represented in the form (2.01) with

$$|m_i^2 - \lambda_i n| \leqslant 2sBn^{1-\beta} \quad (1 \leqslant i \leqslant s).$$

Hence, if j is restricted, by (2.05),

$$|m_i^2-\lambda_i j| \leqslant |m_i^2-\lambda_i n|+\lambda_i|n-j|$$
$$< (2s+1)\,Bn^{1-\beta} < 4s\,Bn^{1-\beta},$$

and so j cannot be exceptional.

LEMMA 4. *If j is restricted, $\mu(j) > Cn^{\gamma}$.*

We define $\nu(j)$ as the number of solutions of (2.02) and (2.03). We can choose the set $h_1, \ldots, h_{s-1}$ to satisfy

$$\lambda_i n - Bn^{1-\beta} < h_i \leqslant \lambda_i n + Bn^{1-\beta} \quad (1 \leqslant i \leqslant s-1) \qquad (3.21)$$

in at least $[2Bn^{1-\beta}]^{s-1}$ ways and $h_1, \ldots, h_{s-1}$ then clearly satisfy (2.03). If we now choose

$$h_s = j-h_1-h_2-\ldots-h_{s-1},$$

we have a solution of (2.02). By (2.05) and (3.21),

$$h_s \geqslant j - \sum_{i=1}^{s-1} (\lambda_i n + Bn^{1-\beta}) \geqslant \lambda_s n - sBn^{1-\beta} > N_s',$$

$$h_s \leqslant j - \sum_{i=1}^{s-1} (\lambda_i n - Bn^{1-\beta}) \leqslant \lambda_s n + (s-1)\,Bn^{1-\beta} < N_s.$$

Hence we have found at least $Cn^{(1-\beta)(s-1)}$ solutions of (2.02) and (2.03); that is,

$$\nu(n) > Cn^{(1-\beta)(s-1)}.$$

If $h_1, \ldots, h_s$ satisfy (2.03), we have

$$\Pi\, h_i^{-\frac{1}{2}} \geqslant \Pi\, N_i'^{-\frac{1}{2}} > Cn^{-\frac{1}{2}s}.$$

Hence $$\mu(j) = 2^{-s} \sum_{h_1, \ldots, h_s} (\Pi\, h_i^{-\frac{1}{2}}) > Cn^{-\frac{1}{2}s}\,\nu(j) > Cn^{\gamma}.$$

3.3. LEMMA 5. *The number of values of j which are at the same time restricted, exceptional and quadratfrei is $O(n^{1-\alpha+\epsilon})$.*

If $s=3$ and j is exceptional, then $j \not\equiv 7 \pmod 8$. Hence, by Lemma 2, for both $s=3$ and $s=4$, the number of restricted, exceptional, and quadratfrei j for which $\mathfrak{S}(j) \leqslant Cj^{-\epsilon}$ is $O(n^{1-\alpha+\epsilon})$. Hence, by Lemma 3, it is sufficient to show that there are only $O(n^{1-\alpha+\epsilon})$ restricted values of j for which

$$r(j) = 0, \quad \mathfrak{S}(j, n\,) > Cj^{-\epsilon}. \qquad (3.31)$$

By (2.05), (3.31), and Lemma 4, we have

$$\sigma^2(j) = \{\mu(j)\mathfrak{S}(j, n^{\alpha})\}^2 > Cj^{2\gamma-2\epsilon} > Cn^{2\gamma-2\epsilon}. \tag{3.32}$$

If this is true for more than $Cn^{1-\alpha+4\epsilon}$ values of j, we have

$$\sum_j \sigma^2(j) > Cn^{2\gamma+1-\alpha+2\epsilon} = Cn^{\omega-\delta+2\epsilon}.$$

But this contradicts Lemma 1. Hence (3.31) is true for only $O(n^{1-\alpha+4\epsilon})$ restricted values of j, that is, since ϵ is any positive number, for only $O(n^{1-\alpha+\epsilon})$ such values.

3.4. We now consider the number of quadratfrei exceptional j less than or equal to N. Let us divide the range $\frac{1}{2}N < j \leqslant N$ into intervals each of length $B(\frac{1}{2}N)^{1-\beta}$. There are at most

$$B^{-1}(\tfrac{1}{2}N)^{\beta} < CN^{\beta}$$

such intervals. If we take the largest integer in each interval to be the n of Lemma 5, we have $n > \frac{1}{2}N$, and so our interval lies within that of (2.05). Hence, by Lemma 5, in each interval there are only

$$O(n^{1-\alpha+\epsilon}) = O(N^{1-\alpha+\epsilon})$$

exceptional quadratfrei values of j. It follows that in the range $\frac{1}{2}N < j \leqslant N$ there are less than

$$CN^{1-\alpha+\epsilon+\beta} = CN^{1-\delta+\epsilon}$$

exceptional quadratfrei j.

Similarly there are less than $C(\frac{1}{2}N)^{1-\delta+\epsilon}$ exceptional quadratfrei j in the range $\frac{1}{4}N < j \leqslant \frac{1}{2}N$, and so on. Hence the total number of exceptional quadratfrei j less than or equal to N is less than

$$C\{N^{1-\delta+\epsilon} + (\tfrac{1}{2}N)^{1-\delta+\epsilon} + (\tfrac{1}{4}N)^{1-\delta+\epsilon} + \dots\}$$
$$= CN^{1-\delta+\epsilon}\left\{1 + \frac{1}{2^{1-\delta+\epsilon}} + \frac{1}{4^{1-\delta+\epsilon}} + \dots\right\} < CN^{1-\delta+\epsilon}.$$

We now replace β by β' and write $\delta' = \alpha - \beta'$, where $0 < \beta' < \alpha$. Since β' satisfies the same condition as β, we have

LEMMA 6. *The number of quadratfrei j less than or equal to N and exceptional with respect to β' is $O(N^{1-\delta'+\epsilon})$.*

3.5. We now remove the restriction that j is quadratfrei. We write $j = j_1{}^2 j_2$, where j_2 is quadratfrei.

LEMMA 7. *If $d > 0$, there are only $O(N^{(1+d)/(2+d)})$ values of $j \leqslant N$ such that*

$$j_2 \leqslant j_1{}^d. \tag{3.51}$$

For the values of j which we are considering we have

$$j_2 \leqslant N/j_1^2, \quad j_2 \leqslant j_1^d, \tag{3.52}$$

and, associated with any j_1, there are at most

$$\min(N/j_1^2,\ j_1^d)$$

values of j for which (3.52) is true. Hence the number of $j \leqslant N$ such that (3.51) is true is at most

$$\sum_{j_1} \min(Nj_1^{-2}, j_1^d) < C\Big(\sum_{j_1 \leqslant N^{1/(2+d)}} j_1^d + \sum_{j_1 > N^{1/(2+d)}} Nj_1^{-2}\Big)$$
$$< CN^{(1+d)/(2+d)}.$$

We now choose

$$\beta' = \beta + \frac{a\delta}{1+a}, \tag{3.53}$$

so that $0 < \beta' < \beta+\delta = a$, and we put

$$d = \frac{2\beta}{\beta'-\beta} \tag{3.54}$$

in Lemma 7. If (3.51) is not satisfied, we have

$$j_1^{2\beta} < j_2^{\beta'-\beta},$$

$$j_1^2 j_2^{1-\beta'} < (j_1^2 j_2)^{1-\beta} = j^{1-\beta}.$$

If this is true and if

$$j_2 = M_1^2 + M_2^2 + M_3^2, \quad |j_2 - \lambda_i M_i^2| < 4sBj_2^{1-\beta'},$$

we have

$$j = (j_1 M_1)^2 + (j_1 M_2)^2 + (j_1 M_3)^2 = m_1^2 + m_2^2 + m_3^2$$

and $$|j - \lambda_i m_i^2| < 4sBj_1^2 j_2^{1-\beta'} < 4sBj^{1-\beta}.$$

We observe also, with reference to the case $s = 3$, that either j and j_2 are both of the form $4^a(8l+7)$, or neither is of this form, since every odd square is congruent to 1 (mod 8). Hence, if (3.51) is not satisfied, j cannot be exceptional with respect to β if j_2 is not exceptional with respect to β'.

The next step is to determine how many $j \leqslant N$ may have j_2 exceptional with respect to β'. Since

$$j_1 = (j/j_2)^{\frac{1}{2}} \leqslant (N/j_2)^{\frac{1}{2}},$$

there are at most $(N/j_2)^{\frac{1}{2}}$ values of j associated with a particular j_2. Hence the number of $j \leqslant N$ associated with j_2 exceptional with respect to β' is at most

$$N^{\frac{1}{2}} \Sigma j_2^{-\frac{1}{2}},$$

where the summation is extended over all $j_2 \leqslant N$ which are exceptional with respect to β'. But, by Lemma 6, there are at most $CN^{1-\delta'+\epsilon}$ such values of j, and so

$$\Sigma j_2^{-\frac{1}{2}} \leqslant \sum_{j \leqslant CN^{1-\delta'+\epsilon}} j^{-\frac{1}{2}} < CN^{\frac{1}{2}-\frac{1}{2}\delta'+\frac{1}{2}\epsilon}.$$

Hence the number of $j \leqslant N$ which have j_2 exceptional with respect to β' is

$$O(N^{1-\frac{1}{2}\delta'+\frac{1}{2}\epsilon}).$$

These j together with those for which (3.51) is satisfied include all the values of $j \leqslant N$ which are exceptional with respect to β. Hence the total number of $j \leqslant N$ exceptional with respect to β is

$$O(N^{1-\frac{1}{2}\delta'+\frac{1}{2}\epsilon}) + O(N^{(1+d)/(2+d)}),$$

where β' and d are defined by (3.53) and (3.54). We have

$$1-\tfrac{1}{2}\delta' = 1-\tfrac{1}{2}\delta+\tfrac{1}{2}(\beta'-\beta) = 1-\tfrac{1}{2}\delta+\frac{\alpha\delta}{2(1+\alpha)} = 1-\frac{\delta}{2(1+\alpha)},$$

$$\frac{1+d}{2+d} = \frac{\beta'+\beta}{2\beta'} = 1-\frac{\beta'-\beta}{2\beta'} < 1-\frac{\beta'-\beta}{2\alpha} = 1-\frac{\delta}{2(1+\alpha)}.$$

Hence the number of exceptional $j \leqslant N$ is

$$O(N^{1-\{\delta/2(1+\alpha)\}+\epsilon}) = O(N^{1-c}),$$

if we choose $$\epsilon < \frac{\delta}{2(1+\alpha)} - c.$$

To complete the proof of Theorem 3, and hence of Theorems 1 and 2, we have to prove Lemmas 1 and 2. The remainder of the paper is devoted to this.

Proof of Lemma 1.

4.1. We write

$$f_i = f_i(x) = \sum_{N_i' \leqslant h^2 \leqslant N_i} x^{h^2},$$

$$\psi_{\rho,\,i} = \psi_{\rho,\,i}(x) = \frac{S_\rho}{2q} \sum_{h=N_i'}^{N_i} h^{-\frac{1}{2}} \left(\frac{x}{\rho}\right)^h.$$

We use Σ to denote summation over all primitive q-th roots of unity and all q satisfying $1 \leqslant q \leqslant n^{\alpha}$. The coefficients of x^j in $\Pi f_i(x)$ and in $\Pi \psi_{\rho, i}(x)$ are $r(j)$ and $q^{-s} S_{\rho}{}^{s} \rho^{-j}$ respectively, and so we have

$$\begin{aligned} \Pi f_i(x) - \Sigma \Pi \psi_{\rho, i}(x) &= \sum_j \{r(j) - \mu(j) \mathfrak{S}(j, n^{\alpha})\} x^j \\ &= \sum_j \sigma(j) x^j. \end{aligned} \tag{4.11}$$

We take the Farey dissection of order

$$l = [n^{\frac{1}{2}(1-\beta)}]$$

of the circumference of the circle $\mathfrak{K}\,(|x| = 1)$. Then, associated with every point ρ on $\mathfrak{K}$ for which $q \leqslant n^{\frac{1}{2}(1-\beta)}$, we have an arc $\mathfrak{M}$. For each ρ we write

$$x = \rho e^{2\pi i y}, \quad |y| \leqslant \tfrac{1}{2}.$$

Then we know that $$|y| < (ql)^{-1}$$

on the arc $\mathfrak{M}$ associated with ρ; while every point for which

$$|y| \leqslant (2ql)^{-1}$$

certainly lies on this $\mathfrak{M}$. Each arc $\mathfrak{M}$ is divided as follows. The part for which

$$|y| \leqslant q^{-\frac{2}{3}} n^{-\frac{2}{3}(1-\beta)}$$

is denoted by $\mathfrak{M}'$, the remainder, if any, by $\mathfrak{M}''$.

We consider three classes of arcs:

(i) arcs $\mathfrak{M}_1$, being the arcs $\mathfrak{M}'$ for which $1 \leqslant q \leqslant n^{\alpha}$,

(ii) arcs $\mathfrak{M}_2$, being the arcs $\mathfrak{M}''$ for which $1 \leqslant q \leqslant n^{\alpha}$,

(iii) arcs $\mathfrak{M}_3$, being the arcs $\mathfrak{M}$ for which $n^{\alpha} < q \leqslant n^{\frac{1}{2}(1-\beta)}$.

We write

$$J_1 = \Sigma \int_{\mathfrak{M}_1} |\Pi f_i - \Pi \psi_{\rho, i}|^2 \, dy, \qquad J_2 = \Sigma \int_{\mathfrak{M}_2} |\Pi f_i|^2 \, dy,$$

$$J_3 = \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} |\Pi f_i|^2 \, dy, \qquad J_4 = n^{2\alpha} \Sigma \int_{\mathfrak{K} - \mathfrak{M}_1} |\Pi \psi_{\rho, i}|^2 \, dy.$$

From (4.11) we have

$$\sum_j \sigma^2(j) = \frac{1}{2\pi} \int_{\mathfrak{K}} |\Pi f_i - \Sigma \Pi \psi_{\rho, i}|^2 \, |dx|. \tag{4.12}$$

We use Σ' to denote summation over all values of ρ for which $q \leqslant n^{\alpha}$, with the exception of the value corresponding to the particular $\mathfrak{M}$ which we are considering. We use the inequality

$$\left| \sum_{h=1}^{H} \xi_h \right|^2 \leqslant H \sum_{h=1}^{H} | \xi_h |^2$$

in the following work. Then

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\mathfrak{K}} | \Pi f_i - \Sigma \Pi \psi_{\rho, i} |^2 |dx| \\
&= \Sigma \int_{\mathfrak{M}_1} | \Pi f_i - \Sigma \Pi \psi_{\rho, i} |^2 dy + \Sigma \int_{\mathfrak{M}_2} | \Pi f_i - \Sigma \Pi \psi_{\rho, i} |^2 dy \\
&\qquad\qquad + \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} | \Pi f_i - \Sigma \Pi \psi_{\rho, i} |^2 dy \\
&\leqslant 2 \Sigma \int_{\mathfrak{M}_1} | \Pi f_i - \Pi \psi_{\rho, i} |^2 dy + 2 \Sigma \int_{\mathfrak{M}_1} | \Sigma' \Pi \psi_{\rho, i} |^2 dy \\
&\qquad\qquad + 2 \, \Sigma \int_{\mathfrak{M}_2} | \Pi f_i |^2 dy + 2 \Sigma \int_{\mathfrak{M}_2} | \Sigma \Pi \psi_{\rho, i} |^2 dy \\
&\qquad\qquad + 2 \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} | \Pi f_i |^2 dy + 2 \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} | \Sigma \Pi \psi_{\rho, i} |^2 dy \\
&\leqslant 2J_1 + 2n^{2\alpha} \Sigma \int_{\mathfrak{M}_1} \Sigma' | \Pi \psi_{\rho, i} |^2 dy + 2J_2 + 2n^{2\alpha} \Sigma \int_{\mathfrak{M}_2} \Sigma | \Pi \psi_{\rho, i} |^2 dy \\
&\qquad\qquad + 2J_3 + 2n^{2\alpha} \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} \Sigma | \Pi \psi_{\rho, i} |^2 dy \\
&= 2(J_1 + J_2 + J_3) + 2n^{2\alpha} \Sigma \int_{\mathfrak{K} - \mathfrak{M}_1} | \Pi \psi_{\rho, i} |^2 dy \\
&= 2(J_1 + J_2 + J_3 + J_4). \qquad (4.13)
\end{aligned}$$

We now proceed to prove that

$$J_1 + J_2 + J_3 + J_4 < C n^{\omega - \delta + (4s+1)\epsilon}. \qquad (4.14)$$

Then we have, by (4.12), (4.13), and (4.14),

$$\sum_{j} \sigma^2(j) < C n^{\omega - \delta + (4s+1)\epsilon}.$$

Since ϵ is any positive number, this is equivalent to Lemma 1.

4.2. We suppose henceforth that x lies on $\mathfrak{K}$ and that $|y| \leqslant \frac{1}{2}$. The following four lemmas are Lemmas 10–13 of Wright (1).

LEMMA 8. $|\psi_{\rho,i}(x)| < Cn^{-\frac{1}{2}}q^{-\frac{1}{2}+\epsilon}\min(n^{1-\beta}, |y|^{-1})$.

LEMMA 9. $|f_i(x) - \psi_{\rho,i}(x)| < Cq^{\frac{1}{2}+\epsilon}\max(1, n^{1-\beta}|y|)$.

LEMMA 10. *If*

$$H_\rho = |\Pi f_i(x) - \Pi \psi_{\rho,i}(x)|,$$

then, for $|y| \leqslant n^{\beta-1}$,

$$H_\rho < Cq^{s\epsilon}(q^{\frac{1}{2}s} + n^{(\frac{1}{2}-\beta)(s-1)}q^{-\frac{1}{2}(s-2)}),$$

and, for $|y| \geqslant n^{\beta-1}$,

$$H_\rho < Cq^{s\epsilon}(q^{\frac{1}{2}s}n^{s(1-\beta)}|y|^s + q^{1-\frac{1}{2}s}n^{\frac{1}{2}(3-s)-\beta}|y|^{2-s}).$$

LEMMA 11. *On* $\mathfrak{M}''$, $|f_i(x)| < Cn^\epsilon(q|y|)^{-\frac{1}{2}-\epsilon}$.

In the course of the proof of Lemma 19 of Wright (**2**), we obtained the following result.

LEMMA 12. $$\int_{\mathfrak{K}} |f_1 f_2|^2 |dx| < Cn^{1-2\beta+\epsilon}.$$

LEMMA 13. *On* $\mathfrak{M}_3$,

$$|f_i(x)| < Cn^{2\epsilon}(n^{\frac{1}{3}(1-\beta)-\frac{1}{6}\alpha} + n^{\frac{1}{2}-\beta-\frac{1}{2}\alpha}).$$

If x lies on $\mathfrak{M}'$, that is, if $|y| \leqslant q^{-\frac{2}{3}}n^{-\frac{2}{3}(1-\beta)}$, we have, by Lemmas 8 and 9,

$$\begin{aligned} |f_i| \leqslant |f_i - \psi_{\rho,i}| + |\psi_{\rho,i}| &\leqslant Cq^\epsilon(q^{\frac{1}{2}} + q^{\frac{1}{2}}n^{1-\beta}|y| + n^{\frac{1}{2}-\beta}q^{-\frac{1}{2}}) \\ &\leqslant Cn^\epsilon(q^{\frac{1}{2}} + n^{\frac{1}{3}(1-\beta)}q^{-\frac{1}{6}} + n^{\frac{1}{2}-\beta}q^{-\frac{1}{2}}) \\ &\leqslant Cn^\epsilon(n^{\frac{1}{3}(1-\beta)}q^{-\frac{1}{6}} + n^{\frac{1}{2}-\beta}q^{-\frac{1}{2}}), \end{aligned}$$

since $q \leqslant n^{\frac{1}{2}(1-\beta)}$.

On $\mathfrak{M}''$, by Lemma 11,

$$|f_i| < Cn^\epsilon(q|y|)^{-\frac{1}{2}-\epsilon} < Cn^{2\epsilon+\frac{1}{3}(1-\beta)}q^{-\frac{1}{6}}.$$

Hence, on $\mathfrak{M}_3$, $(q > n^\alpha)$,

$$\begin{aligned} |f_i| &< Cn^{2\epsilon}(n^{\frac{1}{3}(1-\beta)}q^{-\frac{1}{6}} + n^{\frac{1}{2}-\beta}q^{-\frac{1}{2}}) \\ &< Cn^{2\epsilon}(n^{\frac{1}{3}(1-\beta)-\frac{1}{6}\alpha} + n^{\frac{1}{2}-\beta-\frac{1}{2}\alpha}). \end{aligned}$$

4.3. We write

$$F_1 = n^{\beta-1}\Sigma q^s, \qquad F_2 = n^{s-2-\beta(2s-3)}\Sigma q^{2-s},$$

$$F_3 = n^{\frac{2}{3}(s-1)(1-\beta)}\Sigma q^{-\frac{1}{3}(s+2)}, \quad F_4 = n^{\frac{1}{3}(s-2)-\frac{2}{3}\beta(2s-1)+2\alpha}\Sigma q^{\frac{1}{3}(s-2)},$$

$$g_1 = \beta-1+\alpha(s+2), \quad g_2 = s-2-\beta(2s-3)+\alpha(4-s),$$

$$g_3 = \tfrac{2}{3}(s-1)(1-\beta)+\tfrac{1}{3}\alpha(4-s), \quad g_4 = \tfrac{1}{3}(s-2)-\tfrac{2}{3}\beta(2s-1)+\tfrac{1}{3}\alpha(s+10),$$

$$g_5 = \tfrac{1}{3}(2s-1)-\tfrac{2}{3}\beta(s+1)-\tfrac{1}{3}\alpha(s-2), \quad g_6 = s-1-2\beta(s-1)-\alpha(s-2).$$

If $k \geqslant -2$,

$$\Sigma q^k \leqslant \sum_{1 \leqslant q \leqslant n^\alpha} q^{k+1} < Cn^{\alpha(k+2)+\epsilon},$$

and so, since $s = 3$ or 4, we have

$$F_r < Cn^{g_r+\epsilon} \quad (r = 1, 2, 3, 4). \tag{4.31}$$

By Lemma 10,

$$\int_{\mathfrak{M}_1} H_\rho{}^2\,dy < Cq^{2s\epsilon}\left\{(q^s+n^{(s-1)(1-2\beta)}q^{2-s})\int_0^{n^{\beta-1}} dy\right.$$

$$\left. + q^s n^{2s(1-\beta)}\int_{n^{\beta-1}}^{q^{-\frac{2}{3}}n^{-\frac{2}{3}(1-\beta)}} y^{2s}\,dy + q^{2-s}n^{3-s-2\beta}\int_{n^{\beta-1}}^{q^{-\frac{2}{3}}n^{-\frac{2}{3}(1-\beta)}} y^{4-2s}\,dy\right\}$$

$$< Cn^{2s\epsilon}\{n^{\beta-1}q^s+n^{s-2-\beta(2s-3)}q^{2-s}+n^{\frac{2}{3}(s-1)(1-\beta)}q^{-\frac{1}{3}(s+2)}\}.$$

Hence, on summing over all the $\mathfrak{M}_1$, we have

$$J_1 < Cn^{2s\epsilon}(F_1+F_2+F_3). \tag{4.32}$$

By Lemma 11,

$$\int_{\mathfrak{M}_2} |\Pi f_i|^2\,dy < Cn^{2s\epsilon}q^{-s-2s\epsilon}\int_{q^{-\frac{2}{3}}n^{-\frac{2}{3}(1-\beta)}}^{\infty} y^{-s-2s\epsilon}\,dy$$

$$< Cn^{2s\epsilon}q^{-s-2s\epsilon}(q^{\frac{2}{3}}n^{\frac{2}{3}(1-\beta)})^{s+2s\epsilon-1}$$

$$< Cn^{4s\epsilon+\frac{2}{3}(1-\beta)(s-1)}q^{-\frac{1}{3}(s+2)}.$$

Hence, on summing over all the $\mathfrak{M}_2$, we have

$$J_2 < Cn^{4s\epsilon}F_3. \tag{4.33}$$

By Lemma 8,

$$\int_{\mathfrak{K}-\mathfrak{M}_1} |\Pi\psi_{\rho,i}|^2\,dy < Cn^{-s}q^{-s+2s\epsilon}\int_{q^{-\frac{2}{3}}n^{-\frac{2}{3}(1-\beta)}}^{\infty} y^{-2s}\,dy$$

$$= Cn^{2s\epsilon-s}q^{-s}(q^{\frac{2}{3}}n^{\frac{2}{3}(1-\beta)})^{2s-1}$$

$$= Cn^{\frac{1}{3}(s-2)-\frac{2}{3}\beta(2s-1)}q^{\frac{1}{3}(s-2)}.$$

Summing with respect to $\mathfrak{M}_1$, we have

$$J_4 < CF_4. \tag{4.34}$$

By Lemmas 12 and 13,

$$\begin{aligned} J_3 &< Cn^{2(2s-4)\epsilon}\left(n^{(2s-4)\{\frac{1}{3}(1-\beta)-\frac{1}{6}\alpha\}}+n^{(2s-4)(\frac{1}{2}-\beta-\frac{1}{2}\alpha)}\right)\int_{\mathfrak{K}} |f_i f_2|^2 \, |dx| \\ &< Cn^{4s\epsilon}(n^{g_5}+n^{g_6}). \end{aligned} \tag{4.35}$$

By (4.31) to (4.35), we have

$$J_1+J_2+J_3+J_4 = Cn^{(4s+1)\epsilon}\sum_{r=1}^{6} n^{g_r}. \tag{4.36}$$

In ω and g_r we now write $\alpha-\delta$ for β. If $s=3$, we have $\alpha=\frac{1}{6}$, and so

$$g_1 = -\delta, \quad g_2 = \tfrac{2}{3}+2\delta, \quad g_3 = \tfrac{7}{6}+\tfrac{4}{3}\delta, \quad g_4 = \tfrac{1}{2}+\tfrac{10}{3}\delta,$$

$$g_5 = \tfrac{7}{6}+\tfrac{8}{3}\delta, \quad g_6 = \tfrac{7}{6}+4\delta, \quad \omega = \tfrac{7}{6}+5\delta.$$

Hence

$$g_r \leqslant \omega-\delta \quad (s=3,\ r=1,\ 2,\ \ldots,\ 6). \tag{4.37}$$

If $s=4$, we have $\alpha=\frac{1}{5}$ and so

$$g_1 = \tfrac{2}{5}-\delta, \quad g_2 = 1+5\delta, \quad g_3 = \tfrac{8}{5}+2\delta, \quad g_4 = \tfrac{2}{3}+\tfrac{14}{3}\delta,$$

$$g_5 = \tfrac{23}{15}+\tfrac{10}{3}\delta, \quad g_6 = \tfrac{7}{5}+6\delta, \quad \omega = \tfrac{8}{5}+7\delta.$$

Hence

$$g_r \leqslant \omega-\delta \quad (s=4,\ r=1,\ 2,\ \ldots,\ 6). \tag{4.38}$$

Now (4.14) follows at once from (4.36), (4.37), and (4.38). This completes the proof of Lemma 1.

Proof of Lemma 2. The singular series.

5.1. When $s=4$, Lemma 2 follows at once from the much stronger result

LEMMA 14. *If $s=4$, $4+j$, $Cn<j<Cn$, and n is sufficiently large, we have*

$$\mathfrak{S}(j,\, n\,) > C(\log\log n)^{-1}.$$

We quote certain results from Kloostermann (1). If we put

$$a=b=c=d=1, \quad \nu_1=\nu_2=\nu_3=\nu_4=0, \quad \mu=q-1,$$

in his Lemma 6, we have

$$|A_q| < Cq^{-\frac{9}{8}+\epsilon}(j,\, q)^{\frac{1}{4}}.$$

Hence, for any J,

$$\begin{aligned}|\mathfrak{S}(j)-\mathfrak{S}(j, J)| &\leqslant C \sum_{q>J} |A_q| \\ &\leqslant C \sum_{q>J} q^{-\frac{9}{8}+\epsilon}(j, q)^{\frac{1}{4}} \\ &\leqslant C \sum_{h|j} h^{-\frac{7}{8}+\epsilon} \sum_{Q>J/h} Q^{-\frac{9}{8}+\epsilon} \\ &< C \sum_{h|j} h^{-\frac{7}{8}+\epsilon}(J/h)^{-\frac{1}{8}+\epsilon} \\ &= CJ^{-\frac{1}{8}+\epsilon} \sum_{h|j} h^{-\frac{3}{4}} < Cj^{\epsilon} J^{-\frac{1}{8}+\epsilon}.\end{aligned}$$

It follows from Kloostermann's results, on pp. 448–453 of his paper [see also Wright (**3**), Lemma 1] that

$$\mathfrak{S}(j) > C(\log \log j)^{-1},$$

provided that $4 \nmid j$. Hence

$$\begin{aligned}\mathfrak{S}(j, n^{a}) &\geqslant \mathfrak{S}(j) - |\mathfrak{S}(j)-\mathfrak{S}(j, n^{a})| \\ &> C(\log \log j)^{-1} - Cj^{\epsilon} n^{-a(\frac{1}{8}-\epsilon)} \\ &> C(\log \log n)^{-1} - Cn^{-\frac{1}{8}a+2\epsilon} \\ &> C(\log \log n)^{-1},\end{aligned}$$

provided that $16\epsilon < a$ and that n is sufficiently large.

5.2. When $s = 3$, we encounter much greater difficulties when we endeavour to deal with the singular series. The series for $\mathfrak{S}(j)$ is convergent, but not absolutely so. It is, of course, possible to evaluate A_q (Hardy, **1**), but I have discovered no way of finding a sufficiently good upper bound for

$$|\mathfrak{S}(j, n^{a}) - \mathfrak{S}(j)|$$

to enable me to use the properties of $\mathfrak{S}(j)$. Such a result must depend on finding an upper bound of the type Cn^{-l} for some positive l for

$$\left| \sum_{m=M+1}^{\infty} \left(\frac{-4n}{m}\right) \frac{1}{m} \right|, \tag{5.21}$$

where M is of the same order of magnitude as $n^{\frac{1}{8}}$. It is easy to prove that this sum is $O(\log n)$, but this is, of course, useless. It is, perhaps, not surprising that we should be unable to handle (5.21) directly. The complete series

$$S(d) = \sum_{m=1}^{\infty} \left(\frac{-d}{m}\right) \frac{1}{m}$$

is of extreme importance in the theory of the class-number and has received a great deal of attention, but it is only recently that Siegel (**1**) has been able to show that $S(d) > Cd^{-\epsilon}$. I use his result below.

We adopt an indirect method to prove Lemma 2 for $s = 3$. Let $r_0(j)$ denote the number of representations of j in the form

$$j = m_1^2 + m_2^2 + m_3^2,$$

where m_1, m_2, and m_3 are each positive or zero, and let $R(j)$ denote the number of such representations in which m_1, m_2, and m_3 are positive, negative, or zero; in each case, representations in which the order is different are to be reckoned as distinct. It is obvious that

$$8r_0(j) \geqslant R(j).$$

We now suppose that j is quadratfrei and not congruent to 7 (mod 8); then

$$j \equiv 1,\ 2,\ 3,\ 5, \text{ or } 6 \pmod{8}.$$

Gauss (**1**) and Dirichlet (**1**) have proved that

$$R(j) = 12h(-4j) \quad \text{when} \quad j \equiv 1,\ 2,\ 5,\ 6 \pmod{8},$$

$$R(j) = 8h(-4j) \quad \text{when} \quad j \equiv 3 \pmod{8},$$

where $h(d)$ is the class number. Siegel (**1**) has shown that $h(-4j) > Cj^{\frac{1}{2}-\epsilon}$, provided that $j > 1$. Hence we have

LEMMA 15. *If j is quadratfrei, if $j > 1$, and if $j \not\equiv 7$ (mod 8), then we have*

$$r_0(j) > Cj^{\frac{1}{2}-\epsilon}. \tag{5.22}$$

In § 6 we shall prove

LEMMA 16. *For all except $O(n^{\frac{5}{6}+10\epsilon})$ of the restricted values of j we have*

$$\left| r_0(j) - \frac{\pi\Gamma(j+\frac{3}{2})}{4\Gamma(j+1)}\, \mathfrak{S}(j, n^{\frac{1}{6}}) \right| < Cj^{\frac{1}{2}-2\epsilon}. \tag{5.23}$$

If (5.22) and (5.23) are both true for a particular j, we have

$$\mathfrak{S}(j, n^{\frac{1}{6}}) > \frac{4\Gamma(j+1)}{\pi\Gamma(j+\frac{3}{2})} \{r_0(j) - Cj^{\frac{1}{2}-2\epsilon}\}$$

$$> Cj^{-\epsilon} - Cj^{-2\epsilon} > Cj^{-\epsilon} > Cj^{-10\epsilon},$$

if j is sufficiently large. Hence, by Lemmas 15 and 16, the number of quadratfrei, restricted values of j not congruent to 7 (mod 8) for which

$$\mathfrak{S}(j, n^{\frac{1}{6}}) < Cj^{-10\epsilon}$$

is $O(n^{\frac{5}{6}+10\epsilon})$. Since ϵ is any small positive number and $\alpha = \frac{1}{6}$ when $s = 3$, this is equivalent to Lemma 2.

Proof of Lemma 16.

6.1. We write

$$f = f(x) = \sum_{j=0}^{n} x^{j^2},$$

$$\psi = \psi_\rho(x) = \Gamma(\tfrac{3}{2}) \frac{S_\rho}{q} \left\{1 + \sum_{j=1}^{n} \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots(\frac{1}{2}+j-1)}{j!} \left(\frac{x}{\rho}\right)^j\right\}.$$

Hence $$f^3(x) = 1 + \sum_{j=1}^{n} r_0(j)\, x^j + \text{higher powers of } x,$$

$$\psi_\rho{}^3(x) = \{\Gamma(\tfrac{3}{2})\}^2 \frac{S_\rho{}^3}{q^3} \left\{1 + \sum_{j=1}^{n} \frac{\Gamma(j+\frac{3}{2})}{\Gamma(j+1)} \left(\frac{x}{\rho}\right)^j + \text{higher powers of } x\right\}.$$

Henceforth we use Σ to denote summation over all those ρ for which $q \leqslant n^{\frac{1}{6}}$. We write

$$\rho_0(j) = \{\Gamma(\tfrac{3}{2})\}^2 \frac{\Gamma(j+\frac{3}{2})}{\Gamma(j+1)} \mathfrak{S}(j, n^{\frac{1}{6}}),$$

$$r_0(j) = \rho_0(j) + \sigma_0(j),$$

so that

$$f^3(x) - \Sigma\, \psi_\rho{}^3(x) = 1 + \sum_{j=1}^{n} \sigma_0(j)\, x^j + \text{higher powers of } x,$$

$$\sum_{j=1}^{n} |\sigma_0(j)|^2 \leqslant \frac{1}{2\pi} \int_{\mathfrak{K}} |f^3(x) - \Sigma\, \psi_\rho{}^3(x)|^2 |dx|.$$

We use the Farey dissection of $\mathfrak{K}$ of order $[n^{\frac{1}{2}}]$; the new definition of our arcs $\mathfrak{M}$, $\mathfrak{M}'$, $\mathfrak{M}''$, $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ is obtained by putting $\beta = 0$ in the definitions of § 4.1. We take x on $\mathfrak{K}$, write $x = \rho e^{2\pi i y}$ and take $|y| \leqslant \frac{1}{2}$.

6.2. We shall make use of certain results due to Gelbke (**1**). Throughout his work he is concerned with k-th powers, where $k > 2$, while we, of course, require $k = 2$. However, certain of Gelbke's lemmas remain true when $k = 2$ and the proof is exactly the same. We have then

LEMMA 17. (Gelbke, Satz 8.)

$$|\psi_\rho(x)| < Cq^{-\frac{1}{2}+\epsilon} \min (n^{\frac{1}{2}}, |y|^{-\frac{1}{2}}).$$

LEMMA 18. (Gelbke, Satz 12.)

$$|f(x)-\psi_\rho(x)| < Cq^{\frac{1}{2}+\epsilon} \max (n|y|, 1).$$

LEMMA 19. $$|f^3(x)-\psi_\rho{}^3(x)| < Cq^{3\epsilon}(nq^{-\frac{1}{2}}+qn^3|y|^3).$$

If we consider separately the cases $n|y| \leqslant 1$ and $n|y| > 1$, this is an immediate deduction from Lemmas 17 and 18 and the inequalities $q \leqslant n^{\frac{1}{2}}$ and

$$|f^3-\psi_\rho{}^3| < C\{|\psi_\rho{}^2(f-\psi_\rho)|+|f-\psi_\rho|^3\}.$$

LEMMA 20. *On* $\mathfrak{M}''$,

$$|f(x)| < Cn^\epsilon (q|y|)^{-\frac{1}{2}-\epsilon}.$$

If we put $\beta = 0$, $N_i' = 0$, $N_i = n$ in the proof of Lemma 13 of Wright (1), we have a proof of this lemma.

LEMMA 21. *On* $\mathfrak{M}_3$, $\qquad |f(x)| < Cn^{\frac{5}{12}+2\epsilon}$.

We have $n^{\frac{1}{3}} < q \leqslant n^{\frac{1}{2}}$ and

$$|f(x)| \leqslant |f-\psi_\rho|+|\psi_\rho|.$$

Hence, by Lemmas 17 and 18, if $|y| \leqslant n^{-1}$,

$$|f(x)| < Cq^\epsilon (q^{\frac{1}{2}}+q^{-\frac{1}{2}} n^{\frac{1}{2}}) < Cn^{\frac{5}{12}+\epsilon},$$

and, if $n^{-1} \leqslant |y| \leqslant q^{-\frac{2}{3}} n^{-\frac{2}{3}}$,

$$|f(x)| < Cq^\epsilon (q^{\frac{1}{2}} n|y|+q^{-\frac{1}{2}}|y|^{-\frac{1}{2}})$$
$$< Cq^\epsilon (q^{-\frac{1}{6}} n^{\frac{1}{3}}+q^{-\frac{1}{2}+\epsilon} n^{\frac{1}{2}}) < Cn^{\frac{5}{12}+2\epsilon}.$$

Our result is therefore true if x lies on $\mathfrak{M}'$. But, by Lemma 20, on $\mathfrak{M}''$,

$$|f(x)| < Cn^\epsilon (q|y|)^{-\frac{1}{2}-\epsilon} \leqslant Cn^\epsilon (q^{\frac{1}{3}} n^{-\frac{2}{3}})^{-\frac{1}{2}-\epsilon}$$
$$= Cn^{2\epsilon} q^{-\frac{1}{6}} n^{\frac{1}{3}} < Cn^{\frac{5}{12}+2\epsilon}.$$

LEMMA 22. $$\int_{\mathfrak{R}} |f|^4 |dx| < Cn^{1+\epsilon}.$$

If $r_0'(j)$ is the number of solutions of

$$m_1^2+m_2^2=j, \quad 0 \leqslant m_1 \leqslant n, \quad 0 \leqslant m_2 \leqslant n,$$

then

$$r_0'(j) < Cj^{\epsilon},$$

$$f^2(x) = 1+\sum_{j=1}^{2n} r_0'(j)\, x^j.$$

Hence

$$1+\sum_{j=1}^{2n} r_0'(j) = f^2(1) \leqslant n,$$

and so

$$\int_{\mathfrak{K}} |f|^4 |dx| = 2\pi \left\{1+\sum_{j=1}^{2n} \{r_0'(j)\}^2\right\} < Cn^{1+\epsilon}.$$

6.3. By Lemma 19,

$$\int_{\mathfrak{M}_1} |f^3-\psi_\rho{}^3|^2\, dy < Cn^{3\epsilon}\left(n^2 q^{-1}\int_0^{q^{-\frac{2}{3}}n^{-\frac{2}{3}}} dy + q^3 n^6 \int_0^{q^{-\frac{2}{3}}n^{-\frac{2}{3}}} y^6\, dy\right)$$

$$< Cn^{3\epsilon+\frac{4}{3}} q^{-\frac{5}{3}}.$$

Hence

$$J_1' = \Sigma \int_{\mathfrak{M}_1} |f^3-\psi_\rho{}^3|^2\, dy < Cn^{\frac{4}{3}+3\epsilon}\, \Sigma\, q^{-\frac{5}{3}}. \tag{6.31}$$

By Lemma 20,

$$\int_{\mathfrak{M}_1} |f|^6\, dy < Cn^{6\epsilon} q^{-3-6\epsilon} \int_{q^{-\frac{2}{3}}n^{-\frac{2}{3}}} y^{-3-6\epsilon}\, dy$$

$$= Cn^{6\epsilon} q^{-3-6\epsilon} (q^{\frac{2}{3}} n^{\frac{2}{3}})^{2+6\epsilon}$$

$$= Cn^{\frac{4}{3}+10\epsilon} q^{-\frac{5}{3}-2\epsilon}.$$

Hence

$$J_2 = \Sigma \int_{\mathfrak{M}_2} |f|^6\, dy < Cn^{\frac{4}{3}+10\epsilon}\, \Sigma\, q^{-\frac{5}{3}}. \tag{6.32}$$

By Lemma 17,

$$\int_{\mathfrak{K}-\mathfrak{M}_1} |\psi_\rho|^6\, dy < Cq^{-3+6\epsilon} \int_{q^{-\frac{2}{3}}n^{-\frac{2}{3}}}^{\infty} y^{-3}\, dy < Cn^{\frac{4}{3}} q^{-\frac{5}{3}+6\epsilon}.$$

Hence

$$J_4' = n^{\frac{1}{3}}\, \Sigma \int_{\mathfrak{K}-\mathfrak{M}_1} |\psi_\rho|^6\, dy < Cn^{\frac{5}{3}+6\epsilon}\, \Sigma\, q^{-\frac{5}{3}}. \tag{6.33}$$

Now
$$\Sigma q^{-\frac{5}{3}} \leqslant \sum_{q \leqslant n^{\frac{1}{6}}} q^{-\frac{2}{3}} < Cn^{\frac{1}{18}}.$$

Hence, by (6.31), (6.32), and (6.33),

$$J_1' + J_2' + J_4' < Cn^{\frac{5}{3}+\frac{1}{18}+10\epsilon} < Cn^{\frac{11}{6}}$$

for sufficiently small ϵ. Also, by Lemmas 21 and 22,

$$J_3' = \sum_{\mathfrak{M}_3} \int_{\mathfrak{M}_3} |f|^6\, dy < Cn^{\frac{5}{6}+4\epsilon} \int_{\mathfrak{K}} |f|^4\, dy$$
$$< Cn^{\frac{11}{6}+5\epsilon}.$$

Hence, by calculations similar to those which led to (4.12), we have

$$\sum_{j=1}^{n} \sigma_0^2(j) < C(J_1' + J_2' + J_3' + J_4') < Cn^{\frac{11}{6}+5\epsilon}. \tag{6.34}$$

Since $\{\Gamma(\frac{3}{2})\}^2 = \frac{1}{4}\pi$, (5.23) is true unless

$$|\sigma_0(j)| > Cj^{\frac{1}{2}-2\epsilon},$$

that is, if j is restricted, unless

$$|\sigma_0(j)| > Cn^{\frac{1}{2}-2\epsilon}.$$

If the last inequality is satisfied for more than $Cn^{\frac{5}{6}+10\epsilon}$ restricted values of j, we have

$$\sum_{j=1}^{n} \sigma_0^2(j) > Cn^{\frac{11}{6}+6\epsilon},$$

which contradicts (6.34). Lemma 16 is therefore true.

Results for five or more squares.

7. The following theorem can be proved by methods similar to those of this paper.

THEOREM 4. *If $s \geqslant 5$, then almost all n can be represented in the form*

$$n = m_1^2 + \ldots + m_s^2$$

with
$$|m_i^2 - \lambda_i n| = O(n^{1-\beta}),$$

provided that
$$\beta < \frac{s-1}{4s-1}.$$

References.

P. G. L. Dirichlet, **1**, *Abh. Akad. Wiss. Berlin* (1849), Math., 82–83; *Werke*, 2, 65–66.

C. F. Gauss, **1**, *Disquisitiones Arithmeticae* (1801), Art. 291; *Werke*, 1.

M. Gelbke, **1**, " Zum Waringschen Problem ", *Math. Annalen*, 105 (1931), 637–652.

G. H. Hardy, **1**, " On the representation of a number as a sum of any number of squares, and in particular of five ", *Trans. American Math. Soc.*, 21 (1920), 255–284.

H. D. Kloostermann, **1**, " On the representation of numbers in the form $ax^2+by^2+cz^2+dt^2$ ", *Acta Math.*, 49 (1926), 407–464.

E. Landau, **1**, *Vorlesungen über Zahlentheorie*, 1 (Leipzig, 1927).

C. L. Siegel, **1**, " Über die Classanzahl quadratischer Zahlkörper ", *Acta Arithmetica*, 1 (1935), 83–86.

E. M. Wright, **1**, " The representation of a number as a sum of five or more squares ", *Quart. J. of Math.* (Oxford), 4 (1933), 37–51.

————, **2**, " The representation of a number as a sum of five or more squares (second paper) ", *ibid.*, 4 (1933), 228–232.

————, **3**, " The representation of a number as a sum of four ' almost proportional ' squares ", *ibid.*, 7 (1936), 230–240.

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PROOF THAT EVERY LARGE INTEGER IS THE SUM OF TWO PRIMES AND A SQUARE

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[Received 16 June, 1936.—Read 18 June, 1936.]

1. *Introduction.*

The object of this paper was partly attained by Miss Stanley*, but her result depends on an unproved hypothesis concerning Dirichlet's L functions, namely that none of them has a zero whose real part is greater than $\frac{1}{2}$.

The method used here, an adaptation of the Hardy-Littlewood method, seems applicable to several other problems, *e.g.* those of obtaining asymptotic formulae for the numbers of representations of n as the sum of

(i) r squares and s primes† whenever $r+2s>4$,

(ii) two primes and a k-th power,

(iii) one prime, two squares, and a k-th power.

2. In what follows, h and m denote integers, k, l, n, and q positive integers, p a prime, x, y, u, σ, t, and θ real numbers, and w, τ, z, and s complex numbers. As usual,

$$s=\sigma+it. \tag{2.1}$$

A_1, A_2, ... are suitable (sufficiently large) positive absolute constants.

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* "On the representation of a number as a sum of squares and primes", *Proc. London Math. Soc.* (2), 29 (1929), 122–144, especially Theorem G.

† This problem was solved for $r=4$, $s=1$ by Chowla, *Acta Arithmetica*, 1 (1935), 115–122, and for $r \geqslant 5$ by Walfisz, *ibid.*, 123–132.

Let

$$\nu(n) = \sum_{p_1+p_2+m^2=n} \log p_1 \log p_2, \tag{2.2}$$

$$f_1(\tau) = \sum_p e^{2\pi i p\tau} \log p, \tag{2.3}$$

$$f_2(\tau) = \sum_{m=-\infty}^{\infty} e^{2\pi i m^2 \tau}, \tag{2.4}$$

and

$$f_3(\tau) = f_1^2(\tau) f_2(\tau), \tag{2.5}$$

where

$$\mathfrak{I}\tau > 0. \tag{2.6}$$

Then

$$\nu(n) = \int_{\tau_0}^{\tau_0+1} f_3(\tau)\, e^{-2\pi i n\tau}\, d\tau, \tag{2.7}$$

where τ_0 is any number in the upper half-plane.

We have to show that $\nu(n) \neq 0$ for any sufficiently large n.

3. Henceforth suppose that

$$n > A_1, \tag{3.0}$$

and let

$$u = e^{\log^2 \log n}. \tag{3.1}$$

(3.2). Let $I(h, k)$ denote the interval

$$\frac{h}{k} - \frac{u}{nk} < \theta < \frac{h}{k} + \frac{u}{nk}. \tag{3.20}$$

Then it follows from (3.0) and (3.1) that the intervals $I(h, k)$ corresponding to values of h and k such that

$$k \leqslant u \tag{3.3}$$

and

$$(h, k) = 1 \tag{3.4}$$

are non-overlapping. Take

$$\tau_0 = \frac{u}{n} + \frac{i}{n}, \tag{3.5}$$

so that $\Re\tau_0$ is the right-hand end-point of $I(0,\,1)$. The interval

$$u/n < \theta < u/n+1$$

contains those intervals $I(h,\,k)$ satisfying (3.3) and (3.4) for which

$$0 < h \leqslant k, \tag{3.6}$$

and no point of any other. Hence, by (2.7) and (3.5),

$$\begin{aligned} \nu(n) &= \int_{u/n}^{u/n+1} f_3\left(\frac{i}{n}+\theta\right) e^{2\pi(1-in\theta)}\,d\theta \\ &= \sum_{k\leqslant u}\sum_{h}{}' \int_{I(h,\,k)} f_3\left(\frac{i}{n}+\theta\right) e^{2\pi(1-in\theta)}\,d\theta \\ &\qquad\qquad + \int_E f_3\left(\frac{i}{n}+\theta\right) e^{2\pi(1-in\theta)}\,d\theta, \end{aligned} \tag{3.7}$$

where the dash indicates that h runs through the values satisfying (3.4) and (3.6), and E is the set of those points of the interval $(0,\,1)$ which do not lie in any of the intervals $I(h,\,k)$ satisfying (3.3) and (3.4). From this and (2.3) it follows that

$$\begin{aligned} \int_E \left| f_1^2\left(\frac{i}{n}+\theta\right)\right| d\theta &\leqslant \int_0^1 \left| f_1^2\left(\frac{i}{n}+\theta\right)\right| d\theta \\ &= \sum_p e^{-4\pi p/n} \log^2 p < A_2 n \log n. \end{aligned} \tag{3.8}$$

4. We now investigate $f_2(i/n+\theta)$ for θ in E.

It is known that, corresponding to any θ, there are numbers h and k satisfying (3.4) and the conditions

$$k \leqslant \sqrt{n} \tag{4.1}$$

and

$$\left| 2\theta - \frac{h}{k} \right| \leqslant \frac{1}{k\sqrt{n}}. \tag{4.2}$$

Now there are integers a, b such that

$$ah+bk=-1, \tag{4.3}$$

and it follows from the theory of the theta functions that

$$f_2(\tau) = \vartheta_3(0\,|\,2\tau) = \zeta\vartheta_q\left(0\,\middle|\,\frac{2a\tau+b}{2k\tau-h}\right)(2k\tau-h)^{-\frac{1}{2}}, \tag{4.4}$$

where ζ is an eighth root of unity, and q is 0, 2, or 3. Taking

$$\tau = \frac{i}{n} + \theta,$$

we have, by (4.3),

$$\frac{2a\tau+b}{2k\tau-h} - \frac{a}{k} = \frac{-1}{k(2k\tau-h)} = \frac{-n}{k(2kn\theta-nh+2ki)},$$

and hence, by (4.1) and (4.2),

$$\mathfrak{I}\,\frac{2a\tau+b}{2k\tau-h} = \frac{2n}{n^2(2k\theta-h)^2+4k^2} \geqslant \tfrac{2}{5}.$$

Now the three functions $\vartheta_q(0\,|\,\tau)$ $(q=0,\,2,\,3)$ are bounded for $\mathfrak{I}\tau \geqslant \frac{2}{5}$. Hence, by (4.4),

$$\left| f_2\left(\frac{i}{n}+\theta\right) \right| \leqslant A_3 \left| 2k\theta - h + 2k\,\frac{i}{n} \right|^{-\frac{1}{2}}. \tag{4.5}$$

If θ is in E, it is not in any interval $I(h',\,k')$, where h' and k' satisfy (3.3) and (3.4). It easily follows that θ is not even in any interval $I(h',k')$, where $k' \leqslant u$; for, if it were, it would also be in $I(h'',\,k'')$, where $h'' = h'/(h',k')$ and $k'' = k'/(h',\,k')$, and these would satisfy (3.3) and (3.4). In particular, θ cannot be in $I(h,\,2k)$ if $2k \leqslant u$. From this and (3.2) it follows that we have at least one of the inequalities $2k > u$ and

$$|2k\theta - h| \geqslant \frac{u}{n}.$$

Either of these implies that $|2k\theta - h + 2ki/n| \geqslant u/n$. Hence, by (4.5),

$$\left| f_2\left(\frac{i}{n}+\theta\right) \right| \leqslant A_3 n^{\frac{1}{2}} u^{-\frac{1}{2}} \quad (\theta \text{ in } E), \tag{4.6}$$

and hence, by (2.5) and (3.8),

$$\left| \int_E f_3\left(\frac{i}{n}+\theta\right) e^{2\pi(1-in\theta)}\, d\theta \right| \leqslant A_4 n^{\frac{3}{2}} u^{-\frac{1}{2}} \log n. \tag{4.7}$$

5. Putting

$$\xi_k = e^{2\pi i/k}, \tag{5.0}$$

we have, by (3.2),

$$\int_{I(h,\,k)} f_3\left(\frac{i}{n}+\theta\right) e^{2\pi(1-in\theta)}\, d\theta = e^{2\pi}\, \xi_k^{-nh}\, J(h,\,k), \tag{5.1}$$

where

$$(5.2) \qquad J(h, k) = \int_{-u/(nk)}^{u/(nk)} f_3\left(\frac{i}{n}+\frac{h}{k}+y\right) e^{-2\pi i n y}\, dy.$$

Hence, by (3.7) and (4.7),

$$(5.3) \qquad \left| \nu(n) - e^{2\pi} \sum_{k \leqslant u} \sum_{h}{}' \xi_k^{-nh} J(h, k) \right| \leqslant A_4 n^{\frac{3}{2}} u^{-\frac{1}{2}} \log n.$$

Now let

$$(5.4) \qquad z = n^{-1} - iy,$$

and put

$$(5.5) \qquad f_4(z) = f_1(h/k+iz), \quad f_5(z) = f_2(h/k+iz),$$

so that, by (2.5),

$$(5.6) \qquad f_3\left(\frac{i}{n}+\frac{h}{k}+y\right) = f_4^2(z) f_5(z).$$

6. We now investigate $f_4(z)$, assuming that h and k satisfy (3.3) and (3.4), and that

$$(6.1) \qquad -u(nk)^{-1} \leqslant y \leqslant u(nk)^{-1}.$$

By (5.4), (3.3), and (6.1),

$$(6.2) \qquad k|z| \leqslant 2un^{-1}.$$

(6.3). Let $\Lambda(q)$ be defined as 0 if q is not a power of a prime, and let $\Lambda(p^l) = \log p$.

Let

$$(6.4) \qquad f_6(z) = \sum_{\substack{q \\ (q,\, k)=1}} \xi_k^{hq} \Lambda(q) e^{-2\pi z q}.$$

Then it follows from (5.5), (2.3), and (5.0) that

$$f_6(z) - f_4(z) = -\sum_{p|k} \xi_k^{hp} e^{-2\pi z p} \log p + \sum_{p \nmid k} \log p \sum_{l=2}^{\infty} \xi_k^{hp^l} e^{-2\pi z p^l}.$$

Hence, by (5.4), (3.3), and (3.1),

$$(6.5) \qquad |f_6(z) - f_4(z)| \leqslant \sum_{p|k} \log p + \sum_{l=2}^{\infty} \sum_{p} e^{-2\pi n^{-1} p^l} \log p$$

$$\leqslant \log k + A_5 n^{\frac{1}{2}} \leqslant A_6 n^{\frac{1}{2}}.$$

7. It follows from Mellin's integral formula that

$$(7.1) \qquad e^{-2\pi zq} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)\,(2\pi zq)^{-s}\,ds.$$

Hence, by (6.4),

$$(7.2) \qquad f_6(z) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)\,(2\pi z)^{-s} f_7(s)\,ds,$$

where

$$(7.3) \qquad f_7(s) = \sum_{\substack{q \\ (q,\,k)=1}} \xi_k^{hq}\,\Lambda(q)\,q^{-s}.$$

Let χ denote a character of residues mod k, and let

$$(7.4) \qquad B(\xi_k^h, \chi) = \sum_{m=1}^{k} \chi(m)\,\xi_k^{hm}.$$

Then

$$(7.5) \qquad \xi_k^{hq} = \frac{1}{\phi(k)}\sum_{\chi} \chi(q)\,B(\xi_k^h, \bar{\chi}) \quad [(q,\,k)=1],$$

where ϕ is Euler's function, χ runs through all characters mod k, and $\bar{\chi}$ is the character conjugate to χ. By (7.3) and (7.5),

$$(7.6) \qquad f_7(s) = \frac{1}{\phi(k)}\sum_{\chi} B(\xi_k^h, \bar{\chi})\sum_{q} \chi(q)\,\Lambda(q)\,q^{-s}$$

$$= -\frac{1}{\phi(k)}\sum_{\chi} B(\xi_k^h, \bar{\chi})\,\frac{L'}{L}(s, \chi),$$

where $L(s, \chi)$ is Dirichlet's L function.

8. In what follows, a_1 is a suitable positive absolute constant. (The assertions involving a_1 will be true if a_1 is sufficiently small.)

Let D be the set of those points $s = \sigma + it$ for which

$$\sigma \geqslant 1 - \frac{a_1}{\log u}, \quad |t| \leqslant u^2.$$

Then we have the following results*:—

8.1. If $k \leqslant u$ and χ is a character mod k, and there is a number

$$\rho \geqslant 1 - \frac{2a_1}{\log u}$$

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* These results are implied, for instance, in the lemmas in Page's paper "On the number of primes in an arithmetic progression", *Proc. London Math. Soc.* (2), 39 (1935), 116–141, where further references are given.

such that $L(\rho, \chi)=0$, let us call the function $L(s, \chi)$ an exceptional L function and the number ρ an exceptional zero (with reference, of course, to the number u). Then there is at most one exceptional zero and at most one exceptional L function formed with a primitive character. If this function is $L(s, \chi_0)$, where χ_0 is a primitive character mod k_0, then the only exceptional L functions are those formed with characters equivalent to χ_0, two characters being said to be equivalent if they are equal whenever neither of them is 0.

8.2. If $k \leqslant u$, χ is a non-principal character mod k, and $L(s, \chi)$ is not exceptional, then $L'/L\ (s, \chi)$ is regular in D, and

$$\left|\frac{L'}{L}(s, \chi)\right| \leqslant A_7 \log^3 u \qquad (s \text{ in } D).$$

8.3. If $k \leqslant u$, and χ_1 is the principal character mod k, then

$$\frac{L'}{L}(s, \chi_1)+\frac{1}{s-1}$$

is regular in D, and

$$\left|\frac{L'}{L}(s, \chi_1)+\frac{1}{s-1}\right| \leqslant A_7 \log^3 u \quad (s \text{ in } D).$$

8.4. If $L(s, \chi)$ is exceptional, and ρ is the exceptional zero, then

$$\frac{L'}{L}(s, \chi)-\frac{1}{s-\rho}$$

is regular in D, and

$$\left|\frac{L'}{L}(s, \chi)-\frac{1}{s-\rho}\right| \leqslant A_7 \log^3 u \quad (s \text{ in } D).$$

8.5. Any real zero ρ of an L function formed with a character mod k $(k>1)$ satisfies the inequality

$$\rho \leqslant 1-\frac{a_1}{k^{\frac{1}{2}} \log^2 k}.$$

8.6. The exceptional L functions are formed with *real* characters.

8.7. If $k \leqslant u$, and χ is any character mod k, then

$$\left|\frac{L'}{L}(s, \chi)\right| \leqslant A_7 \log^3 |t| \quad (\sigma \geqslant 1,\ |t| \geqslant u^2).$$

9. Let us first consider a value of k such that none of the L functions formed with characters mod k is exceptional.

Let χ_1 be the principal character mod k.

It is known* that

$$B(\xi_k{}^h, \chi_1) = \mu(k), \tag{9.0}$$

and, for any character χ mod k,

$$|B(\xi_k{}^h, \chi)| \leqslant k^{\frac{1}{2}}. \tag{9.1}$$

Also

$$|\Gamma(s)| \leqslant A_8 e^{-\frac{1}{2}\pi|t|}(1+|t|)^{\sigma-\frac{1}{2}} \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 2), \tag{9.2}$$

and, by (5.4) and (6.2),

$$|\operatorname{am} z| = \tfrac{1}{2}\pi - \operatorname{arc\,sin}(n^{-1}|z|^{-1}) \leqslant \tfrac{1}{2}\pi - n^{-1}|z|^{-1} \leqslant \tfrac{1}{2}\pi - \tfrac{1}{2}u^{-1}. \tag{9.3}$$

By (9.2) and (9.3),

$$|\Gamma(s)(2\pi z)^{-s}| \leqslant A_8 |z|^{-\sigma} e^{-\frac{1}{2}u^{-1}|t|}(1+|t|)^{\sigma-\frac{1}{2}} \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 2). \tag{9.4}$$

Let

$$\sigma_0 = 1 - \frac{a_1}{\log u}, \tag{9.5}$$

and let C be the contour consisting of the straight lines from $1-i\infty$ to $1-iu^2$, thence to σ_0-iu^2, thence to σ_0+iu^2, thence to $1+iu^2$, and thence to $1+i\infty$.

Put

$$f_8(s) = \Gamma(s)(2\pi z)^{-s}\left(f_7(s) + \frac{\mu(k)}{\phi(k)(s-1)}\right). \tag{9.6}$$

Then, by (7.6), 8.2, 8.3, and (9.0), $f_8(s)$ is regular in D. It is also regular in the half-plane $\sigma \geqslant 1$, and, by (7.6), 8.7, and (9.4), it tends to zero uniformly in the strip $1 \leqslant \sigma \leqslant 2$ as $t \to \pm\infty$. Hence

$$\int_{2-i\infty}^{2+i\infty} f_8(s)\,ds = \int_C f_8(s)\,ds. \tag{9.7}$$

Also, by Mellin's integral formula,

$$\int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s)(2\pi z)^{-s}}{s-1}\,ds = \int_{2-i\infty}^{2+i\infty} \Gamma(s-1)(2\pi z)^{-s}\,ds$$

$$= (2\pi z)^{-1}\int_{1-i\infty}^{1+i\infty} \Gamma(s')(2\pi z)^{-s'}\,ds' = iz^{-1}e^{-2\pi z},$$

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* Landau, *Vorlesungen über Zahlentheorie*, 1 (1927), 189 (Theorem 222), and 3 (1927), 333.

so that (since $|z| < 1$)

$$\left| \int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s)(2\pi z)^{-s}}{s-1}\, ds - iz^{-1} \right| < A_9. \tag{9.8}$$

By (7.2), (9.6), (9.7), and (9.8),

$$\left| f_6(z) - \frac{1}{2\pi i} \int_C f_8(s)\, ds - \frac{\mu(k)}{\phi(k)\, 2\pi z} \right| < \frac{A_9}{\phi(k)}. \tag{9.9}$$

10. We have, by the definition of C (§ 9),

$$\int_C f_8(s)\, ds = \sum_{m=1}^{6} I_m, \tag{10.0}$$

where

$$I_1 = \int_{1-i\infty}^{1-iu^2}, \quad I_2 = \int_{1-iu^2}^{\sigma_0 - iu^2}, \quad I_3 = \int_{\sigma_0-iu^2}^{\sigma_0}, \tag{10.1}$$

$$I_4 = \int_{\sigma_0}^{\sigma_0+iu^2}, \quad I_5 = \int_{\sigma_0+iu^2}^{1+iu^2}, \quad I_6 = \int_{1+iu^2}^{1+i\infty},$$

the integrand being $f_8(s)$ throughout.

By (7.6), (9.1), and 8.7,

$$|f_7(1+it)| \leqslant A_7 k^{\frac{1}{2}} \log^3 t \leqslant A_{10} k^{\frac{1}{2}} t^{\frac{1}{2}} \quad (t \geqslant u^2).$$

Hence, by (9.6) and (9.4),

$$|f_8(1+it)| \leqslant A_{11} |z|^{-1} k^{\frac{1}{2}} t e^{-\frac{1}{2}u^{-1}t} \quad (t \geqslant u^2),$$

and hence, by (10.1), (5.4), and (3.3),

$$|I_6| \leqslant A_{11} |z|^{-1} k^{\frac{1}{2}} \int_{u^2}^{\infty} t e^{-\frac{1}{2}u^{-1}t}\, dt \tag{10.2}$$

$$= 2A_{11} |z|^{-1} k^{\frac{1}{2}} u^2 (u+2) e^{-\frac{1}{2}u} < A_{12} n u^{-2}.$$

By (7.6) and (9.0),

$$-\phi(k) f_7(s) + \frac{\mu(k)}{s-1} = \mu(k) \left(\frac{L'}{L}(s, \chi_1) + \frac{1}{s-1} \right) + \sum_{\chi \neq \chi_1} B(\xi_k^h, \bar{\chi}) \frac{L'}{L}(s, \chi).$$

Hence, by 8.3, (9.1), and 8.2,

$$\left| f_7(s) - \frac{\mu(k)}{\phi(k)(s-1)} \right| \leqslant A_7 k^{\frac{1}{2}} \log^3 u \quad (s \text{ in } D),$$

and hence, by (9.6), (9.4), (3.3), and (5.4),

$$(10.3) \qquad |f_8(s)| \leqslant A_{13} k^{\frac{1}{2}} \log^3 u \,.\, |z|^{-\sigma} e^{-\frac{1}{2}u^{-1}|t|} (1+|t|)^{\sigma-\frac{1}{2}}$$

$$\leqslant A_{14} u^2 n^{\sigma} e^{-\frac{1}{2}u^{-1}|t|} \qquad (s \text{ on } C, \ |t| \leqslant u^2).$$

In particular, if $\sigma_0 \leqslant \sigma \leqslant 1$, it follows from (10.3) that

$$|f_8(\sigma+iu^2)| \leqslant A_{14} u^2 n e^{-\frac{1}{2}u}.$$

Hence, by (10.1) and (9.5),

$$(10.4) \qquad |I_5| < A_{15} n u^{-2}.$$

By (10.1), (10.3), (9.5), and (3.1),

$$(10.5) \qquad |I_4| \leqslant A_{14} u^4 n^{\sigma_0} = A_{14} u^4 n e^{-a_1 \log n/\log^2 \log n}$$

$$\leqslant A_{16} n u^{-2}.$$

By (10.2), (10.4), and (10.5),

$$|I_4+I_5+I_6| < A_{17} n u^{-2},$$

and similarly

$$|I_1+I_2+I_3| < A_{17} n u^{-2}.$$

Hence, by (10.0),

$$(10.6) \qquad \left| \int_C f_8(s)\, ds \right| < 2A_{17} n u^{-2}.$$

By (6.5), (9.9), and (10.6),

$$(10.7) \qquad \left| f_4(z) - \frac{\mu(k)}{\phi(k)\, 2\pi z} \right| < A_{18} n u^{-2}.$$

From this and (6.2) it follows that

$$(10.8) \qquad |f_4(z)| < \frac{A_{19}}{\phi(k)\,|z|}.$$

11. To obtain an inequality for $f_5(z)$ on the assumption that (3.3), (3.4), (5.4), and (6.1) are satisfied, we note that (3.4), (4.1), and (4.2) imply (4.5). Hence the conditions

$$(11.0) \qquad (h', k') = 1,$$

$$(11.1) \qquad k' \leqslant \sqrt{n},$$

and

(11.2) $$\left|2\theta-\frac{h'}{k'}\right| \leqslant \frac{1}{k'\sqrt{n}}$$

imply

(11.3) $$\left|f_2\left(\frac{i}{n}+\theta\right)\right| \leqslant A_3\left|2k'\theta-h'+2k'\frac{i}{n}\right|^{-\frac{1}{2}}.$$

If k is odd, put $h'=2h$, $k'=k$, and if k is even, put $h'=h$, $k'=\frac{1}{2}k$. Then, in either case,

(11.4) $$\frac{h'}{k'}=2\frac{h}{k},$$

and it follows from (3.4) that (11.0) is satisfied, and from (3.3), in view of (3.0) and (3.1), that (11.1) is also satisfied. Now take

(11.5) $$\theta=\frac{h}{k}+y.$$

Then it follows from (6.1), (3.1), and (11.4) that (11.2) is also satisfied. Hence we have (11.3). Now, by (5.4), (5.5), (11.5) and (11.4), we have

(11.6) $$f_5(z)=f_2\left(\frac{i}{n}+\theta\right)$$

and $$2k'\theta-h'+2k'\frac{i}{n}=2k'iz.$$

Hence, by (11.3),

(11.7) $$|f_5(z)| \leqslant A_3(k|z|)^{-\frac{1}{2}}.$$

By (5.6), (10.8), (11.7), and (5.4),

(11.8) $$\left|f_3\left(\frac{i}{n}+\frac{h}{k}+y\right)\right| < A_{20}\{\phi(k)\}^{-2}k^{-\frac{1}{2}}(n^{-2}+y^2)^{-\frac{5}{4}},$$

and hence, by (5.2),

(11.9) $$|J(h,k)| \leqslant A_{20}\{\phi(k)\}^{-2}k^{-\frac{1}{2}}\int_{-\infty}^{\infty}(n^{-2}+y^2)^{-\frac{5}{4}}\,dy$$
$$=A_{21}n^{\frac{3}{2}}\{\phi(k)\}^{-2}k^{-\frac{1}{2}} < A_{22}n^{\frac{3}{2}}k^{-\frac{9}{4}}.$$

12. Now suppose that there is an exceptional L function $L(s,\chi_0)$, where χ_0 is a primitive character mod k_0, and ρ is the exceptional zero. Then, by

(8.1) and (8.5,)

$$(12.0) \qquad 1-\frac{2a_1}{\log u} \leqslant \rho \leqslant 1-\frac{a_1}{k_0^{\frac{1}{2}} \log^2 k_0},$$

so that $2k_0^{\frac{1}{2}} \log^2 k_0 \geqslant \log u$, and hence, by (3.0) and (3.1),

$$(12.1) \qquad k_0 > \log u.$$

Suppose that k is a multiple of k_0. Then there is exactly one character mod k equivalent to χ_0. Let this character be χ_2. Instead of $f_8(s)$, we now use the function

$$\Gamma(s)\,(2\pi z)^{-s}\left(f_7(s)-\frac{\mu(k)}{\phi(k)(s-1)}+\frac{B(\xi_k^h, \chi_2)}{\phi(k)(s-\rho)}\right),$$

and instead of (10.7), but in the same way, we obtain

$$(12.2) \qquad \left| f_4(z)-\frac{\mu(k)}{\phi(k)\,2\pi z}+\frac{B(\xi_k^h, \chi_2)}{\phi(k)\,2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s)(2\pi z)^{-s}}{s-\rho}\, ds \right| < A_{18}\, n u^{-2}.$$

Now

$$\int_{2-i\infty}^{2+i\infty} \frac{\Gamma(s)(2\pi z)^{-s}}{s-\rho}\, ds = 2\pi i\, \Gamma(\rho)(2\pi z)^{-\rho}+\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)(2\pi z)^{-s}}{s-\rho}\, ds,$$

and, by (9.4) and (5.4),

$$\left| \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s)(2\pi z)^{-s}}{s-\rho}\, ds \right| \leqslant A_{23} |z|^{-\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}u^{-1}t}\, dt \leqslant 2A_{23}\, n^{\frac{1}{2}} u.$$

Also* $\qquad |B(\xi_k^h, \chi_2)| \leqslant k_0^{\frac{1}{2}}.$

Hence, by (12.2) and (6.2),

$$(12.3) \qquad |f_4(z)| < \frac{A_{24}\, k_0^{\frac{1}{2}}}{\phi(k)\,|z|}.$$

From this and (5.6), (11.7), and (5.4) it follows that

$$(12.4) \qquad \left| f_3\left(\frac{i}{n}+\frac{h}{k}+y\right) \right| < A_{25}\, k_0\, k^{-\frac{9}{4}} (n^{-2}+y^2)^{-\frac{5}{4}}.$$

Hence, by (5.2),

$$(12.5) \qquad |J(h, k)| < A_{26}\, n^{\frac{3}{2}}\, k_0\, k^{-\frac{9}{4}} \quad (k_0 | k).$$

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* Landau, *loc. cit.*, 3 (1927), 333.

13. Put

$$U_k = \sum_h{}' \xi_k^{-nh} J(h, k), \tag{13.0}$$

so that, by (5.3),

$$\Big| \nu(n) - e^{2\pi} \sum_{k \leqslant u} U_k \Big| \leqslant A_4 n^{\frac{3}{2}} u^{-\frac{1}{2}} \log n. \tag{13.1}$$

If there are no exceptional L functions, then, by (13.0) and (11.9),

$$|U_k| < A_{22} n^{\frac{3}{2}} k^{-\frac{5}{4}} \tag{13.2}$$

for any $k \leqslant u$. If there are exceptional L functions, we have (13.2) only for those values of k which are not multiples of k_0, but, by (13.0) and (12.5),

$$|U_k| < A_{26} n^{\frac{3}{2}} k_0 k^{-\frac{5}{4}} \quad (k_0 | k),$$

and hence, by (12.1),

$$\Big| \sum_{\substack{k \leqslant u \\ k_0 | k}} U_k \Big| < A_{26} n^{\frac{3}{2}} k_0 \sum_{l=1}^{\infty} (k_0 l)^{-\frac{5}{4}} = A_{27} n^{\frac{3}{2}} k_0^{-\frac{1}{4}} < A_{27} n^{\frac{3}{2}} (\log u)^{-\frac{1}{4}}.$$

It follows that, in either case,

$$\Big| \sum_{\log u < k \leqslant u} U_k \Big| < A_{27} n^{\frac{3}{2}} (\log u)^{-\frac{1}{4}} + A_{22} n^{\frac{3}{2}} \sum_{k > \log u} k^{-\frac{5}{4}}$$

$$< A_{28} n^{\frac{3}{2}} (\log u)^{-\frac{1}{4}}.$$

Hence, by (13.1) and (3.1),

$$\Big| \nu(n) - e^{2\pi} \sum_{k \leqslant \log u} U_k \Big| < A_{29} n^{\frac{3}{2}} (\log u)^{-\frac{1}{4}}. \tag{13.3}$$

14. Suppose that

$$k \leqslant \log u. \tag{14.0}$$

Then, by 8.1 and (12.1), none of the L functions formed with characters mod k is exceptional, and it follows that (10.7) holds.

By (10.7) and (10.8),

$$\Big| f_4{}^2(z) - \frac{\mu^2(k)}{\phi^2(k)\,(2\pi z)^2} \Big| < \frac{A_{30}\, n u^{-2}}{\phi(k)\, |z|},$$

and hence, by (5.6) and (11.7),

$$(14.1)\qquad \left| f_3\left(\frac{i}{n}+\frac{h}{k}+y\right)-\frac{\mu^2(k)\,f_5(z)}{\phi^2(k)(2\pi z)^2}\right| < A_{31}\,nu^{-2}\,k^{-1}\,|z|^{-\frac{3}{2}}.$$

Now, by (5.5), (2.4), and (5.0),

$$(14.2)\qquad f_5(z)=\sum_{m=-\infty}^{\infty}\xi_k^{hm^2}\,e^{-2\pi m^2 z}=\sum_{m}\xi_k^{hm^2}\int_{m^2}^{\infty}2\pi z e^{-2\pi xz}\,dx$$

$$=2\pi z\int_0^{\infty}e^{-2\pi xz}\sum_{|m|\leqslant\sqrt{x}}\xi_k^{hm^2}\,dx,$$

and it is easily seen that

$$(14.3)\qquad \left|\sum_{|m|\leqslant\sqrt{x}}\xi_k^{hm^2}-\frac{2\sqrt{x}}{k}\,S_{h,\,k}\right|\leqslant 2k,$$

where

$$(14.4)\qquad S_{h,\,k}=\sum_{m=1}^{k}\xi_k^{hm^2}.$$

By (14.2), (14.3), (5.4), and (6.2),

$$\left|f_5(z)-4\pi z k^{-1}\,S_{h,\,k}\int_0^{\infty}e^{-2\pi xz}\,x^{\frac{1}{2}}\,dx\right|$$

$$\leqslant 4\pi k\,|z|\int_0^{\infty}e^{-2\pi x/n}\,dx=2nk\,|z|\leqslant 4u,$$

and it is known that

$$\int_0^{\infty}e^{-2\pi xz}\,x^{\frac{1}{2}}\,dx=(2\pi z)^{-\frac{3}{2}}\,\Gamma(\tfrac{3}{2})=2^{-\frac{5}{2}}\,\pi^{-1}\,z^{-\frac{3}{2}},$$

so that

$$(14.5)\qquad |f_5(z)-2^{-\frac{1}{2}}\,z^{-\frac{1}{2}}\,k^{-1}\,S_{h,\,k}|\leqslant 4u.$$

Hence, by (5.4) and (3.1),

$$\left|\frac{f_5(z)}{\phi^2(k)(2\pi z)^2}-\frac{S_{h,\,k}}{\phi^2(k)\,k\pi^2(2z)^{\frac{5}{2}}}\right|<A_{32}\,uk^{-1}\,|z|^{-2}$$

$$\leqslant A_{32}\,n^{\frac{1}{2}}\,uk^{-1}\,|z|^{-\frac{3}{2}}<A_{33}\,nu^{-2}\,k^{-1}\,|z|^{-\frac{3}{2}},$$

and hence, by (14.1),

$$(14.6)\qquad \left| f_3\left(\frac{i}{n}+\frac{h}{k}+y\right)-\frac{\mu^2(k)\,S_{h,\,k}}{\phi^2(k)\,k\pi^2(2z)^{\frac{5}{2}}}\right|<A_{34}\,nu^{-2}\,k^{-1}\,|z|^{-\frac{3}{2}}.$$

From this and (5.2) and (5.4) it follows that

$$(14.7) \qquad |J(h,k)-J_1(h,k)| < A_{34}\, n u^{-2} k^{-1} \int_{-\infty}^{\infty} (n^{-2}+y^2)^{-\frac{3}{4}}\, dy$$

$$= A_{35}\, n^{\frac{3}{2}} u^{-2} k^{-1},$$

where

$$(14.8) \qquad J_1(h,k) = \frac{\mu^2(k)\, S_{h,k}}{\phi^2(k)\, k\pi^2\, 2^{\frac{5}{2}}} \int_{-u/(nk)}^{u/(nk)} (n^{-1}-iy)^{-\frac{5}{2}}\, e^{-2\pi i n y}\, dy.$$

15. Put

$$(15.0) \qquad J_2(h,k) = \frac{\mu^2(k)\, S_{h,k}}{\phi^2(k)\, k\pi^2\, 2^{\frac{5}{2}}} \int_{-\infty}^{\infty} (n^{-1}-iy)^{-\frac{5}{2}}\, e^{-2\pi i n y}\, dy.$$

It is known that

$$(15.1) \qquad |S_{h,k}| \leqslant \sqrt{(2k)},$$

and

$$\int_{u/(nk)}^{\infty} y^{-\frac{5}{2}}\, dy = \tfrac{2}{3} u^{-\frac{3}{2}} (nk)^{\frac{3}{2}}.$$

Hence, by (15.0) and (14.8),

$$|J_2(h,k)-J_1(h,k)| \leqslant n^{\frac{3}{2}} u^{-\frac{3}{2}} k\phi^{-2}(k),$$

and hence, by (14.7),

$$(15.2) \qquad |J(h,k)-J_2(h,k)| < A_{36}\, n^{\frac{3}{2}} u^{-\frac{3}{2}}.$$

Now (with the substitution $w = 2\pi - 2\pi i n y$)

$$e^{2\pi} \int_{-\infty}^{\infty} (n^{-1}-iy)^{-\frac{5}{2}}\, e^{-2\pi i n y}\, dy = -i \int_{2\pi-i\infty}^{2\pi+i\infty} w^{-\frac{5}{2}} (2\pi n)^{\frac{3}{2}}\, e^{w}\, dw$$

$$= (2\pi)^{\frac{5}{2}}\, n^{\frac{3}{2}} / \Gamma(\tfrac{5}{2}) = 2^{\frac{9}{2}}\, 3^{-1} \pi^2 n^{\frac{3}{2}}.$$

Hence, by (15.0),

$$(15.3) \qquad e^{2\pi} J_2(h,k) = \tfrac{4}{3} n^{\frac{3}{2}} \frac{\mu^2(k)\, S_{h,k}}{\phi^2(k)\, k}.$$

Putting

$$(15.4) \qquad V_k = \frac{\mu^2(k)}{\phi^2(k)\, k} \sum_h{}' S_{h,k}\, \xi_k^{-nh},$$

we have, by (13.0), (15.2), and (15.3),

$$|e^{2\pi} U_k - \tfrac{4}{3} n^{\frac{3}{2}} V_k| < e^{2\pi} A_{36}\, k n^{\frac{3}{2}} u^{-\frac{3}{2}},$$

and hence, by (13.3),

$$\left| \nu(n) - \tfrac{4}{3} n^{\frac{3}{2}} \sum_{k \leqslant \log u} V_k \right| < A_{37} n^{\frac{3}{2}} (\log u)^{-\frac{1}{4}}. \tag{15.5}$$

Now let

$$S(n) = \sum_{k=1}^{\infty} V_k. \tag{15.6}$$

By (15.4) and (15.1),

$$|V_k| \leqslant 2k^{-\frac{1}{2}}/\phi(k) < A_{38} k^{-\frac{5}{4}},$$

so that

$$\left| \sum_{k > \log u} V_k \right| < A_{39} (\log u)^{-\frac{1}{4}}.$$

Hence, by (15.5), (15.6), and (3.1),

$$\left| \nu(n) - \tfrac{4}{3} n^{\frac{3}{2}} S(n) \right| < A_{40} n^{\frac{3}{2}} (\log \log n)^{-\frac{1}{2}}. \tag{15.7}$$

It follows from formula (3.212) of Miss Stanley's paper that $S(n) > a_2$, where a_2 is a positive absolute constant. Hence, by (15.7),

$$\nu(n) \sim \tfrac{4}{3} n^{\frac{3}{2}} S(n). \tag{15.8}$$

This implies $\nu(n) \neq 0$ for any sufficiently large n, which was to be proved.

(15.8) is weaker than Miss Stanley's Theorem C, but it is all that she needs of it to deduce Theorem G.

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BEHAVIOUR OF THE COEFFICIENTS IN A TRIGONOMETRICAL INTEGRAL

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[Received 21 August, 1936.—Read 12 November, 1936.]

1. If a_s, b_s have continuous integrals which are subject to integration by parts in connection with any function of bounded variation in every interval $(0, p)$, for $0 \leqslant p < \infty$, then, if

$$\int_0^\infty (a_s \cos sx - b_s \sin sx)\, ds \tag{1}$$

converges in a set of density exceeding $\frac{1}{2}$ in any interval symmetrical with respect to the origin,

$$\int_\sigma^{\sigma+u} a_s\, ds, \quad \int_\sigma^{\sigma+u} b_s\, ds \to 0 \tag{2}$$

as $\sigma \to \infty$, uniformly in $0 \leqslant u \leqslant 1$, and if it oscillates finitely in a set of density exceeding $\frac{1}{2}$ in any interval symmetrical with respect to the origin,

$$\int_\sigma^{\sigma+u} a_s\, ds, \quad \int_\sigma^{\sigma+u} b_s\, ds = O(1) \tag{3}$$

as $\sigma \to \infty$, uniformly in $0 \leqslant u \leqslant 1$.

2. The set in which

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds$$

has either of the behaviours described above has density greater than $\frac{1}{2}$ in the interval considered, since this is symmetrical with respect to the origin. Thus

$$\int_0^\infty a_s \cos sx\, ds, \quad \int_0^\infty b_s \sin sx\, ds$$

have the behaviour in question in a set of positive density in the interval considered.

In the case of convergence, as $\sigma \to \infty$, in a set of positive density in the interval considered,

$$(1) \qquad \int_{\sigma}^{\sigma+u} a_s \cos sx\, ds, \quad \int_{\sigma}^{\sigma+u} b_s \sin sx\, ds \to 0$$

uniformly in $0 \leqslant u \leqslant 1$. In the case of finite oscillation, as $\sigma \to \infty$,

$$(2) \qquad \int_{\sigma}^{\sigma+u} a_s \cos x\, ds, \quad \int_{\sigma}^{\sigma+u} b_s \sin sx\, ds = O(1)$$

uniformly in $0 \leqslant u \leqslant 1$.

Let η be any positive number less than 1. If sx lies in any one of the intervals $(p\pi - \cos^{-1}\eta,\ p\pi)$, $(p\pi,\ p\pi + \cos^{-1}\eta)$, where p is any integer,

$$|\cos sx| \geqslant \eta$$

and $\cos sx$ is monotonic in sx.

Let λ be any positive number. If $\sigma \leqslant s \leqslant \sigma+\lambda$, σ being positive, sx lies in $(p\pi - \cos^{-1}\eta,\ p\pi)$ if x lies in the common part of

$$\left(\frac{p\pi - \cos^{-1}\eta}{\sigma},\ \frac{p\pi}{\sigma}\right), \quad \left(\frac{p\pi - \cos^{-1}\eta}{\sigma+\lambda},\ \frac{p\pi}{\sigma+\lambda}\right).$$

This common part exists if

$$\frac{p\pi - \cos^{-1}\eta}{\sigma+\lambda} \leqslant \frac{p\pi}{\sigma}, \quad \frac{p\pi - \cos^{-1}\eta}{\sigma} \leqslant \frac{p\pi}{\sigma+\lambda}.$$

It is sufficient that

$$\lambda \leqslant \frac{\cos^{-1}\eta}{|p\pi/\sigma|}$$

and the length of the common part is then not less than

$$\frac{\cos^{-1}\eta - (|p\pi/\sigma|\lambda)}{\sigma+\lambda}.$$

The case of the other interval can be treated similarly.

Let $(-g, g)$ be the interval in which the conditions associated with (1, 2) are realized. For

$$\left(\frac{p\pi - \cos^{-1}\eta}{\sigma},\ \frac{p\pi}{\sigma}\right), \quad \left(\frac{p\pi}{\sigma},\ \frac{p\pi + \cos^{-1}\eta}{\sigma}\right)$$

to lie in this interval, it is sufficient that

$$|p|\pi < g\sigma - \tfrac{1}{2}\pi.$$

This is the case, for all sufficiently large values of σ, if

$$\left|\frac{p\pi}{\sigma}\right| < g. \tag{3}$$

Subject p to this restriction.

If now

$$\lambda < \frac{\cos^{-1}\eta}{g}, \tag{4}$$

the common part in question certainly exists.

The number of common parts corresponding to values of p satisfying (3) is $4[g\sigma/\pi]-2$, which is not less than $4g\sigma/\pi-6$.

The common parts in question cover a portion of $(-g, g)$ of measure not less than

$$\left(\frac{4g\sigma}{\pi}-6\right)\frac{\cos^{-1}\eta-g\lambda}{\sigma+\lambda},$$

which is greater than $\dfrac{4kg}{\pi}(\cos^{-1}\eta-g\lambda)$

for all sufficiently large values of σ, if λ is sufficiently small, where k is any positive number strictly less than 1.

Suppose that (2, 1) holds.

Let μ be the average density in $(-g, g)$ of the set concerned. Choose $k > 1-\mu$ and take $\eta < \cos\dfrac{1-\mu}{2k}\pi$. Then the average density of the portion of $(-g, g)$ covered by the common parts in question is greater than

$$\frac{1}{2g}\,\frac{4kg}{\pi}\cos^{-1}\eta = \frac{2k}{\pi}\cos^{-1}\eta > 1-\mu.$$

They contain a sub-set of the set in which (2, 1) holds, of average density not less than $\dfrac{2k}{\pi}\cos^{-1}\eta-(1-\mu)$. Denote this by μ_1.

Take any positive value of λ satisfying (3), sufficiently small to make the above hold, and not greater than 1. Then

$$\int_{\sigma}^{\sigma+u} a_s \cos sx\,ds \to 0$$

uniformly in $0 \leqslant u \leqslant \lambda$.

Take any positive sequence $\{\sigma_n\}$ tending to infinity with n and with its terms sufficiently large for (3) to hold.

Let C_n be the portion of the common parts corresponding to the value σ_n of σ for which (2, 1) holds. Then, d indicating density,

$$\frac{1}{2g} C_n > \mu_1,$$

so that
$$d \overline{\lim}\, C_n \geqslant \mu_1 > 0,$$

and hence $\overline{\lim}\, C_n$ exists.

Let ξ be a point of $\overline{\lim}\, C_n$. Then there is a sub-sequence $\{\sigma_{n_r}\}$ of $\{\sigma_n\}$ which tends to infinity with r, and is such that C_{n_r} contains ξ for all sufficiently large values of r.

For these values of r,

$$|\cos s\xi| \geqslant \eta$$

and $\cos s\xi$ is monotonic in s when $\sigma_{n_r} \leqslant s \leqslant \sigma_{n_r}+\lambda$. Thus, if $0 \leqslant u \leqslant \lambda$, and one of s_1, s_2 is an end-point of $(\sigma_{n_r}, \sigma_{n_r}+\lambda)$ and the other is within this interval, end-points being included,

$$\left| \int_{\sigma_{n_r}}^{\sigma_{n_r}+u} a_s\, ds \right| \leqslant \frac{1}{\eta} \left| \int_{s_1}^{s_2} a_s \cos s\xi\, ds \right|$$
$$\to 0,$$

uniformly in $0 \leqslant u \leqslant \lambda$.

Suppose that
$$\int_{\sigma}^{\sigma+u} a_s\, ds$$

does not tend to 0 uniformly in $0 \leqslant u \leqslant \lambda$, as $\sigma \to \infty$. Then

$$\left\| \int_{\sigma}^{\sigma+u} a_s\, ds \right\|$$

does not tend to 0 as $\sigma \to \infty$, where the sign $\|\quad\|$ refers to the upper bound of the modulus in the range $0 \leqslant u \leqslant \lambda$. Since, as is easily seen, the integral concerned is continuous in σ over the range $0 \leqslant \sigma < \infty$ of σ, a sequence $\{\sigma_n\}$, tending to infinity with n, can be found, without the axiom of selection, such that

$$\left\| \int_{\sigma_n}^{\sigma_n+u} a_s\, ds \right\|$$

tends to a limit other than 0; but this makes

$$\left\| \int_{\sigma_{n_r}}^{\sigma_{n_r}+u} a_s\, ds \right\|$$

tend to a limit other than 0, which contradicts what has been proved above. We must have

$$\int_\sigma^{\sigma+u} a_s\,ds \to 0$$

uniformly in $0 \leqslant u \leqslant \lambda$. It is an immediate deduction that λ can be replaced by 1. The first half of (1, 2) is established.

The second half of (1, 2) can be established in exactly the same way.

The property (1, 3) is proved on exactly the same lines, tendency to zero being replaced by finite oscillation.

3. If either of the conditions in § 1 holds,

$$\int_c^\infty \frac{a_s}{s^{1+a}}\,ds, \qquad \int_c^\infty \frac{b_s}{s^{1+a}}\,ds,$$

where $c > 0$, converge for $a > 0$.

We have, if $c \leqslant \sigma \leqslant \tau < \infty$,

$$\int_\sigma^\tau \frac{a_s}{s^{1+a}}\,ds = \int_\sigma^{\sigma+1} + \int_{\sigma+1}^{\sigma+2} + \ldots + \int_{\sigma+p-1}^{\sigma+p} + \int_{\sigma+p}^{\tau} \frac{a_s}{s^{1+a}}\,ds,$$

where $p = [\tau - \sigma]$,

$$= \frac{1}{\sigma^{1+a}} \int_\sigma^{\sigma+u_1} + \frac{1}{(\sigma+1)^{1+a}} \int_{\sigma+1}^{\sigma+1+u_2} + \ldots$$
$$+ \frac{1}{(\sigma+p-1)^{1+a}} \int_{\sigma+p-1}^{\sigma+p-1+u_p} + \frac{1}{(\sigma+p)^{1+a}} \int_{\sigma+p}^{\sigma+p+u_p} a_s\,ds,$$

where $0 \leqslant u_1, u_2, \ldots, u_{p+1} \leqslant 1$. If, now,

$$\sup \left| \int_s^{s+u} a_t\,dt \right| = M_\sigma \quad (s \geqslant \sigma,\ 0 \leqslant u \leqslant 1),$$

we have

$$\left| \int_\sigma^\tau \frac{a_s}{s^{1+a}}\,ds \right| \leqslant M_\sigma \left\{ \frac{1}{\sigma^{1+a}} + \frac{1}{(\sigma+1)^{1+a}} + \ldots + \frac{1}{(\sigma+p)^{1+a}} \right\}$$
$$\to 0$$

as $\sigma, \tau \to \infty$. Thus the first of the integrals mentioned converges; and similarly for the second.

4. The above result is the best that can be obtained, in that a cannot be replaced by 0. To see this, it is sufficient to consider (the integrals being ordinary)

$$\int_0^\infty \frac{\cos sx + \sin sx}{\log s}\,ds,$$

which converges save at the origin. Here

$$\int_c^\infty \frac{a_s}{s}\,ds, \qquad \int_c^\infty \frac{b_s}{s}\,ds$$

both diverge.

5. If the conditions of § 1 hold, the set in question being replaced by any set of positive measure,

$$\int_c^\infty \left(\frac{a_s}{s^{1+\alpha}}\cos sx - \frac{b_s}{s^{1+\alpha}}\sin sx\right) ds, \tag{1}$$

where $c > 0$, converges uniformly in any bounded range of x.

Since the set has positive measure it has points of unit density. Let ξ be one of them.

In $(\xi-h,\ \xi+h)$ the average density is greater than $\frac{1}{2}$ for all sufficiently small values of h. Let g be one of them.

In $(\xi-g,\ \xi+g)$ the average density is greater than $\frac{1}{2}$. Write $x=\xi+\theta$. We have

$$\int_0^\infty (a_s\cos sx - b_s \sin sx)\,ds$$
$$= \int_0^\infty \{(a_s\cos s\xi - b_s\sin s\xi)\cos s\theta - (a_s \sin s\xi + b_s\cos s\xi)\sin s\theta\}\,ds,$$

both being convergent or both finitely oscillating.

The value of θ for which the second integral converges or oscillates finitely, as the case may be, has density greater than $\frac{1}{2}$ in $(-g,\ g)$, which is symmetrical with respect to the origin. Thus

$$\int_\sigma^{\sigma+u} (a_s\cos s\xi - b_s\sin s\xi)\,ds, \qquad \int_\sigma^{\sigma+u} (a_s\sin s\xi + b_s\cos s\xi)\,ds$$

both tend to zero or both remain bounded, as the case may be, as $\sigma \to \infty$, uniformly in $0 \leqslant u \leqslant 1$.

Write $$A_u = \int_\sigma^{\sigma+u} (a_s\cos s\xi - b_s\sin s\xi)\,ds.$$

Then

$$\int_\sigma^{\sigma+u} \frac{a_s\cos s\xi - b_s\sin s\xi}{s^{1+\alpha}}\cos s\theta\,ds$$
$$= \frac{A_u}{(\sigma+u)^{1+\alpha}}\cos(s+u)\theta + \int_\sigma^{\sigma+u} A_{s-\sigma}\frac{s\theta\sin s\theta + (1+\alpha)\cos s\theta}{s^{2+\alpha}}\,ds.$$

If, now, M_σ is the upper bound of $|A_u|$ in $0 \leqslant u \leqslant 1$,

$$\left| \int_\sigma^{\sigma+u} \frac{a_s \cos s\xi - b_s \sin s\xi}{s^{1+a}} \cos s\theta \, ds \right| \leqslant \frac{M_\sigma}{\sigma^{1+a}} + \frac{M_\sigma |\theta|}{\sigma^{1+a}} + \frac{(1+a) M_\sigma}{\sigma^{2+a}}$$

$$\leqslant \frac{1}{\sigma^{1+a}} M_\sigma (2+a+|\theta|) \quad (\sigma > 1)$$

for $0 \leqslant u \leqslant 1$.

By an argument similar to that of § 3, the uniform convergence of

$$\int_0^\infty \frac{a_s \cos s\xi - b_s \sin s\xi}{s^{1+a}} \cos s\theta \, ds$$

can be proved in any finite range of θ; and similarly for

$$\int_0^\infty \frac{a_s \sin s\xi + b_s \cos s\xi}{s^{1+a}} \sin s\theta \, d\theta.$$

The uniform convergence of (1) is now given in any finite range of x, ξ being fixed.

6. This result cannot be improved upon in that a cannot be replaced by 0.

Let any bounded interval be given. Take any value ξ of x strictly within this range and write

$$a_s = \frac{\cos s\xi}{\log s}, \qquad b_s = -\frac{\sin s\xi}{\log s} \quad (s \geqslant 1),$$

$$a_s = 0, \qquad b_s = 0 \quad \text{(otherwise)}.$$

We have

$$\int_0^\infty \left(\frac{a_s}{s} \cos sx - \frac{b_s}{s} \sin sx \right) ds = \int_1^\infty \left(\frac{\cos s\xi}{s \log s} \cos sx + \frac{\sin s\xi}{s \log s} \sin sx \right) ds$$

$$= \int_1^\infty \frac{\cos s(x-\xi)}{s \log s} \, ds$$

$$= \int_1^\infty \frac{\cos s\theta}{s \log s} \, ds,$$

where $x = \xi + \theta$.

Now, if $\theta = 0$,

$$\int_\sigma^{e^\sigma} \frac{\cos s\theta}{s \log s} \, ds = \int_\sigma^{e^\sigma} \frac{ds}{s \log s} = \log \sigma - \log \log \sigma \to \infty$$

as $\sigma \to \infty$.

But ξ lies in the bounded interval. The convergence cannot be uniform.

7. If the integral (1, 1) converges in a set of positive measure, there is a whole interval in which

$$\int_{\sigma}^{\sigma+u}(a_s\cos sx - b_s\sin sx)\,ds, \qquad \int_{\sigma}^{\sigma+u}(a_s\sin sx + b_s\cos sx)\,ds \to 0$$

uniformly for $0 \leqslant u \leqslant 1$. If it oscillates finitely in a set of positive measure, there is a whole interval in which

$$\int_{\sigma}^{\sigma+u}(a_s\cos sx - b_s\sin sx)\,ds, \qquad \int_{\sigma}^{\sigma+u}(a_s\sin sx + b_s\cos sx)\,ds = O(1)$$

uniformly for $0 \leqslant u \leqslant 1$.

We have shown that there is an interval $(\xi - g,\ \xi + g)$ in which the set in which the integral (1, 1) has the behaviour in question has density greater than $\frac{1}{2}$. Now the density in $(\xi_1 - g,\ \xi_1 + g)$ is a continuous function of ξ_1 and so is greater than $\frac{1}{2}$ in all sufficiently small intervals centred at ξ. By what is proved in § 5,

$$\int_{\sigma}^{\sigma+u}(a_s\cos s\xi_1 - b_s\sin s\xi_1)\,ds, \qquad \int_{\sigma}^{\sigma+u}(a_s\sin s\xi_1 + b_s\cos \xi_1)\,ds$$

tend to zero or remain bounded, as the case may be, as $s \to \infty$, uniformly in $0 \leqslant u \leqslant 1$. This is what is required.

8. If the integral (1, 1) converges as described in § 1,

$$a_s,\ b_s \to 0 \quad (C,\ +);$$

if it remains bounded as described in § 1,

$$a_s,\ b_s = O(1) \quad (C,\ +),$$

where $(C,\ +)$ refers to convergence by Cesàro means, with the last term left out, of any positive order.

Where convergence holds, we have to show that

$$\frac{1}{\tau}\int_0^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1}a_s\,ds, \qquad \frac{1}{\tau}\int_0^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1}b_s\,ds \to 0$$

for every positive value of k.

If $k < 1$, we have, for $0 \leqslant \sigma,\ \sigma+u \leqslant \tau-1$, with $0 \leqslant u \leqslant 1$,

$$\int_{\sigma}^{\sigma+u}\left(1-\frac{s}{\tau}\right)^{k-1}a_s\,ds = \left(1-\frac{\sigma+u}{\tau}\right)^{k-1}\int_{\sigma+v}^{\sigma+u}a_s\,ds$$

for some value of v satisfying $0 \leqslant v \leqslant u$ and so $0 \leqslant v \leqslant 1$. Thus

$$\left|\int_{\sigma}^{\sigma+u}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right| \leqslant \left(1-\frac{\sigma+u}{\tau}\right)^{k-1} M_{\sigma},$$

for $0 \leqslant u \leqslant 1$, M_{σ} being as in § 3,

If now $0 \leqslant \kappa < \infty$, for $\tau \geqslant \kappa+1$,

$$\int_{\kappa}^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds = \int_{\kappa}^{\kappa+1}+\int_{\kappa+1}^{\kappa+2}+\ldots+\int_{\kappa+p-1}^{\kappa+p}+\int_{\kappa+p}^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds,$$

where $p=[\tau-1-\kappa]$,

$$\left|\int_{\kappa}^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right|$$

$$\leqslant M_{\kappa}\left\{\left(1-\frac{\kappa+1}{\tau}\right)^{k-1}+\left(1-\frac{\kappa+2}{\tau}\right)^{k-1}+\ldots\right.$$

$$\left.+\left(1-\frac{\kappa+p}{\tau}\right)^{k-1}+\left(1-\frac{\tau-1}{\tau}\right)^{k-1}\right\}$$

$$\leqslant M_{\kappa}\left\{\int_{\kappa+1}^{\kappa+p+1}\left(1-\frac{v}{\tau}\right)^{k-1} dv+\frac{1}{\tau^{k-1}}\right\}$$

$$\leqslant M_{\kappa}\left\{\int_{\kappa+1}^{\tau}\left(1-\frac{v}{\tau}\right)^{k-1} dv+\frac{1}{\tau^{k-1}}\right\}$$

$$= M_{\kappa}\,\tau\left\{\frac{1}{k}\left(1-\frac{\kappa+1}{\tau}\right)^{k}+\frac{1}{\tau^{k}}\right\}$$

$$\leqslant M_{\kappa}\,\tau\left(\frac{1}{k}+\frac{1}{\tau^{k}}\right).$$

This gives

$$\left|\frac{1}{\tau}\int_{\kappa}^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right| \leqslant M_{\kappa}\left(\frac{1}{k}+\frac{1}{\tau^{k}}\right).$$

On the other hand, if $\tau \geqslant \kappa+1$,

$$\frac{1}{\tau}\int_{0}^{\kappa}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds = \frac{1}{\tau}\left(1-\frac{\kappa}{\tau}\right)^{k-1}\int_{\kappa_1}^{\kappa} a_s\,ds$$

for some number κ_1 satisfying $0 \leqslant \kappa_1 \leqslant \kappa$.

If P_{κ} is the upper bound of $\left|\int_0^{\sigma} a_s\,ds\right|$ in the range $0 \leqslant \sigma \leqslant \kappa$, we have

$$\left|\frac{1}{\tau}\int_{0}^{\kappa}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right| \leqslant \frac{2P_{\kappa}}{\tau}\left(1-\frac{\kappa}{\tau}\right)^{k-1}.$$

We have now

$$\left|\frac{1}{\tau}\int_0^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right| \leqslant M_\kappa\left(\frac{1}{k}+\frac{1}{\tau^k}\right)+\frac{2P_\kappa}{\tau}\left(1-\frac{\kappa}{\tau}\right)^{k-1}.$$

Make $\tau\to\infty$; then

$$\overline{\lim_{\tau\to\infty}}\left|\frac{1}{\tau}\int_0^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right| \leqslant \frac{M_\kappa}{k}.$$

Make $\kappa\to\infty$; then

$$\overline{\lim_{\tau\to\infty}}\left|\frac{1}{\tau}\int_0^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right| \leqslant 0,$$

which is only possible if

$$\frac{1}{\tau}\int_0^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\to 0$$

as $\tau\to\infty$.

If $k\geqslant 1$, for $0\leqslant\sigma$, if $0\leqslant u\leqslant 1$,

$$\int_\sigma^{\sigma+u}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds=\left(1-\frac{\sigma}{\tau}\right)^{k-1}\int_\sigma^{\sigma+v} a_s\,ds$$

for some value of v satisfying $0\leqslant v\leqslant u$ and so $0\leqslant v\leqslant 1$,

$$\left|\int_\sigma^{\sigma+u}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right|\leqslant M_\sigma,$$

and at once, for $\tau\geqslant\kappa+1$,

$$\begin{aligned}\left|\int_\kappa^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right| &\leqslant M_\kappa(\tau-\kappa)\\ &\leqslant M_\kappa\tau.\end{aligned}$$

If $\tau\geqslant\kappa+1$,

$$\frac{1}{\tau}\int_0^{\kappa}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds=\frac{1}{\tau}\int_0^{\kappa_2} a_s\,ds$$

for some number κ_2 satisfying $0\leqslant\kappa_2\leqslant\kappa$,

$$\left|\frac{1}{\tau}\int_0^{\kappa}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right|\leqslant\frac{P_\kappa}{\tau},$$

$$\left|\frac{1}{\tau}\int_0^{\tau-1}\left(1-\frac{s}{\tau}\right)^{k-1} a_s\,ds\right|\leqslant M_\kappa+\frac{P_\kappa}{\tau},$$

and the required result can be obtained as before.

Replacing a_s by b_s we have the second of the results desired; and similarly for the case in which the integral remains bounded.

9. The integral (1, 1) may converge in a set of the kind described in § 1 without

$$a_s,\ b_s \to 0 \tag{1}$$

as $s \to \infty$. It may oscillate finitely without

$$a_s,\ b_s = O(1) \tag{2}$$

as $s \to \infty$.

We can go further than this. The integral (1, 1) may converge uniformly in the whole range of x without either (1) or (2).

For consider

$$a_s = b_s = n \quad [n \leqslant s \leqslant n + (1/n^3)],$$

$$a_s = b_s = 0 \quad \text{(otherwise)},$$

where n takes all positive integral values. Then, if $\sigma < p$,

$$\int_\sigma^\infty |a_s \cos sx - b_s \sin sx|\, ds \leqslant \int_p^\infty \{|a_s| + |b_s|\}\, ds$$

$$\leqslant 2\left(\frac{p}{p^3} + \frac{p+1}{(p+1)^3} + \ldots + \frac{p+r}{(p+r)^3} + \ldots\right)$$

$$\leqslant 2\left(\frac{1}{p^2} + \frac{1}{(p+1)^2} + \ldots + \frac{1}{(p+r)^2} + \ldots\right) \to 0$$

as $\sigma \to \infty$, independently of x. Thus the integral (1, 1) converges uniformly over the whole range of x.

But for $n \leqslant s \leqslant n + (1/n^3)$,

$$a_s,\ b_s = n \to \infty.$$

This gives what is required.

10. It may be noticed that, if (1, 1) converges for all values of x,

$$\int_0^\infty a_s\, ds \tag{1}$$

converges. If (1, 1) oscillates finitely for all values of x, (1) oscillates finitely.

For the convergence or finite oscillation must hold when $x = 0$. We notice, however, that (1, 1) may converge or oscillate finitely for all values of x without

$$\int_0^\infty b_s\,ds \tag{2}$$

oscillating finitely.

For consider

$$a_s = 0,$$

$$b_s = \frac{1}{s} \quad (s \geqslant 1),$$

$$= 0 \quad \text{(otherwise)}.$$

We have

$$\int_0^\infty (a_s \cos sx - b_s \sin sx)\,ds = -\int_1^\infty \frac{\sin sx}{s}\,ds,$$

which converges for all values of s. But (2) reduces to

$$\int_1^\infty \frac{ds}{s},$$

which diverges.

11. It may be remarked that the above results are non-translative. The property of integrals expressed by

$$\int_{a+h}^{b+h} f(t-h)\,dt = \int_a^b f(t)\,dt,$$

or any equivalent, is not required.

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A GENERALIZATION OF THE KUMMER 16_6 CONFIGURATION (II)

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[Received 28 September, 1936.—Read 12 November, 1936.]

The interest of Part I of this paper* centred round the "Kummer-type" 64_8 configuration in [7], comprising sixty-four points lying by sets of eight in sixty-four solids, which themselves lie by sets of eight in sixty-four primes. In this part, although the more general Kummer-type figure of 2^{2m} points in $[2^m-1]$ and its associated spaces finds a place, the centre of interest has shifted to the "Weddle-type" configuration in $[2m-1]$ on which the Kummer figure can be represented in the same way as the 16_6 figure of [3] can be represented on the Weddle manifold in [3].

The configurations have been discussed very fully by Coble (**2**, **3**, **4**)† using hyperelliptic ϑ-functions (with the Weddle-type configuration as a foundation), and there is not therefore the need to enter here into the detail that was necessary in **1**. The outlook of this Part is still geometrical, and it leads to results of a kind rather different from those obtained by Coble. In particular, explicit equations are obtained for the generalized Weddle manifold, and with them a formula for its order which confirms the values found by Coble in special cases.

Combining the results obtained from the functional and geometrical methods, the discussion of the Weddle manifolds is now reasonably complete, but on the geometrical side at least there is much lacking yet in the treatment of the Kummer manifolds. In this paper little more is given than an indication of the way in which the geometrical treatment could be developed; the greatest hope for further development would seem to come from finding the equations that would enable us to identify the hyperelliptic Kummer figure as a special Barrau figure*.

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* Reference **1**. Clarendon symbols **1**, **2**, ... are used in the text to refer to the papers cited at the end of this paper.

† I regret that I had overlooked **2** when Part I was written; there is fortunately very little overlapping, since the two treatments are quite different. Using Coble's work I should have been able to remove the doubt which is expressed in **1**, p. 337, about the connection of the figure with hyperelliptic ϑ-functions, since Coble, working with the ϑ-functions, obtains the same configuration.

1. *The group of reciprocal inversion transformations*†.

Given $n+3$ points in $[n]$ there exists a unique "reciprocal inversion" transformation for which $n+1$ of the points are the base simplex, and the other two points are transformed one into the other. Thus, if $W_0, \ldots, W_n$ is the simplex of reference ($y_0 W_0 + \ldots + y_n W_n$ being a general point of the space), and W_u and W_v are the points‡

$$W_u = W_0 + W_1 + \ldots + W_n$$

and

$$W_v = a_0 W_0 + a_1 W_1 + \ldots + a_n W_n,$$

the transformation τ_{uv} which interchanges W_u and W_v is

$$(\tau_{uv}) \qquad \rho y_\lambda' = a_\lambda / y_\lambda.$$

We denote similarly by $\tau_{\lambda\mu}$ the transformation which interchanges W_λ and W_μ, and which has the remaining points as the base simplex. The equations of typical transformations are

$$(\tau_{0v}) \qquad \rho y_0' = a_0, \quad \rho y_\lambda' = (a_\lambda y_0 - a_0 y_\lambda)/(y_0 - y_\lambda),$$

$$(\tau_{0u} = \tau_{0v}\tau_{vu}) \qquad \rho y_0' = 1, \quad \rho y_\lambda' = a_\lambda (y_0 - y_\lambda)/(a_\lambda y_0 - a_0 y_\lambda),$$

$$(\tau_{01}) \qquad \rho y_0' = a_0 y_1, \quad \rho y_1' = a_1 y_0,$$

$$\rho y_\lambda' = \begin{vmatrix} a_0 y_0 & a_1 y_1 & a_\lambda y_\lambda \\ y_0 & y_1 & y_\lambda \\ a_0 & a_1 & a_\lambda \end{vmatrix} \Bigg/ \begin{vmatrix} y_0 & y_1 & y_\lambda \\ 1 & 1 & 1 \\ a_0 & a_1 & a_\lambda \end{vmatrix}.$$

These transformations satisfy the relations

$$\tau_{\lambda\mu}^2 = \mathbf{I}, \quad \tau_{\lambda\mu}\tau_{\mu\nu}\tau_{\nu\lambda} = \mathbf{I}, \quad \tau_{\lambda\mu} = \tau_{\mu\lambda},$$

and thus define a group of transformations $\tau_{\lambda_1 \ldots \lambda_r}$, with the properties (i) symbols with the same suffixes, whatever their order, refer to the same transformation, (ii) no symbol contains the same suffix twice, (iii) the product of any number of the transformations is the transformation of which the symbol contains the suffixes of all the component transformations, pairs of equal suffixes being deleted. We shall write $\tau_{(r)}$ for $\tau_{\lambda_1 \dots \lambda_r}$, the suffix (r) indicating that the symbol has some r different suffixes λ_i.

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* *Cf.* **1**, 336–7.

† **1**, 315; **2**, 444, § 3; **3**, 183, § 1. I use the notation of **1** for the transformations, namely τ_{ij}, which is the same as Coble's I_{ij}.

‡ The points here called W_0, W_1, ..., W_n, W_u, W_v correspond to the points denoted in **1** by W_1', W_2', ..., W_6', W_7', W_8'.

The system obtained when n is even is quite distinct from that obtained when n is odd. Thus

(i) If n is odd, there is an even number of digits from which the suffixes may be selected. The number of suffixes in every symbol is even, and there is a unique transformation

$$\tau = \tau_{01 \dots nuv}.$$

(ii) If n is even, the number of digits is odd, and there are symbols having odd numbers of suffixes as well as even numbers. There are $n+3$ fundamental transformations $\tau_0, \tau_1, \dots, \tau_n, \tau_u, \tau_v$, which satisfy the relations

$$\tau_\lambda^2 = \mathbf{I}, \quad \tau_0 \tau_1 \dots \tau_n \tau_u \tau_v = \mathbf{I}, \quad \tau_\lambda \tau_\mu = \tau_\mu \tau_\lambda,$$

$$\tau_{\lambda_1 \dots \lambda_r} = \tau_{\lambda_1} \dots \tau_{\lambda_r}.$$

If $\alpha\beta \dots \delta$ and $\alpha' \beta' \dots \delta'$ are complementary sets of suffixes (*i.e.*, sets exhausting between them the numbers $01 \dots nuv$), we have

(i) if n is odd: $\tau_{\alpha\beta \dots \delta} = \tau \tau_{\alpha' \beta' \dots \delta'}$;

(ii) if n is even: $\tau_{\alpha\beta \dots \delta} = \tau_{\alpha' \beta' \dots \delta'}$.

A transformation can always be represented by a symbol in which an even number of suffixes appears; the formulation of results is considerably simplified if the convention is made that $\tau_{(r)}$ (whatever the actual parity of r) stands always for the operation expressed with an even number of suffixes. This I suppose done in all that follows, unless the contrary is explicitly stated.

2. *The principal primals*[*].

The principal primals are the manifolds into which the (neighbourhoods of) the fundamental points are transformed. The following examples are typical[†]:

$\tau_{01} W_0 = W_1$.

$\tau_{01} W_2 = \Pi_{012}$, namely the prime $[W_3, W_4, \dots, W_n, W_u, W_v]$.

$\tau_{01} \Pi_{123} = \Pi_{023}$.

---

[*] 2, 445, the "*P*-loci".

[†] The transformations are involutory, so that if $\tau_{(r)} \Gamma = \Delta$, then also $\tau_{(r)} \Delta = \Gamma$.

$\tau_{01}\Pi_{234}=\Pi_{01234}$, namely the quadric cone, vertex $[W_5, \ldots, W_n, W_u, W_v]$, that can be drawn through $W_0, W_1, \ldots, W_4$.

$\tau_{01}\Pi_{134\ldots r}=\Pi_{034\ldots r}$, r being even.

$\tau_{01}\Pi_{23\ldots r}=\Pi_{0123\ldots r}$, where $\Pi_{0123\ldots r}$ is the cone of order $\frac{1}{2}r$ with vertex $[W_{r+1}, \ldots, W_n, W_u, W_v]$ which projects the locus of chordal $[\frac{1}{2}r-2]$'s of the normal rational curve that can be drawn through the projections of the $r+1$ points $W_0, W_1, \ldots, W_r$. The equation of this primal follows shortly.

For the other transformations of the group we have $\tau_{(r)}\Pi_{(s)}=\Pi_{(r)(s)}$, where the common suffixes in the sets (r) (s) are cancelled. If the convention that $\tau_{(r)}$ has always an even number of suffixes is adhered to, then $\Pi_{(s)}$ has always an odd number of suffixes.

The symbol for the general point of the normal rational curve through $W_0, W_1, \ldots, W_n, W_u, W_v$ is $\Sigma(\theta-a_\lambda)^{-1}W_\lambda$, $\lambda=0, \ldots, n$, so that a general point of the locus of chordal $[t-1]$'s of the curve is $\Sigma\phi_\alpha(\theta_\alpha-a_\lambda)^{-1}W_\lambda$, $\alpha=0, \ldots, t-1$. The equations of this manifold, obtained by the elimination of the parameters θ_λ, ϕ_α, are

$$\|y_\lambda,\ a_\lambda y_\lambda,\ a_\lambda^2 y_\lambda,\ \ldots,\ a_\lambda^t y_\lambda,\ 1,\ a_\lambda,\ a_\lambda^2,\ \ldots,\ a_\lambda^{t-1}\|_{2t}=0,$$

the matrix containing $n+1$ rows and $2t+1$ columns, of which t are constants and the rest are linear functions of the coordinates. We can apply this form of equation directly to the cones $\Pi_{(r)}$, and we find the four types below*, for $r=2s+1$.

(i) The symbol contains both suffixes u and v: $\Pi_{uv01\ldots 2s-2}$

$$|y_\gamma,\ a_\gamma y_\gamma,\ \ldots,\ a_\gamma^{s-1}y_\gamma,\ 1,\ a_\gamma,\ \ldots,\ a_\gamma^{s-2}|=0 \quad (\gamma=0, \ldots, 2s-2).$$

(ii) Neither suffix u nor v present: $\Pi_{01\ldots 2s}=\tau_{uv}\Pi_{uv012\ldots 2s}$

$$|y_\gamma,\ a_\gamma y_\gamma,\ \ldots,\ a_\gamma^{s-1}y_\gamma,\ 1,\ a_\gamma,\ \ldots,\ a_\gamma^{s}|=0 \quad (\gamma=0, \ldots, 2s).$$

(iii) u present, but not v: $\Pi_{u01\ldots 2s-1}$.

$$|y_\gamma,\ a_\gamma y_\gamma,\ \ldots,\ a_\gamma^{s-1}y_\gamma,\ a_\gamma,\ a_\gamma^2,\ \ldots,\ a_\gamma^{s}|=0 \quad (\gamma=0, \ldots, 2s-1).$$

(iv) v present, but not u: $\Pi_{v01\ldots 2s-1}=\tau_{uv}\Pi_{u01\ldots 2s-1}$

$$|a_\gamma y_\gamma,\ a_\gamma^2 y_\gamma,\ \ldots,\ a_\gamma^{s}y_\gamma,\ 1,\ a_\gamma,\ \ldots,\ a_\gamma^{s-1}|=0 \quad (\gamma=0, \ldots, 2s-1).$$

---

* It should be noticed that these forms of equations depend only on r and not on n. In each case the numbers of rows and columns are equal, and there is a single equation.

3. *The equations of the transformations.*

Suppose that the equations of the transformation $\tau_{(r)}$ are

$$\rho y_\lambda = f_\lambda(y_0', y_1', \ldots, y_n').$$

Then f_λ must be a function such that the prime $y_\lambda = 0$, *i.e.* $\Pi_{uv\lambda}$, is transformed into $\Pi_{uv\lambda(r)}$ and the point W_λ is transformed into $\Pi_{\lambda(r)}$. Thus f_λ must be some constant multiple of $\Pi_{uv\lambda(r)}/\Pi_{\lambda(r)}$. The transformations are therefore typified by

(i) Suffixes u and v both present*: $\tau_{uv01\ldots r}$

$$\rho y_\lambda' = \Pi_{\lambda 01\ldots r}/\Pi_{uv\lambda 01\ldots r},$$

where it is to be understood that, if $\lambda = 0, \ldots, r$, the repeated suffix is cancelled.

(ii) Neither suffix u nor v present: $\tau_{01\ldots r}$

$$y_\lambda' = a_\lambda \Pi_{uv\lambda 01\ldots r}/\Pi_{\lambda 01\ldots r}.$$

(iii) u present, but not v: $\tau_{u01\ldots r}$

$$\rho y_\lambda' = \Pi_{v\lambda 01\ldots r}/\Pi_{u\lambda 01\ldots r}.$$

(iv) v present, but not u:

$$\rho y_\lambda' = a_\lambda \Pi_{u\lambda 01\ldots r}/\Pi_{v\lambda 01\ldots r}.$$

4. *The fundamental involution in* $[2m-1]$.

The equations of the fundamental involution $\tau = \tau_{01\ldots nuv}$ in $[n]$, when n is odd and equal to $2m-1$, are

$$\rho y_\lambda' = \varpi_{\lambda uv}/\varpi_\lambda \quad (\lambda = 0, \ldots, 2m-1), \tag{1}$$

where $\varpi_{\lambda uv}$ and ϖ_λ are now written instead of $\Pi_{01\ldots\lambda-1,\lambda+1\ldots n}$, and $\Pi_{uv01\ldots\lambda-1,\lambda+1\ldots n}$ respectively. The equation of $\varpi_{\lambda uv}$ is given by the vanishing of the determinant obtained from the matrix

$$|y_\gamma,\ a_\gamma y_\gamma,\ \ldots,\ a_\gamma^{m-2} y_\gamma,\ 1,\ a_\gamma,\ \ldots,\ a_\gamma^{m-1}| \quad (\gamma = 0, \ldots, 2m-1)$$

---

* The constant multiple is here unity. This and the other multiples, which must be powers of a_λ, can be verified by counting the "dimensions" of the terms of the equations in the quantities a_λ.

by suppressing the row for which $\gamma = \lambda$, and that of ϖ_λ by the vanishing of the corresponding determinant in

$$|y_\gamma,\ a_\gamma y_\gamma,\ \ldots,\ a_\gamma^{m-1} y_\gamma,\ 1,\ a_\gamma,\ \ldots,\ a_\gamma^{m-2}|.$$

The equations of the locus of united points of the involution are

$$||\varpi_{\lambda uv},\ y_\lambda \varpi_\lambda||_1 = 0. \tag{2}$$

But we have the following identities:

$$\sum_\lambda a_\lambda^\alpha \Pi_{\lambda uv} \equiv 0 \quad (\alpha = 0, \ldots, m-1)$$

and

$$\Sigma a_\lambda^\alpha y_\lambda \Pi_\lambda \equiv 0,$$

so that the number of independent equations in the matrix above is $m-1$ only. Thus the dimension of the locus is $m+1 = \frac{1}{2}(n+1)$.

There are the further identities

$$\Sigma a_\lambda^\beta y_\lambda \Pi_{\lambda uv} \equiv 0 \quad (\beta = 0, \ldots, m-2),$$

and from these, using equations 2, we derive the equations

$$\Sigma a_\lambda^\beta y_\lambda^2 \Pi_\lambda = 0,$$

$m-1$ in number, and independent, so that part of the locus represented by them is the united locus. Similarly we obtain

$$\Sigma a_\lambda^\beta y_\lambda^{-1} \Pi_{\lambda uv} = 0.$$

These equations are different from the first set, but are contained with them in the matrix below. Hence

The equations of the manifold ψ', the hyperelliptic analogue in* $[2m-1]$ *of the Weddle surface in* $[3]$, *are expressible in the form:*

$$||y_\gamma^2,\ a_\gamma y_\gamma^2,\ \ldots,\ a_\gamma^{m-2} y_\gamma^2,\ y_\gamma,\ a_\gamma y_\gamma,\ \ldots,\ a_\gamma^{m-1} y_\gamma,\ 1,\ a_\gamma,\ \ldots,\ a_\gamma^{m-2}||_{2m-1} = 0.$$

Thus the order of ψ' is the sum of the products $m-1$ at a time of the set of numbers consisting of $m-1$ 2's, m 1's, and $m-1$ zeros†, namely

$$2^{m-1} + m(m-1)\,2^{m-2} + \binom{m}{2}\binom{m-1}{2}2^{m-3} + \ldots + \binom{m}{m-2}\binom{m-1}{m-2}2 + \binom{m}{m-1}.$$

---

* Coble calls this manifold W_m (2, 457), and finds the orders to be 19 and 96 when $m = 3$ and 4 (2, 489, 3, 218). See also 1, 329, § 19, for the case $m = 3$.

† Segre, *Encyk. der Math. Wiss.*, III C 7, 929.

5. *The fundamental involutions in* [2*m*].

In [$2m$] there are $2m+3$ fundamental involutions $\tau_0, \ldots, \tau_{2m}, \tau_u, \tau_v$, each of which gives rise to a manifold of united points. We find the equations of the manifolds associated with τ_u and τ_v, which will be of more symmetrical form than the others. For τ_v we have, in the notation of § 4,

$$\rho y_\lambda' = \varpi_{\lambda v}/\varpi_{\lambda u} \quad (\lambda = 0, \ldots, 2m),$$

where $\varpi_{\lambda v}$ is written for $\Pi_{u01\ldots\lambda-1, \lambda+1, \ldots, 2m}$, so that the locus Γ_v of united points is

$$\| \varpi_{\lambda v}, \; y_\lambda \varpi_{\lambda u} \| = 0,$$

and from these we derive, as before, the equations

$$\| y_\lambda^2, \; a_\lambda y_\lambda^2, \; \ldots, \; a_\lambda^{m-1} y_\lambda^2, \; y_\lambda, \; a_\lambda y_\lambda, \; \ldots, \; a_\lambda^{m-1} y_\lambda, \; a_\lambda, \; a_\lambda^2, \; \ldots, \; a_\lambda^m \| = 0.$$

Γ_v is thus of dimension m.

The equations of τ_u are

$$\rho y_\lambda' = a_\lambda \varpi_{\lambda u}/\varpi_{\lambda v},$$

and those of the manifold Γ_u are

$$\| y_\lambda^2, \; a_\lambda y_\lambda^2, \; \ldots, \; a_\lambda^{m-1} y_\lambda^2, \; a_\lambda y_\lambda, \; a_\lambda^2 y_\lambda, \; \ldots, \; a_\lambda^m y_\lambda, \; a_\lambda, \; a_\lambda, \; \ldots, \; a_\lambda^m \| = 0.$$

Using the Segre formula again, we find that the order of these manifolds is

$$2^m + m^2 \, 2^{m-1} + \binom{m}{2}^2 2^{m-2} + \ldots + \binom{m}{m-1}^2 2 + 1.$$

As a slight check on the forms of these equations it may be noticed that Γ_v transforms into itself under τ_{uv}, and therefore under all the transformations $\tau_{(r)}$. And this we expect, since, if P_v is a point of Γ_v, then

$$\tau_{uv} P_v = \tau_u \tau_v P_v = \tau_u P_v,$$

while $\tau_v(\tau_u P_v) = (\tau_u P_v)$, so that $\tau_{uv} P_v$ is also a point of Γ_v. Further, a point P_{uv} common to Γ_u and Γ_v must be self-conjugate in τ_{uv}. There are only 2^{2m} such points, namely $\Sigma(\pm \sqrt{a_\lambda} W_\lambda)$. These are, of course, additional to the points W_λ themselves, which are multiple on all the manifolds.

6. *The generalized Segre-type and hyperelliptic Kummer-type manifolds.*

The hyperelliptic 64_8 was obtained by projecting the Segre-type manifold Ω in [13] from an osculating [5] (**1**, § 13); in the same way the general

Kummer-type configuration can be obtained by projecting a general Segre-type manifold from a high-tangent space. The Segre-type manifold is defined thus: we take $\binom{2m+1}{m}$ points

$$A_{a_0 a_1 \dots a_m} \equiv (-1)^{m+1} A_{a_0' a_1' \dots a_m'},$$

where $a_0, a_1, \dots, a_m, a_0', a_1', \dots, a_m' = 0, 1, \dots, 2m-1, u, v$, and are all different*, connected by the $\binom{2m+2}{m}$ syzygies

$$\Sigma A_{ab_1 b_2 \dots b_m} = 0, \tag{1}$$

each of which involves $m+2$ points (*cf.* **1**, § 2). With these syzygies, and no others, it can be proved that $\binom{2m}{m} - \binom{2m}{m-2}$ of the points are independent†. As in **1**, it is more convenient to work only with those points among whose suffixes v is included while u is not (**1**, 297), and we write

$$V_{l_1 \dots l_m} \equiv A_{vl_1 \dots l_m} \equiv (-1)^{m+1} A_{ul_1' \dots l_m'} \equiv (-1)^{m+1} U_{l_1' \dots l_m'}, \tag{2}$$

where $l_1, \dots, l_m, l_1', \dots, l_m' = 0, \dots, 2m-1$.

These points are connected by the $\binom{2m}{m-2}$ syzygies

$$\Sigma V_{\lambda\lambda' l_3 \dots l_m} = 0, \tag{3}$$

all of which are independent, and these are equivalent to

$$\Sigma U_{\mu\mu' l_3' \dots l_m'} = 0 \quad \text{or} \quad \Sigma V_{\lambda_1' \dots \lambda_m'} = 0. \tag{4}$$

Ω is the locus of points

$$\Sigma y_{\lambda_1} y_{\lambda_2} \dots y_{\lambda_m} V_{\lambda_1 \lambda_2 \dots \lambda_m},$$

so that, when $\Sigma y_\lambda W_\lambda$ is taken to be a point of $[2m-1]$, we obtain a representation of Ω by primals of order m having $(m-1)$-ple points at the $2m+1$ points $W_0, \dots, W_{2m-1}, W_u = \Sigma W_\lambda$‡.

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* We assume always that different symbols denote distinct suffixes; following the convention used in 1 on page 297, we use a_i, a_i, b_i, etc., when the range of the suffixes includes u and v, and λ_i, l_i, etc., when the range is from 0 to $2m-1$ only. The summation signs always refer to the Greek suffixes, the Latin suffixes being regarded as fixed in each sum, while the Greek suffixes take all the remaining values in turn.

† A proof from another point of view is given later.

‡ We can prove, by a process similar to that adopted in §7, that the freedom of these primals is $\binom{2m}{m} - \binom{2m}{m-2} - 1$.

The hyperelliptic Kummer-type manifold, Υ, is the projection of Ω from the $(m-2)$-th tangent space $\mathfrak{V}$ at a general point $\Sigma a_{\lambda_1} \dots a_{\lambda_m} V_{\lambda_1 \dots \lambda_m}$[*]. Thus in the space containing Υ we have the additional syzygies

$$\Sigma a_\lambda a_{\lambda'} V_{\lambda\lambda' l_3 \dots l_m} = 0, \tag{5}$$

which, since the sets of syzygies 3 and 4 are equivalent, are equivalent to

$$\Sigma a_\lambda^{-1} a_{\lambda'}^{-1} U_{\lambda\lambda' l_3 \dots l_m} = 0, \quad \textit{i.e.} \text{ to } \quad \Sigma a_\lambda^{-1} a_{\lambda'}^{-1} V_{\mu_1 \dots \mu_m} = 0. \tag{6}$$

Υ is represented on $[2m-1]$ by the primals Σ' of order m which have an additional $(m-1)$-ple point at $W_v \equiv \Sigma a_\lambda W_\lambda$, *i.e.* by a system that is transformed into itself by all the operations $\tau_{(r)}$ associated with the $2m+2$ points $W_0, \dots, W_{2m-1}, W_u, W_v$. Σ' contains $W_i W_j$ as an $(m-2)$-ple line, $W_i W_j W_k$ as an $(m-3)$-ple plane, ..., $W_{i_0} W_{i_1} \dots W_{i_r}$ as an $(m-r-1)$-ple $[r]$ (*cf.* **1**, § 16). As in the cases $m = 2, 3$, every point of Υ is the projection of two points of Ω, namely of the points represented on the $[2m-1]$ by P' and $\tau P'$ (*cf.* **1**, 335). That is, every line which meets $\mathfrak{V}$ and meets Ω once necessarily meets Ω in a further point. The united points of this involution on Ω are obtained from the tangent lines which meet $\mathfrak{V}$; these project into the manifold Ψ[†] which is represented on the $[2m-1]$ by the locus of united points of the involution $(P', \tau P')$, namely by the Weddle-type manifold Ψ'.

7. *The* $(2^{2m})_{2m+2}$ *configuration in* $[2^m-1]$.

The $[m+r-1]$, $[W_0, W_1, \dots, W_{m+r-1}]$ in the $[2m-1]$ represents on Υ a manifold of which the prime sections are represented by primals of order m with $m+r$ fixed $(m-1)$-ple points, and $m-r+2$ fixed collinear $(r-1)$-ple points, namely the points where the $[m+r-1]$ meets the $[m-r]$'s through the sets of $m-r+1$ of the points $W_{m+r}, W_{m+r+1}, \dots, W_{2m-1}, W_u, W_v$. The reciprocal inverse of this system in the $[m+r-1]$, with regard to the simplex of $(m-1)$-ple points, is a system of primals Σ'' of order r with $2m+2$ fixed $(r-1)$-ple points all lying on a normal rational curve c^{m+r-1}. It is to be proved that

The freedom of primals of order r in $[m+r-1]$ *with* $2m+2$ *fixed* $(r-1)$*-ple points lying on a normal rational curve is* $\binom{m}{1} + \binom{m}{2} + \dots + \binom{m}{r} = \nu_r$[‡].

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[*] **1**, §§ 13, *et seq.*

[†] Ψ is the manifold denoted by K_m in **2**, 427, § 16.

[‡] A different proof is given in **3**, 196, Theorem 6.

Denote by $v_{m,r}$ a primal of this type, $v_{m,r}=0$ being its equation, and by $f_{m,r}$ its freedom. Let the curve r^{m+r-1} degenerate into a curve r^{m+r-2} lying in the prime $z=0$, and a line l meeting it, but not lying in $z=0$. Of the $2m+2$ points take $2m$ on r^{m+r-2} and the other two on l, and consider first a manifold $v_{m-1,r}$ lying in $z=0$, with its multiple points on r^{m+r-2}. The equation of any primal in the $[m+r-1]$ of order r and through $v_{m-1,r}$ may be written in the form

$$v_{m-1,r}+z\phi=0,$$

where ϕ is any primal of order $r-1$ in the $[m+r-1]$. Now in order that this primal should belong to the family $v_{m,r}$, ϕ must have $(r-2)$-ple points at the $2m$ points on r^{m+r-2} and $(r-1)$-ple points at the two points on l. It is therefore a cone with l as vertex. Projecting ϕ from l we obtain a manifold of the type $v_{m-1,r-1}$, with its $2m$ multiple points assigned on the projection of r^{m+r-2}. On the other hand, the section by $z=0$ of any manifold $v_{m,r}$ which has its multiple points on $r^{m+r-2}+l$ is $v_{m-1,r}$ (that is, provided that two of the multiple points lie on l, the other cases can be dealt with by more elementary methods and will be found to produce families of lower freedom), and thus

$$f_{m,r}=f_{m-1,r}+f_{m-1,r-1}+1.$$

When $r=1$, the manifolds are unrestricted primes, so that $f_{r,1}=r$; when $r=2$, they are quadrics, with $2m+2$ fixed points, so that

$$f_{m,2}=\binom{m}{1}+\binom{m}{2}.$$

From this the result follows.

Thus, corresponding to the $[m+r-1]$, we obtain a manifold on Ω which lies in a space of dimension ν_r. We can build up the complete system of such spaces as in the table in **1**, 325, and we find that it forms a N_{2m+2} *system in* $[\nu]$, *consisting of m sets of N spaces of dimension* ν_r, *one set for each of r from* 0 *to* $m-1$, *arranged in pairs that are duals, namely those of dimensions* ν_r *and* ν_{m-r-1}, *where*

$$N=2^{2m},\quad \nu=\nu_m=2^m-1,\quad \textit{and}\quad \nu_r=\binom{m}{1}+\binom{m}{2}+\ldots+\binom{m}{r}.$$

To keep the notation directly in contact with that used in the representation, we use symbols with a single suffix and any odd numbers of suffixes to denote primes*, and we have then to distinguish between the cases of m odd and m even, since the points of the figure have numbers of suffixes of the same parity as m. The range of the suffixes in the table below is in each row 0, ..., $2m-1$, u, v, and in the case of elements which involve $m+1$ suffixes, any two symbols which have complementary sets of suffixes represent the same element. The families of spaces are:

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* *Cf.* **1**, foot-note p. 323. The primes are represented by the points W_a, and by the primals $\Pi_{(r)}$, the symbols for which always have odd numbers of suffixes.

(i) m even (typified by the ordinary 16_6 figure).

Primes: $\Delta_{\alpha}^{(m-1)},\ \Delta_{\alpha\beta\gamma}^{(m-1)},\ \ldots,\ \Delta_{\alpha_0\alpha_1\ldots\alpha_m}^{(m-1)} = \Delta_{\alpha_0'\alpha_1'\ldots\alpha_m'}^{(m-1)}.$

$[\nu-m-1]$'s: $\Delta^{(m-2)},\ \Delta_{\alpha\beta}^{(m-2)},\ \ldots,\ \Delta_{\alpha_0\ldots\alpha_{m-1}}^{(m-2)}.$

$\left[\nu-\binom{m}{1}-\binom{m}{2}-\ldots-\binom{m}{2i}-1\right]$'s:

$\Delta_{\alpha}^{(m-2i-1)}$, etc., as for primes.

$\left[\nu-\binom{m}{1}-\binom{m}{2}-\ldots-\binom{m}{2i+1}-1\right]$'s:

$\Delta^{(m-2i-2)}$, etc., as for $[\nu-m-1]$'s.

$[m]$'s: $\Delta_{\alpha}^{(1)}$, etc., as for primes.

Points*: $\Delta^{(0)},\ \Delta_{\alpha\beta}^{(0)},\ \ldots,\ \Delta_{\alpha_0\ldots\alpha_{m-1}}^{(0)}.$

(ii) m odd (typified by the 64_8 configuration).

The first and last rows are:

Primes: $\Delta^{(m-1)}$, etc.

Points: $\Delta_{\alpha}^{(0)}$, etc.

The fundamental distinction between the two cases is that when m is odd the pairs of spaces that are duals have the same numbers of suffixes, and one of the sets form their own duals, while when m is even the pairs which are duals have different numbers of suffixes. Each space $\Delta_{(r)}^{(i)}$ lies in $2m+2$ of the spaces $\Delta_{(s)}^{(i+1)}$ and contains $2m+2$ of the spaces $\Delta_{(s)}^{(i+1)}$, namely in each case those of the families whose sets of suffixes differ from the set (r) only in containing one extra member or one member fewer. In particular the $[m]\,\Delta_{(r)}^{(1)}$ contains $2m+2$ of the points $\Delta_{(s)}^{(0)}$, these points lying on a normal rational curve. (*Cf.* **1**, 301, at the top, and 318.)

By working downwards we find that the space $\Delta^{(k)}$ contains all the spaces $\Delta_{(r)}^{(k-h)}$ in the symbols for which there are h suffixes or fewer; *i.e.*, the number of spaces $\Delta_{(r)}^{(k-h)}$ lying in a space $\Delta_{(s)}^{(k)}$ is

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* The relation of this notation to the $A_{\alpha\beta\ldots}$ notation is similar to that explained in **1**, 323, foot-note.

$$\binom{2m+2}{h}+\binom{2m+2}{h-2}+\binom{2m+2}{h-4}+\ldots+\binom{2m+2}{\mu},$$

where $\mu=0$, or 1, according as h is even or odd. This number may also be written as $\sum_{i=0}^{h}\binom{2m+1}{i}$. In particular, each prime $\Delta_{(s)}^{(m-1)}$ contains all the points $\Delta_{(r)}^{(0)}$ except a certain set of $\binom{2m+1}{m}$ of them.

8. *The transformations in* $[2^m-1]$ *corresponding to* $\tau_{\lambda\mu}$ *in* $[2m-1]$.

The inversion τ_{uv} in $[2m-1]$ interchanges the points $\Sigma y_\lambda W_\lambda$ and $\Sigma a_\lambda y_\lambda^{-1} W_\lambda$, and thus corresponds to the transformation T_{uv} in $[2^m-1]$ that interchanges $\Sigma y_{\lambda_1}\ldots y_{\lambda m} V_{\lambda_1\ldots\lambda m}$ and $\Sigma a_{\lambda_1}\ldots a_{\lambda m} y_{\lambda_1}^{-1}\ldots y_{\lambda m}^{-1} V_{\lambda_1\ldots\lambda m}$. The effect of T_{uv} is to interchange the parts played by the suffixes u and v (*cf.* **1**, 334), and it is thus an involutory collineation. It is to be proved that

T_{uv} is the harmonic inversion in the two $[2^{m-1}-1]$'s $\mathfrak{T}_{uv}$ and $\mathfrak{T}'_{uv}$ defined by the sets of points

$$b_{\lambda_1}\ldots b_{\lambda m} V_{\lambda_1\ldots\lambda m}+b_{\lambda_1'}\ldots b_{\lambda m'} V_{\lambda_1'\ldots\lambda m'}$$

and

$$b_{\lambda_1}\ldots b_{\lambda m} V_{\lambda_1\ldots\lambda m}-b_{\lambda_1'}\ldots b_{\lambda m'} V_{\lambda_1'\ldots\lambda m'},$$

where $b_\lambda=\sqrt{a_\lambda}$*.

The actual syzygies that we obtain below connecting the points of these two sets do not make it clear that a set lies in a $[2^{m-1}-1]$, but T_{uv} is an involutory collineation, and therefore necessarily an harmonic inversion. Thus, if we establish that $\mathfrak{T}_{uv}$ and $\mathfrak{T}'_{uv}$ do meet the join of any pair of conjugate points in the $[2^m-1]$, and harmonically separate these points, it follows, since they must be of the same dimension, that their dimension is $2^{m-1}-1$.

Write

$$B_{\lambda_1\ldots\lambda_m}\equiv B_{\lambda_1'\ldots\lambda_m'}\equiv b_{\lambda_1}\ldots b_{\lambda_m} V_{\lambda_1\ldots\lambda_m}+b_{\lambda_1'}\ldots b_{\lambda_m'} V_{\lambda_1'\ldots\lambda_m'} \tag{1}$$

and $B'_{\lambda_1\ldots\lambda_m}$ for the corresponding point of $\mathfrak{T}'_{uv}$, then

$$\Sigma b_\lambda b_{\lambda'} B_{\lambda\lambda' l_3\ldots l_m}=0, \tag{2}$$

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* It does not matter which sign is taken for each of the roots, since, if an odd number of the signs is changed, $\mathfrak{T}_{uv}$ and $\mathfrak{T}'_{uv}$ are interchanged, while an even number of changes leaves them unaltered.

since

$$\Sigma\, b_\lambda b_{\lambda'} b_{l_3} \dots b_{l_m} B_{\lambda\lambda' l_3 \dots l_m} \equiv \Sigma\, a_\lambda a_{\lambda'} a_{l_3} \dots a_{l_m} V_{\lambda\lambda' l_3 \dots l_m} + \kappa \Sigma\, V_{\lambda_1' \dots \lambda_m'},$$

where $\kappa = b_0 b_1 \dots b_{2m-1}$, and these two sums provide syzygies (6.5 and 6.4). Similarly we have

$$\Sigma\, b_\lambda^{-1} b_{\lambda'}^{-1} B_{\lambda\lambda' l_3 \dots l_m} = 0. \tag{3}$$

To prove the harmonic property we have to show that

$$\Sigma\, x_{\lambda_1 \dots \lambda_m} V_{\lambda_1 \dots \lambda_m} \quad \text{and} \quad \Sigma\, a_{\lambda_1'} \dots a_{\lambda_m'} x_{\lambda_1 \dots \lambda_m} V_{\lambda_1' \dots \lambda_m'}$$

are conjugate; this follows since the two points can be written as

$$\frac{1}{2\kappa} \{ (b_{\lambda_1'} \dots b_{\lambda_m'} x_{\lambda_1 \dots \lambda_m} + b_{\lambda_1} \dots b_{\lambda_m} x_{\lambda_1' \dots \lambda_m'}) B_{\lambda_1 \dots \lambda_m}$$
$$\pm (b_{\lambda_1'} \dots b_{\lambda_m'} x_{\lambda_1 \dots \lambda_m} - b_{\lambda_1} \dots b_{\lambda_m} x_{\lambda_1' \dots \lambda_m'}) B'_{\lambda_1 \dots \lambda_m} \}.$$

Explicit forms for the other harmonic inversions are not obtainable so easily, but it can be verified, for example, that $\mathfrak{I}_{v0}$ is the harmonic inversion in the spaces defined by the sets of points

$$c_{\gamma_1} \dots c_{\gamma_m} A_{\gamma_1 \dots \gamma_m v} \pm c_{\gamma_1'} \dots c_{\gamma_{m-1}'} c_0 A_{\gamma_1' \dots \gamma_{m-1}' uv},$$

where $c_\gamma^2 = a_\gamma - a_0$, and $c_0^2 = -a_0$.

Of more interest than the collineations would be the correlations of the figure into itself (the transformations S_g, **1**, 332). All these can be obtained from the collineations when one of them is known, say, in particular, the polarity with regard to the quadric S when m is odd (as in the case of the 64_8), and with regard to one the null systems S_λ when m is even (as in the case of the 16_6).

9. *The section of the* $(2^{2m})_{2m+2}$ *figure by* $\mathfrak{I}_{uv}$.

The Barrau configuration* consists of the set of 2^{2m} points in $[2^m-1]$ with symbols $\Sigma \pm g_\delta Z_\delta$, $\delta = 0, \dots, 2^m-1$, where the sets g_δ are all permutations of the same 2^m numbers, and, of the possible sets of signs, only those belonging to a certain matrix are used. For this paragraph we assume, what is almost certainly true, that the $(2^{2m})_{2m+2}$ figure is a special

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* *Cf.* **1**, 336.

Barrau figure in which the g_δ are not 2^{2m} independent quantities, but all depend on the $2m$ numbers a_λ.

One of the harmonic inversions of the Barrau figure into itself is that with regard to the two $[2^{m-1}-1]$'s, $z_0=0, \ldots, z_{h-1}=0$, and $z_h=0, \ldots, z_{2h-1}=0$ $(h=2^{m-1})$. If we write the coordinates of the points symbolically as $\begin{pmatrix} + & + \\ + & - \end{pmatrix}\begin{pmatrix} \mathbf{g} & \mathbf{g}' \\ \mathbf{g}' & \mathbf{g} \end{pmatrix}$, where each symbol stands for a "Barrau" matrix with 2^{m-1} elements in each row and column, then the pairs of conjugate points in this collineation are $(+\mathbf{g}++\mathbf{g}', +\mathbf{g}-+\mathbf{g}')$ and $(+\mathbf{g}'++\mathbf{g}, +\mathbf{g}'-+\mathbf{g})$, and the points in which the joins of conjugate pairs meet the first of the two spaces above form the two independent Barrau sets $+\mathbf{g}$ and $+\mathbf{g}'$ (with $z_h=0, \ldots, z_{2h-1}=0$). *Thus the section of the Barrau configuration of 2^{2m} points by one of the $[2^{m-1}-1]$'s of a pair with regard to which the figure is its own harmonic inverse is two Barrau 2^{2m-2} configurations, depending on the same group of transformations, but defined by two distinct sets of 2^{m-1} numbers.*

Applying this to the hyperelliptic configuration, we find

(i) *The section of the $(2^{2m})_{2m+2}$ configuration by $\mathfrak{T}_{uv}$ is two $(2^{2m-2})_{2m}$ configurations*. One of them is derived from all the points $A_{(r)}$ with suffixes which include either u or v, but not both, and the other from the remaining points.*

These two are closely associated, since between them they depend on only the $2m$ numbers a_λ, whereas if the difference between them were the same as that between the general Barrau configurations, they would depend on two sets of $2m-2$ numbers. This association may be expressed as

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* It should be possible to establish this result directly from the syzygies 2 and 3, but I have not been able to do so even in the case $m=3$. In that case we obtain from the 64_8 in the solid $\mathfrak{T}_{uv}$ thirty-two points forming two Kummer 16_6 configurations, which are associated in some way. The thirty-two points are constructible from the ten points $B_{\lambda\mu\nu} \equiv B_{\lambda'\mu'\nu'}$, $\lambda, \mu, \nu, \lambda', \mu', \nu' = 1, 2, 3, 4, 5, 6$, connected by the two sets of six syzygies

$$\Sigma\, b_\lambda b_\mu B_{\lambda\mu n} = 0,$$

$$\Sigma\, b_\lambda^{-1} b_\mu^{-1} B_{\lambda\mu n} = 0.$$

The six points $B_{\rho\sigma 5}$, $\rho, \sigma = 1, \ldots, 4$, are connected by the relation

$$\Sigma\,(b_\rho b_\sigma b_5 - b_{\rho'} b_{\sigma'} b_6) B_{\rho\sigma 5} = 0,$$

but I have not been able to reduce the second syzygy connecting these to the form

$$\Sigma\, g_\rho g_\sigma (b_\rho b_\sigma b_5 - b_{\rho'} b_{\sigma'} b_6) B_{\rho\sigma 5} = 0,$$

which is required for the standard set of syzygies for the 16_6 configuration (*cf.* 1, 315).

follows:

(ii) *Each* $[m-1]$ *of one configuration contains one point of the other*.*

What we have in effect in the $[2^{m-1}-1]$ is a $(2^{2m-1})_{2m+1}$ configuration†.

To prove these results it is most convenient to take the points of the figure in $[2^m-1]$ to be A, $A_{o\beta}$, ..., $A_{(2r)}$, ... $(\alpha, \beta = 0, \ldots, 2m-1, u, v)$, and to denote by $B_{\lambda\mu\ldots\rho}$ and $C_{\lambda\mu\ldots\sigma}$ the points in which $\mathfrak{J}_{uv}$ meets the lines $A_{\lambda\mu\ldots\sigma u}A_{\lambda\mu\ldots\rho v}$ and $A_{\lambda\mu\ldots\sigma}A_{\lambda\mu\ldots\sigma uv}$ respectively $(\lambda, \mu, \ldots, \rho, \sigma = 0, \ldots, 2m-1$; B has an odd number of suffixes, and C has an even number). The $[m]$'s $\Delta^{(1)}_{\lambda\mu\ldots\rho}$, $\Delta^{(1)}_{\lambda\mu\ldots\rho uv}$ both contain the points $A_{\lambda\mu\ldots\rho u}$ and $A_{\lambda\mu\ldots\rho v}$, and therefore contain also $B_{\lambda\mu\ldots\rho}$; $\Delta^{(1)}_{\lambda\mu\ldots\sigma u}$ and $\Delta^{(1)}_{\lambda\mu\ldots\sigma v}$ similarly contain $C_{\lambda\mu\ldots\sigma}$. Further, it can be proved (*e.g.*, by taking a space $\Delta^{(1)}_{(2m+1)}$ which can be defined by a set of the points $V_{\lambda_1\ldots\lambda_m}$ of § 8) that the complete intersection of $\mathfrak{J}_{uv}$ with one of these $[m]$'s is the single point. This provides an alternative definition of the points B and C, namely: $B_{\lambda\mu\ldots\rho}$ is the point in which $\mathfrak{J}_{uv}$ meets $\Delta^{(1)}_{\lambda\mu\ldots\rho}$ and $\Delta^{(1)}_{\lambda\mu\ldots\rho uv}$, and $C_{\lambda\mu\ldots\sigma}$ is the point in which $\mathfrak{J}_{uv}$ meets $\Delta^{(1)}_{\lambda\mu\ldots\sigma u}$ and $\Delta^{(1)}_{\lambda\mu\ldots\sigma v}$.

The $\left[m+\binom{m}{2}\right]\Delta^{(2)}_{\lambda\mu\ldots\sigma}$ contains all the $[m]$'s $\Delta^{(1)}_{\lambda\mu\ldots\sigma\beta}$, and therefore contains the $2m$ points $B_{\lambda\mu\ldots\sigma\rho}$, and also $C_{\lambda\mu\ldots\sigma}$. From this the proof may be completed. It will be sufficient to indicate the line of proof when $m=4$.

The 256_{10} figure in $[15]$ consists of:

256 points: A, $A_{\alpha\beta}$, $A_{\alpha\beta\gamma\delta}$,

256 $[4]$'s: $\Delta^{(1)}_{\alpha}$, $\Delta^{(1)}_{\alpha\beta\gamma}$, $\Delta^{(1)}_{\alpha\beta\gamma\delta\epsilon} \equiv \Delta^{(1)}_{\alpha'\beta'\gamma'\delta'\epsilon'}$,

256 $[10]$'s: $\Delta^{(2)}$, $\Delta^{(2)}_{\alpha\beta}$, $\Delta^{(2)}_{\alpha\beta\gamma\delta}$,

256 primes: $\Delta^{(3)}_{\alpha}$, $\Delta^{(3)}_{\alpha\beta\gamma}$, $\Delta^{(3)}_{\alpha\beta\gamma\delta\epsilon}$ $(\alpha, \beta, \ldots = 0, \ldots, 7, u, v)$.

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* Since there are only $2m$ numbers involved, it seems likely that the point lies on the normal rational curve in the $[m-1]$ on which lie the $2m$ points of the first configuration (*cf.* **1**, 318 at the top, and 301 at the top).

† The existence of a configuration of this type is not unexpected, since the configuration can be obtained from the operation of the group of transformations on two points which lie on the manifold which gives the hyperelliptic configurations [**1**, 337 (iv)], and each in a $[m-1]$ of the configuration defined by the other. For example, if the group associated with a 16_6 figure in [3] operates on a point which lies in the conic in one of the planes we obtain altogether a 32_7 configuration, in which the seven points in each plane lie on a conic, and the seven planes through each point envelope a quadric cone.

From these we obtain in the [7] $\mathfrak{J}_{uv}$

$$64 \text{ points } B_\lambda,\ B_{\lambda\mu\nu},$$

$$64 \text{ points } C,\ C_{\lambda\mu},\ C_{\lambda\mu\nu\sigma} \quad (\lambda,\ \mu,\ \ldots = 0,\ \ldots,\ 7).$$

$\Delta^{(2)}$ contains the ten [4]'s $\Delta_\lambda^{(1)}$, $\Delta_u^{(1)}$, $\Delta_v^{(1)}$, and therefore contains the eight points B_λ, and the single point C. From the syzygies it can be proved that the eight points lie in a solid, say $D^{(1)}$; then, either C lies in this solid, or else the [7] $\mathfrak{J}_{uv}$ meets the [10] $\Delta^{(2)}$ in a [4]. The prime $\Delta_0^{(3)}$ contains the twenty-nine points B_0, B_ρ, $B_{0\rho\rho'}$, and the eight points C, $C_{0\rho}$ ($\rho, \rho' = 1, \ldots, 7$), the first of which sets lies in a [6] and the second in a [3]. Now $\mathfrak{J}_{uv}$ is a [7], and it does not lie entirely in $\Delta_0^{(3)}$ (for, if it did, so would $\mathfrak{J}'_{uv}$, and the harmonic inversion would break down), and thus must meet $\Delta_0^{(3)}$ only in the [6] containing the twenty-nine points; *i.e.*, the [3] containing the eight points lies in the [6].

Denote this [6] by $D_0^{(2)}$: it contains the thirty-seven points above. Each [6] $\Delta_\rho^{(2)}$ contains a similar set of thirty-seven points, and we find that the points common to the eight spaces are the 8, B_λ, and the single point C. But the meet of the eight spaces is the solid $D^{(1)}$; we have now proved that it also contains C.

Thus, if we denote by $D^{(r-1)}_{\lambda\mu\ldots\rho}$, $E^{(r-1)}_{\lambda\mu\ldots\sigma}$ the sections by $\mathfrak{J}_{uv}$ of $\Delta^{(r)}_{\lambda\mu\ldots\rho}$ and $\Delta^{(r)}_{\lambda\mu\ldots\sigma u}$, we find as an alternative form for (ii): $D^{(r)}_{\lambda\mu\ldots\rho}$ contains the space $E^{(r-2)}_{\lambda\mu\ldots\rho}$ and $E^{(r-1)}_{\lambda\mu\ldots\sigma}$ contains $D^{(r-2)}_{\lambda\mu\ldots\sigma}$.

10. *The figure corresponding to the system in* [2m].

The system of reciprocal inversions defined by $2m+3$ points in [2m] has no such simple representation as has the corresponding figure in a space of an odd number of dimensions; for such a representation we have to find a family of primals, such as the family Σ' of § 6, which transform into each other under the operations $\tau_{(r)}$. Only in the plane is there a simple family of this type; the five base points in the plane define a family of cubics, passing through them, which invert into each other. These curves represent the prime sections of a del Pezzo quartic surface in [4] (the surface common to two general quadrics), and the five principal transformations τ_0, τ_1, τ_2, τ_u, τ_v correspond to the harmonic inversions of the surface into itself, with regard to the vertices and solids of the self polar simplex. But, given seven points in [4], for example, the simplest self-reciprocal system consists of primals of order 5 with triple points at the seven base points, and such a system has freedom at least* 20. Likewise, the simplest system in $[2m]$ consists of primals of order $2m+1$ on which the $2m+3$ base points are $(2m-1)$-ple.

References.

1. Room, " A generalization of the Kummer 16_6 configuration (I) ", *Proc. London Math. Soc.* (2), 37 (1933), 292–337.
2. Coble, " A generalization of the Weddle surface, of its Cremona group, and of its parametric representation in terms of hyperelliptic theta functions ", *American J. of Math.*, 52 (1930), 439–500.
3. Coble and Chanler, " The geometry of the Weddle manifold W_p ", *American J. of Math*. 57 (1935), 183–218.
4. Coble, " The geometry of the Weddle manifold W_p ", *Bull. American Math. Soc.*, 41 (1935), 209–222.

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* The freedom of all quintic primals in [4] is 125, and assigning seven triple points imposes at most 105 conditions.

A TABLE OF PARTITIONS (II)

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[Received 7 September, 1936.—Read 12 November, 1936.]

1. In an earlier paper*, I have given a table for $p(n)$, the number of unrestricted partitions of n, for values of $n \leqslant 300$. Here I give a table for the next 300 values. The work was undertaken in order to verify the value of $p(599)$ obtained by Lehmer† by means of the remarkable asymptotic formula due to Hardy and Ramanujan.

The method used in the work of computation was described in § 5 of the previous paper*. The work was considerably simplified by typing out the entries on uniform slips of paper about half of an inch in width. The work of arranging the slips for addition was done by my younger brother Dinanath. To avoid all chances of error several checks were applied. The idea of applying these checks occurred quite late, however. When the work had been carried to $p(362)$, it was discovered that an error had crept in at $p(328)$. The rest of the entries up to $p(362)$ were then corrected with the help of formula (E) below.

2. Let $p(n)$ denote the correct and $q(n)$ an incorrect value of the number of unrestricted partitions of n. Let $p(n) = q(n) + w$; then, if there has been no further error in the calculation of the succeeding entries, we have

$$p(n+m) = q(n+m) + w \,.\, p(m), \tag{E}$$

where $m \geqslant 0$ and w is a positive or negative integer. It will be noticed that my value of $p(599)$ agrees with that given by Lehmer. This removes the slight element of doubt expressed by Lehmer in a foot-note to his paper.

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* *Proc. London Math. Soc.* (2), 39 (1935), 142–149.

† *Journal London Math. Soc.*, 11 (1936), 114–118.

n	$p(n)$	n	$p(n)$
301	9 93097 23924 03501	351	298 33006 30627 58076
302	10 65733 12325 48839	352	318 55597 37883 29084
303	11 43554 20778 22104	353	340 12281 00485 77428
304	12 26921 80192 29465	354	363 11751 20481 10005
305	13 16221 78950 57704	355	387 63253 29190 29223
306	14 11866 26652 80005	356	413 76618 09333 42362
307	15 14295 27388 57194	357	441 62298 19293 58437
308	16 23978 65358 29663	358	471 31406 42683 98780
309	17 41418 01331 47295	359	502 95756 65060 00020
310	18 67148 82996 00364	360	536 67907 03106 91121
311	20 01742 67625 76945	361	572 61205 88980 37559
312	21 45809 60373 52891	362	610 89840 37518 84101
313	23 00000 66554 87337	363	651 68887 99972 06959
314	24 65010 61508 30490	364	695 14371 34589 46040
315	26 41580 76335 66326	365	741 43315 98840 81684
316	28 30502 03409 96003	366	790 73811 96494 11319
317	30 32618 19898 42964	367	843 25078 85625 28427
318	32 48829 33514 66654	368	899 17534 83960 88349
319	34 80095 48694 40830	369	958 72869 79123 38045
320	37 27440 57767 48077	370	1022 14122 83673 45362
321	39 91956 55269 99991	371	1089 65764 44243 99782
322	42 74807 80359 54696	372	1161 53783 48499 62850
323	45 77235 85435 78028	373	1238 05779 41191 25085
324	49 00564 36352 37875	374	1319 51059 97274 73500
325	52 46204 42288 28641	375	1406 20744 65614 84054
326	56 15660 21128 74289	376	1498 47874 35905 81081
327	60 10534 98396 66544	377	1596 67527 44907 56791
328	64 32537 46091 14550	378	1701 16942 79758 13525
329	68 83488 59460 73850	379	1812 35649 97394 72950
330	73 65328 78618 50339	380	1930 65607 23504 65812
331	78 80125 53026 66615	381	2056 51347 53366 33805
332	84 30081 56362 25119	382	2190 40133 24237 65131
333	90 17543 49805 49623	383	2332 82119 85438 92336
334	96 45011 01922 02760	384	2484 30529 42654 18180
335	103 15146 63217 35325	385	2645 41834 06887 63701
336	110 30786 04252 92772	386	2816 75950 32179 42792
337	117 94949 15461 13972	387	2998 96444 77364 52194
338	126 10851 78337 96355	388	3192 70751 84335 32826
339	134 81918 06233 01520	389	3398 70404 13581 60275
340	144 11793 65278 73832	390	3617 71276 38676 04423
341	154 04359 73795 76030	391	3850 53843 46674 29186
342	164 63747 91657 61044	392	4098 03453 56265 94791
343	175 94355 98104 22753	393	4361 10617 07622 84114
344	188 00864 70522 92980	394	4640 71312 46996 23515
345	200 88255 62876 83159	395	4937 87309 67881 91655
346	214 61829 97432 86299	396	5253 66512 44169 75163
347	229 27228 68712 17150	397	5589 23320 25954 04488
348	244 90453 74553 82406	398	5945 79011 47078 74597
349	261 57890 73511 44125	399	6324 62148 25042 94325
350	279 36332 84837 02152	400	6727 09005 17410 41926

n	$p(n)$	n	$p(n)$
401	7154 64022 26539 42321	451	1 42573 13615 53474 04229
402	7608 80284 33398 79269	452	1 51112 26207 19173 13678
403	8091 20027 64844 65581	453	1 60152 90524 45537 15585
404	8603 55175 93486 55060	454	1 69723 95104 64580 40965
405	9147 67906 88591 17602	455	1 79855 91645 39582 67598
406	9725 51251 37420 21729	456	1 90581 04044 26519 31034
407	10339 09726 71239 47241	457	2 01933 37928 51146 88629
408	10990 60006 37759 26994	458	2 13948 90703 27330 69132
409	11682 31627 71923 17780	459	2 26665 62143 58313 45565
410	12416 67740 31511 90382	460	2 40123 65561 39251 92081
411	13196 25896 69254 35702	461	2 54365 39575 85741 99975
412	14023 78888 35188 47344	462	2 69435 60521 29549 94471
413	14902 15629 03099 48968	463	2 85381 55524 19619 86287
414	15834 42088 44881 87770	464	3 02253 16287 25766 36605
415	16823 82278 71392 35544	465	3 20103 13615 29932 90544
416	17873 79296 96898 76004	466	3 38987 12724 95254 32549
417	18987 96426 73316 64557	467	3 58963 89376 81628 76613
418	20170 18301 88059 33659	468	3 80095 46876 31205 98477
419	21424 52136 02556 36320	469	4 02447 33986 17114 75160
420	22755 29021 65800 25259	470	4 26088 63801 56524 13417
421	24167 05302 14413 63961	471	4 51092 33635 50960 99864
422	25664 64021 38377 14846	472	4 77535 45970 81641 15593
423	27253 16454 62304 21739	473	5 05499 30531 42046 29558
424	28938 03725 70847 98150	474	5 35069 67535 16072 62125
425	30724 98514 70950 51099	475	5 66337 12186 58055 99675
426	32620 06861 74102 32189	476	5 99397 20478 23018 52926
427	34629 70071 39035 75934	477	6 34350 76365 37870 28583
428	36760 66724 18315 27309	478	6 71304 20389 67318 07232
429	39020 14800 02372 59665	479	7 10369 79823 66282 38005
430	41415 73920 71023 58378	480	7 51666 00419 49931 25591
431	43955 47717 05181 16534	481	7 95317 79841 47582 32180
432	46647 86328 42292 67991	482	8 41457 02874 28236 49455
433	49501 89040 94051 50715	483	8 90222 78495 19280 88294
434	52527 07072 91082 40605	484	9 41761 78911 49976 98055
435	55733 46514 46362 86656	485	9 96228 80660 85734 11012
436	59131 71430 91696 18645	486	10 53787 07886 24553 46513
437	62733 07137 60430 79215	487	11 14608 77893 64264 84248
438	66549 43656 69662 97367	488	11 78875 49115 57358 02646
439	70593 39364 65621 35510	489	12 46778 71600 12729 19665
440	74878 24841 94708 86233	490	13 18520 40161 22702 33223
441	79418 06934 64434 02240	491	13 94313 50322 44478 16939
442	84227 73040 77294 99781	492	14 74382 57204 03639 53132
443	89322 95632 13536 45667	493	15 58964 37499 49778 06173
444	94720 37025 78934 71820	494	16 48308 54706 61724 38760
445	1 00437 54417 17528 47604	495	17 42678 27774 77609 81187
446	1 06493 05190 52391 18581	496	18 42351 03350 31598 91466
447	1 12906 52519 91961 03354	497	19 47619 31798 76580 64007
448	1 19698 71278 27202 05954	498	20 58791 47204 28849 01563
449	1 26891 54269 09814 18000	499	21 76192 51543 92874 61625
450	1 34508 18800 15729 23840	500	23 00165 03257 43239 95027

n	$p(n)$	n	$p(n)$
501	24 31070 10430 92873 27876	551	362 70732 34887 12851 28752
502	25 69288 28837 70982 89281	552	382 37963 52094 31772 37554
503	27 15220 65077 22453 13220	553	403 10029 41640 92445 87122
504	28 69289 85080 24006 62045	554	424 92419 40439 77208 72600
505	30 31941 28246 44131 32742	555	447 90905 29390 70180 09131
506	32 03644 27509 62020 70012	556	472 11555 61416 03980 40338
507	33 84893 35624 43498 44341	557	497 60750 60435 44327 57376
508	35 76209 57999 81546 53671	558	524 45197 94774 63136 27407
509	37 78141 92403 51715 37110	559	552 71949 28608 51377 15955
510	39 91268 75895 81641 18300	560	582 48417 55275 18680 50007
511	42 16199 39350 46400 98482	561	613 82395 16416 17753 18496
512	44 53575 69957 09409 47378	562	646 82073 11154 29433 80454
513	47 04073 82100 21758 42062	563	681 56060 99653 62361 72174
514	49 68405 97048 81263 19775	564	718 13408 05683 95962 03570
515	52 47322 31892 38787 93976	565	756 63625 22960 90554 43637
516	55 41612 98201 31139 36133	566	797 16708 30334 31305 21599
517	58 52110 10892 13016 61040	567	839 83162 21064 08800 02321
518	61 79690 07823 80848 08000	568	884 74026 51749 58179 81253
519	65 25275 80654 96527 88115	569	932 00902 16664 36541 87580
520	68 89839 17540 95423 85648	570	981 75979 53603 39713 12388
521	72 74403 58255 17333 77346	571	1034 12067 87556 37109 92446
522	76 80046 62371 60943 32553	572	1089 22626 18906 73929 56037
523	81 07902 91152 74741 24146	573	1147 21795 63086 04086 58491
524	85 59167 03843 77167 36150	574	1208 24433 49032 05642 37125
525	90 35096 69082 90059 15201	575	1272 46148 84055 19665 62301
526	95 37015 92199 02400 21538	576	1340 03339 93172 51535 97473
527	100 66318 59178 75001 06586	577	1411 13233 41252 99127 46558
528	106 24471 98151 20750 20731	578	1485 93925 46811 98901 97615
529	112 13020 59252 16951 11580	579	1564 64424 96608 28174 48060
530	118 33590 13800 63004 16410	580	1647 44698 70734 03875 84240
531	124 87891 73736 75218 03652	581	1734 55718 88238 00960 95248
532	131 77726 32347 45246 12308	582	1826 19512 83905 68239 19887
533	139 04989 27324 55673 35012	583	1922 59215 27207 81295 26277
534	146 71675 27284 07832 32475	584	2023 99122 95062 90955 80175
535	154 79883 42897 07610 68959	585	2130 64752 10488 42071 60129
536	163 31822 63872 97014 93803	586	2242 82898 59904 68310 34631
537	172 29817 23061 72107 20599	587	2360 81701 02330 51309 45921
538	181 76312 89039 08614 35034	588	2484 90706 84458 62614 13858
539	191 73882 88568 74544 84110	589	2615 40941 76124 06422 65710
540	202 25234 60440 91512 66221	590	2752 64982 41493 41732 06642
541	213 33216 42221 17085 70598	591	2896 97032 61882 71229 74972
542	225 00824 91557 73561 65493	592	3048 73003 26997 53660 31783
543	237 31212 43734 63701 38355	593	3208 30596 12029 53866 10807
544	250 27695 07282 12791 46420	594	3376 09391 59006 51695 60935
545	263 93760 99500 53829 68154	595	3552 50940 81500 27025 58187
546	278 33079 23887 98493 85687	596	3737 98862 12843 68527 09430
547	293 49508 91513 39863 74841	597	3932 98942 18788 32511 57044
548	309 47108 88521 74751 01876	598	4137 99241 96672 78329 78027
549	326 30147 92016 32340 60900	599	4353 50207 84031 73482 70000
550	344 03115 36720 50509 43160	600	4580 04788 00814 43085 53622

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TWO TABLES OF PARTITIONS

By G. N. Watson.

[Received October 30, 1936.—Read November 12, 1936.]

Let
$$\chi(x) = \prod_{n=0}^{\infty} (1+x^{2n+1}),$$
and let the expansions of $\chi(x)$ and $1/\chi(-x)$ in powers of x be
$$\chi(x) = \sum_{n=0}^{\infty} q_0(n)\, x^n, \qquad \frac{1}{\chi(-x)} = \sum_{n=0}^{\infty} q(n)\, x^n.$$

Then $q_0(n)$ is evidently the number of partitions of n into odd integers without repetitions, while $q(n)$ is the number of unrestricted partitions of n into odd integers. Further, from Euler's relation*
$$\frac{1}{\chi(-x)} = \prod_{n=1}^{\infty} (1+x^n),$$
it is clear that $q(n)$ is also the number of partitions of n into integers without repetitions.

I have computed tables of $q_0(n)$ and $q(n)$ up to $n = 400$. My computations were considerably simplified by making use of certain formulae of elliptic functions in conjunction with the existing table of $p(n)$, the number of unrestricted partitions of n, up to $n = 200$ which was constructed by MacMahon and published by Hardy and Ramanujan†.

If
$$f(-x) = \prod_{n=1}^{\infty} (1-x^n),$$

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* L. Euler, *Introductio in analysin infinitorum*, 1 (1748), 271.

† G. H. Hardy and S. Ramanujan, "Asymptotic formulae in combinatory analysis", *Proc. London. Math. Soc.* (2), 17 (1918), 75–115. This paper also included a table of $q(n$ up to $n = 100$, which was computed by H. B. C. Darling. Mr. Darling had extended his table up to $n = 200$, and he kindly lent me a copy of it not long before his death.

it is evident that the expansion of $1/f(-x)$ in powers of x is

$$\frac{1}{f(-x)} = \sum_{n=0}^{\infty} p(n)\, x^n.$$

The formulae of elliptic functions which I used are the following:

$$\chi^6(x) = \frac{2x^{\frac{1}{4}}}{\sqrt{(kk')}}, \qquad \chi^6(-x) = \frac{2x^{\frac{1}{4}}k'}{\sqrt{k}},$$

$$f^6(-x^2) = \frac{2kk'K^3}{\pi^3\sqrt{x}}, \qquad f^6(-x^4) = \frac{k^2 k'^{\frac{1}{2}} K^3}{2\pi^3 x};$$

in these formulae, the standard notations of elliptic functions are used, except that, to avoid confusion, x is used in place of the more usual q to denote the expression exp $(-\pi K'/K)$. These formulae are to be found in any treatise on elliptic functions.

From the formulae quoted, it is evident that

$$\chi^2(x) f^2(-x^4) = \frac{K\sqrt{k}}{\pi x^{\frac{1}{4}}} = \frac{\vartheta_2(0,\, x)\,\vartheta_3(0,\, x)}{2x^{\frac{1}{4}}},$$

$$\frac{f^2(-x^2)}{\chi^2(-x)} = \frac{K\sqrt{k}}{\pi x^{\frac{1}{4}}} = \frac{\vartheta_2(0,\, x)\,\vartheta_3(0,\, x)}{2x^{\frac{1}{4}}};$$

and, from the expressions for the theta functions as infinite products, it follows immediately that

$$\frac{\vartheta_2(0,\, x)\,\vartheta_3(0,\, x)}{2x^{\frac{1}{4}}} = \left\{\frac{\vartheta_2(0,\, \sqrt{x})}{2x^{\frac{1}{8}}}\right\}^2 = \left\{\sum_{n=0}^{\infty} x^{\frac{1}{2}n(n+1)}\right\}^2.$$

We now see at a glance that

$$\chi(x) = \left\{\sum_{n=0}^{\infty} x^{\frac{1}{2}n(n+1)}\right\}\left\{\sum_{n=0}^{\infty} p(n)\, x^{4n}\right\}$$

and that
$$\frac{1}{\chi(-x)} = \left\{\sum_{n=0}^{\infty} x^{\frac{1}{2}n(n+1)}\right\}\left\{\sum_{n=0}^{\infty} p(n)\, x^{2n}\right\}.$$

Since $\Sigma\, x^{\frac{1}{2}n(n+1)}$ is a power series with gaps, the use of the last two formulae makes it possible to derive the values of $q_0(n)$ and $q(n)$ with extreme rapidity from a table of values of $p(n)$. My tables were constructed in this way, and I checked them by evaluating

$$\sum_{n=0}^{400} (-)^n q_0(n)\, q(400-n),$$

which is evidently the coefficient of x^{400} in the expansion of the product

$$\left\{\sum_{n=0}^{\infty} (-)^n q_0(n)\, x^n\right\}\left\{\sum_{n=0}^{\infty} q(n)\, x^n\right\}.$$

so that the coefficient in question is zero. This check seemed adequate since it involved all the entries in my tables except $q(398)$; and I carried out the complete computation of $q(398)$ on two separate occasions.

It is to be remarked that the formulae given above show that a table of $p(n)$ up to $n=m$ makes it possible to compute $q_0(n)$ and $q(n)$ up to $n=4m+3$ and $n=2m+1$ respectively. My tables could therefore be extended considerably by means of Gupta's table* of $p(n)$ up to $n=300$ (of which I first saw the manuscript at the time when I was computing my own tables) and his more recent table† of $p(n)$ up to $n=600$; but, being of the opinion that my tables go as far as is likely to be required, I have resisted the temptation to make such extensions.

A few words about my object in constructing the tables may not be out of place. I had in mind the construction of the "irrational modular equation" of order n, when $n+1$ is a multiple of 8 and is fairly large. Examples of such modular equations for small values of n have been constructed by Weber‡ and by some subsequent writers. These modular equations are equations connecting A and B, where

$$2A = x^{-\frac{1}{24}(n+1)}\chi(x)\chi(x^n) + (-)^{\frac{1}{8}(n+1)}x^{-\frac{1}{24}(n+1)}\chi(-x)\chi(-x^n) + \frac{(-)^{\frac{1}{8}(n+1)}2x^{\frac{1}{12}(n+1)}}{\chi(-x^2)\chi(-x^{2n})},$$

$$B = x^{-\frac{1}{12}(n+1)}\chi(-x^2)\chi(-x^{2n}) + \frac{2x^{\frac{1}{24}(n+1)}}{\chi(-x)\chi(-x^n)} + \frac{(-)^{\frac{1}{8}(n+1)}x^{\frac{1}{24}(n+1)}}{\chi(x)\chi(x^n)},$$

their degrees in A and B depending upon the values of n and of its divisors.

Finally it should be stated that the functions $q_0(n)$ and $q(n)$ do not seem to possess any simple congruence properties akin to those of $p(n)$. The only congruence properties that I have noticed are consequences of the result that

$$\sum_{n=0}^{\infty} q(n)x^n + \sum_{m=-\infty}^{\infty} (-)^m x^{\frac{1}{2}m(3m+1)} = \frac{1+\chi(-x)f(-x)}{\chi(-x)}$$

$$= \frac{1+\vartheta_4(0,x)}{\chi(-x)} = 2\,\frac{1+\sum_{n=1}^{\infty}(-)^n x^{n^2}}{\chi(-x)},$$

whence it follows that $q(n)$ is even except when n is of the form $\frac{1}{2}m(3m+1)$, $(m=0, \pm 1, \pm 2, \ldots)$; and that $q_0(n)$ and $p(n)$ are of the same parity.

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* H. Gupta, *Proc. London Math. Soc.* (2), 39 (1935), 142–149.

† H. Gupta, *Proc. London Math. Soc.* (2), 42 (1937), 546–549.

‡ H. Weber, *Lehrbuch der Algebra*, 3 (Braunschweig, 1908), 265–274.

n	$q_0(n)$	$q(n)$	n	n	$q_0(n)$	$q(n)$	n	n	$q_0(n)$	$q(n)$	n
0	1	1	0								
1	1	1	1	51	107	4097	51	101	2720	4 83330	101
2	0	1	2	52	117	4582	52	102	2871	5 25016	102
3	1	2	3	53	125	5120	53	103	3040	5 70078	103
4	1	2	4	54	133	5718	54	104	3223	6 18784	104
5	1	3	5	55	144	6378	55	105	3405	6 71418	105
6	1	4	6	56	157	7108	56	106	3590	7 28260	106
7	1	5	7	57	168	7917	57	107	3797	7 89640	107
8	2	6	8	58	178	8808	58	108	4022	8 55906	108
9	2	8	9	59	192	9792	59	109	4244	9 27406	109
10	2	10	10	60	209	10880	60	110	4472	10 04544	110
11	2	12	11	61	223	12076	61	111	4727	10 87744	111
12	3	15	12	62	236	13394	62	112	5000	11 77438	112
13	3	18	13	63	255	14848	63	113	5272	12 74118	113
14	3	22	14	64	276	16444	64	114	5553	13 78304	114
15	4	27	15	65	294	18200	65	115	5863	14 90528	115
16	5	32	16	66	312	20132	66	116	6194	16 11388	116
17	5	38	17	67	335	22250	67	117	6528	17 41521	117
18	5	46	18	68	361	24576	68	118	6871	18 81578	118
19	6	54	19	69	385	27130	69	119	7247	20 32290	119
20	7	64	20	70	408	29927	70	120	7652	21 94432	120
21	8	76	21	71	437	32992	71	121	8057	23 68800	121
22	8	89	22	72	471	36352	72	122	8474	25 56284	122
23	9	104	23	73	501	40026	73	123	8933	27 57826	123
24	11	122	24	74	530	44046	74	124	9420	29 74400	124
25	12	142	25	75	568	48446	75	125	9912	32 07086	125
26	12	165	26	76	609	53250	76	126	10422	34 57027	126
27	14	192	27	77	647	58499	77	127	10975	37 25410	127
28	16	222	28	78	686	64234	78	128	11564	40 13544	128
29	17	256	29	79	732	70488	79	129	12162	43 22816	129
30	18	296	30	80	784	77312	80	130	12778	46 54670	130
31	20	340	31	81	833	84756	81	131	13446	50 10688	131
32	23	390	32	82	881	92864	82	132	14157	53 92550	132
33	25	448	33	83	939	1 01698	83	133	14878	58 02008	133
34	26	512	34	84	1004	1 11322	84	134	15623	62 40974	134
35	29	585	35	85	1065	1 21792	85	135	16430	67 11480	135
36	33	668	36	86	1126	1 33184	86	136	17284	72 15644	136
37	35	760	37	87	1199	1 45578	87	137	18152	77 55776	137
38	37	864	38	88	1279	1 59046	88	138	19053	83 34326	138
39	41	982	39	89	1355	1 73682	89	139	20021	89 53856	139
40	46	1113	40	90	1433	1 89586	90	140	21045	96 17150	140
41	49	1260	41	91	1523	2 06848	91	141	22092	103 27156	141
42	52	1426	42	92	1621	2 25585	92	142	23174	110 86968	142
43	57	1610	43	93	1717	2 45920	93	143	24335	118 99934	143
44	63	1816	44	94	1814	2 67968	94	144	25565	127 69602	144
45	68	2048	45	95	1925	2 91874	95	145	26819	136 99699	145
46	72	2304	46	96	2048	3 17788	96	146	28117	146 94244	146
47	78	2590	47	97	2166	3 45856	97	147	29510	157 57502	147
48	87	2910	48	98	2286	3 76256	98	148	30977	168 93952	148
49	93	3264	49	99	2425	4 09174	99	149	32478	181 08418	149
50	98	3658	50	100	2574	4 44793	100	150	34037	194 06016	150

n	$q_0(n)$	$q(n)$	n	n	$q_0(n)$	$q(n)$	n
151	35698	207 92120	151	201	3 26197	5173 61670	201
152	37449	222 72512	152	202	3 39953	5494 62336	202
153	39246	238 53318	153	203	3 54357	5834 73184	203
154	41106	255 40982	154	204	3 69381	6195 03296	204
155	43087	273 42421	155	205	3 84898	6576 67584	205
156	45175	292 64960	156	206	4 00988	6980 87424	206
157	47315	313 16314	157	207	4 17826	7408 90786	207
158	49534	335 04746	158	208	4 35371	7862 12446	208
159	51895	358 39008	159	209	4 53497	8341 94700	209
160	54376	383 28320	160	210	4 72297	8849 87529	210
161	56923	409 82540	161	211	4 91945	9387 48852	211
162	59567	438 12110	162	212	5 12409	9956 45336	212
163	62370	468 28032	163	213	5 33564	10558 52590	213
164	65313	500 42056	164	214	5 55491	11195 55488	214
165	68343	534 66624	165	215	5 78393	11869 49056	215
166	71483	571 14844	166	216	6 02245	12582 38720	216
167	74806	610 00704	167	217	6 26890	13336 40710	217
168	78299	651 39008	168	218	6 52435	14133 83026	218
169	81889	695 45358	169	219	6 79107	14977 05768	219
170	85612	742 36384	170	220	7 06857	15868 61606	220
171	89552	792 29676	171	221	7 35539	16811 16852	221
172	93679	845 43782	172	222	7 65273	17807 51883	222
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174	1 02342	962 14550	174	224	8 28540	19973 57056	224
175	1 06996	1026 14114	175	225	8 61889	21149 65120	225
176	1 11872	1094 20549	176	226	8 96441	22392 29960	226
177	1 16899	1166 58616	177	227	9 32460	23705 13986	227
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180	1 33358	1412 31780	180	230	10 48746	28103 94454	230
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183	1 51915	1707 27424	183	233	11 78881	33284 23936	233
184	1 58688	1818 10744	184	234	12 25403	35207 06304	234
185	1 65673	1935 82642	185	235	12 73839	37236 75326	235
186	1 72923	2060 84096	186	236	13 24158	39379 02688	236
187	1 80542	2193 58315	187	237	13 76183	41639 89458	237
188	1 88505	2334 51098	188	238	14 30065	44025 67324	238
189	1 96727	2484 10816	189	239	14 86137	46543 00706	239
190	2 05253	2642 88462	190	240	15 44381	49198 87992	240
191	2 14204	2811 38048	191	241	16 04580	52000 62976	241
192	2 23561	2990 16608	192	242	16 66921	54955 97248	242
193	2 33215	3179 84256	193	243	17 31778	58073 01632	243
194	2 43227	3381 04630	194	244	17 99101	61360 27874	244
195	2 53737	3594 44904	195	245	18 68692	64826 71322	245
196	2 64703	3820 75868	196	246	19 40760	68481 72604	246
197	2 76026	4060 72422	197	247	20 15678	72335 19619	247
198	2 87776	4315 13602	198	248	20 93430	76397 50522	248
199	3 00085	4584 82688	199	249	21 73805	80679 55712	249
200	3 12928	4870 67746	200	250	22 57009	85192 80128	250

n	$q_0(n)$	$q(n)$	n	n	$q_0(n)$	$q(n)$	n
251	23 43473	89949 26602	251	301	141 43101	12 07425 10607	301
252	24 33185	94961 58208	252	302	146 38805	12 69025 30816	302
253	25 25904	1 00243 00890	253	303	151 51662	13 33663 83848	303
254	26 21877	1 05807 47264	254	304	156 81873	14 01485 59930	304
255	27 21578	1 11669 59338	255	305	162 29196	14 72642 18618	305
256	28 24974	1 17844 71548	256	306	167 94525	15 47292 17536	306
257	29 31833	1 24348 95064	257	307	173 79234	16 25601 42890	307
258	30 42436	1 31199 20928	258	308	179 83605	17 07743 43642	308
259	31 57269	1 38413 23582	259	309	186 07429	17 93899 64242	309
260	32 76326	1 46009 65705	260	310	192 51642	18 84259 79304	310
261	33 99377	1 54008 01856	261	311	199 17779	19 79022 32212	311
262	35 26698	1 62428 82560	262	312	206 06195	20 78394 72390	312
263	36 58841	1 71293 59744	263	313	213 16645	21 82593 94656	313
264	37 95820	1 80624 90974	264	314	220 50208	22 91846 82870	314
265	39 37360	1 90446 44146	265	315	228 08578	24 06390 52286	315
266	40 83793	2 00783 03620	266	316	235 92107	25 26472 94208	316
267	42 35730	2 11660 75136	267	317	244 00630	26 52353 25352	317
268	43 93156	2 23106 91192	268	318	252 35352	27 84302 35904	318
269	45 55821	2 35150 17984	269	319	260 98054	29 22603 40224	319
270	47 24097	2 47820 61070	270	320	269 89217	30 67552 32574	320
271	48 98608	2 61149 71540	271	321	279 08711	32 19458 41664	321
272	50 79385	2 75170 53882	272	322	288 57810	33 78644 88192	322
273	52 66173	2 89917 72486	273	323	298 38518	35 45449 47722	323
274	54 59346	3 05427 58738	274	324	308 51395	37 20225 12608	324
275	56 59619	3 21738 19904	275	325	318 96309	39 03340 57172	325
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281	70 15885	4 38791 78240	281	331	389 15013	51 99993 15040	331
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283	75 32903	4 86265 19094	283	333	415 67333	57 18543 13990	333
284	78 05213	5 11828 44672	284	334	429 56720	59 96286 87918	334
285	80 86491	5 38689 49522	285	335	443 91437	62 87095 13216	335
286	83 77243	5 66911 97084	286	336	458 72374	65 91563 14788	336
287	86 78385	5 96562 52987	287	337	473 99572	69 10312 43770	337
288	89 90032	6 27710 98024	288	338	489 75003	72 43991 92576	338
289	93 11875	6 60430 42088	289	339	506 01532	75 93279 10200	339
290	96 44512	6 94797 40554	290	340	522 80104	79 58881 23110	340
291	99 88954	7 30892 09120	291	341	540 10914	83 41536 64940	341
292	103 45290	7 68798 39744	292	342	557 96152	87 42016 06890	342
293	107 13259	8 08604 19136	293	343	576 38873	91 61123 94270	343
294	110 93525	8 50401 45750	294	344	595 40240	95 99699 92704	344
295	114 87140	8 94286 47940	295	345	615 00555	100 58620 35461	345
296	118 94267	9 40360 04868	296	346	635 22169	105 38799 77632	346
297	123 14650	9 88727 65938	297	347	656 08490	110 41192 60918	347
298	127 48980	10 39499 71456	298	348	677 60860	115 66794 79970	348
299	131 98445	10 92791 76298	299	349	699 79635	121 16645 56454	349
300	136 63248	11 48724 72064	300	350	722 67480	126 91829 24648	350

n	$q_0(n)$	$q(n)$	n	n	$q_0(n)$	$q(n)$	n
351	746 28130	132 93477 19190	351	376	1644 35295	414 85655 73659	376
352	770 63052	139 22769 71520	352	377	1696 25615	433 84690 00206	377
353	795 72796	145 80938 18816	353	378	1749 72341	453 68080 55808	378
354	821 60323	152 69267 15868	354	379	1804 82430	474 39464 06976	379
355	848 29651	159 89096 56578	355	380	1861 59616	496 02629 40968	380
356	875 82514	167 41824 09148	356	381	1920 06235	518 61523 80864	381
357	904 19649	175 28907 55072	357	382	1980 28173	542 20259 26436	382
358	933 44250	183 51867 38752	358	383	2042 33138	566 83119 27092	383
359	963 60772	192 12289 32216	359	384	2106 25380	592 54565 72864	384
360	994 71227	201 11827 04478	360	385	2172 07527	619 39246 14094	385
361	1026 76439	210 52205 02772	361	386	2239 86180	647 42001 16480	386
362	1059 80015	220 35221 50410	362	387	2309 69820	676 67872 37064	387
363	1093 86863	230 62751 50210	363	388	2381 63120	707 22110 32064	388
364	1128 99166	241 36750 01278	364	389	2455 69265	739 10183 03854	389
365	1165 18046	252 59255 33946	365	390	2531 95577	772 37784 71936	390
366	1202 47505	264 32392 51488	366	391	2610 51230	807 10844 79444	391
367	1240 92817	276 58376 86784	367	392	2691 41605	843 35537 42947	392
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370	1363 47784	316 77000 44480	370	395	2948 77465	961 85031 43424	395
371	1406 85466	331 38466 77248	371	396	3039 71937	1004 83241 32444	396
372	1451 56078	346 65347 41118	372	397	3133 33260	1049 67982 04736	397
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ADDENDUM

SOME RELATIONS BETWEEN THE DE LA VALLEE POUSSIN AND ABEL METHODS OF SUMMABILITY.

J. M. HYSLOP.

[*Proc. London Math. Soc.* (2), 40 (1936), 449–467.]

It has been pointed out to me by Prof. J. M. Whittaker that Theorem VIII of my paper is a particular case of a result due to T. H. Gronwall, *Annals of Math.*, 33 (1932), 101–117.

BIBL.
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